MATRIX OPERATORS ON SOME NEW MATRIX DOMAINS IN DIFFERENCE SEQUENCE SPACES

A THESIS

submitted as a partial fulfillment of the requirements for the award of the Master Degree in

MATHEMATICS

by

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under the supervision of

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DEDICATION

To my parents, for their all time support and prayers.

My wife and my sons Amar and Abdullah for their support and love.

CERTIFICATE

This is to certify that the work presented in this thesis entitled

" MATRIX OPERATORS ON SOME NEW MATRIX DOMAINS IN DIFFERENCE SEQUENCE SPACES "

is an authentic and original research work carried out by

Mr. Omar Hussain Salem Al-Sabri

under my supervision and submitted to the Department of Mathematics, Faculty of Education and Science - Rada'a, Al Baydha University, as a partial fulfillment of the requirements for the award of the Master Degree in Mathematics.

To the best of my knowledge and belief, the present work has fulfilled the prescribed conditions given in the academic ordinances and regulations of Al Baydha University and it has not been submitted before to other university for the award of any degree.

(SUPERVISOR)

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(February, 2022)

EXAMINING COMMITTEE REPORT

After reading the thesis entitlded " **Matrix Operators On Some New Matrix Domains In Difference Sequence Spaces** " and examining the researcher **Mr. Omar Hussain Salem Al-Sabri** in its contents, we find it is adequate as a thesis for award of the **Master Degree** in **Mathematics**.

On Saturday, 26 March 2022

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OMAR H. AL-SABRI

ABSTRACT

In this thesis, we have introduced the new spaces of λ-difference sequences by means of the classical sequence spaces of bounded and convergent difference sequences. Further, we have studied the isomorphic, algebraic and topological properties of our new spaces with their Schauder bases and inclusion relations. Besides, we have obtained their dual spaces. Moreover, we have concluded some new results characterizing certain classes of matrix operators acting on, into and between our spaces. Furthermore, several important and new results have been obtained and discussed as particular cases of our main results.

PREFACE

Functional analysis was founded by S. Banach, M. Fréchet, H. Hahn, D. Hilbert, F. Hausdorff, F. Riesz and others. These names have become synonymous with the tools of functional analysis as a branch of mathematical analysis including a study of abstract spaces and their operators and transformations, such as the Fourier transform, which provides a general framework for finding solutions of various problems in the applied mathematics and physics. The most general types of these abstract spaces are those infinite dimensional spaces, and the sequence spaces are the most important spaces of these types, known as infinite dimensional analysis. Especially in summability theory which encompasses a variety of fields and has many applications in various subjects. For instance, in numerical analysis, approximation theory, operator theory and the theory of differential equations and orthogonal series with their special functions. Thus, many mathematicians have done a lot of work in this field of sequence spaces and studied their matrix transformations which have been applied in all other areas of mathematics. So, we have selected this area for research and study.

In the present thesis, the main contribution is to introduce some new sequence spaces and study their topological properties, inclusion relations, Schauder bases, dual spaces and certain classes of matrix operators on these new sequence spaces.

My thesis is divided into five chapters and the main results in the last four chapters have been published in two research papers as mentioned at the beginning of each chapter, and the first paper has been presented in the 2$^{\text{nd}}$ conference of Albaydha University (2021). The materials of this exposition are organized as follows:

Chapter 1 is an introductory chapter to present a short survey on some basic definitions, notations and preliminary results which are already known in the literature of the theory of sequence spaces and their matrix transformations with the historical and theoretical background of this area.

In **Chapter 2**, we have introduced the new λ-difference sequence spaces of bounded, convergent and null sequences, and studied their algebraic, topological and isomorphic properties with constructing their Schauder bases.

Chapter 3 is devoted to establish some interesting inclusion relations between our new spaces and the classical sequence spaces, and some particular cases of equalities and strict inclusions will be discussed with important examples.

In **Chapter 4**, we have deduced the Köthe-Toeplitz duals of our new λ-sequence spaces defined in terms of difference sequences.

Chapter 5 is devoted to characterize the related classes of matrix operators acting on, into and between our new spaces, and some known or new results will be deduced as particular cases.

For more utility, we hope for the reader's familiarity with the basic concepts of our subject. Thus, for further knowledge in our notions, we refer the reader to [56] for basic idea of sequences and series, to [17, 34] for elementary concepts of functional analysis, to [10, 16, 35, 67] for the notions of sequence spaces and to [9, 52] for the particular sequence spaces of λ-type.

The obtained facts are those remarks, examples, lemmas or theorems, which are presented throughout this thesis as paragraphs and every paragraph is specified with triple decimal numbering. The first number indicates the chapter, the second represents the section, and the third refers to the number of current paragraph. For example, the form 3.2.1 refers to the first paragraph (remark, example, lemma or theorem) appearing in Section 2 of Chapter 3.

At the end of this monograph, we have given an exhaustive list of the relevant references to the literature presented in this thesis. All results stated without proof are cited and can be found in the references given before the statements.

Besides, I hope for the reader's forgiveness if there is any typing mistake which may appear here or there throughout this simple work. Despite all efforts to make this thesis free from such errors, there may be some still left.

Finally, I have to acknowledge with sincere thanks those persons whom I have met due to my good fortune. First of all, I am irrevocably indebted to the great man and supervisor Prof. Abdullah K. Noman, Department of Mathematics, Faculty of Education and Science, Albaydha University, for his wonderful guidance, boundless support and incessant help which have inculcated in me the interest and motivation to undertake the research in this field. It has been only the providential that has enabled me to get a lifetime opportunity to work under his scholarly supervision.

OMAR HUSSAIN AL-SABRI

(Januery, 2022)

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Chapter 1

INTRODUCTION

1 INTRODUCTION

The most general types of abstract spaces are those infinite dimensional spaces, and the sequence spaces are the most important spaces of these types, known as the infinite dimensional analysis. In this first chapter, we display a historical and theoretical survey of our field concerning the theory of sequence spaces and their matrix transformations with a short note on some basic definitions, notations and preliminary results which are already known in the literature of this field. This introductory chapter is divided into three sections, the first is devoted to the theoretical background, the second is for the research methodology and the last is to present some preliminary results which will be needed in the next chapters.

1.1 Theoretical Background

In this section, we display a historical background for the theory of sequence spaces and matrix transformations, and we give a short survey on some basic concepts of this area with certain previous studies.

1.1.1 Historical Survey

In 1828, Abel wrote ”the divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever”. Such was the authority of Cauchy, Abel and their successors that divergent series were, in Hardy’s words, ‘gradually banished from analysis’. It was not until 1890, when Cesàro published a paper about the multiplication of series that, according to Hardy, ‘for the first time a ”theory of divergent series” is formulated explicitly’. Indeed, Cesàro’s idea was quickly picked up and applied fruitfully to Fourier series. One of the early successes was a beautiful theorem of Fejer's which we state after first giving the simplest form of Cesàro's idea [16].

The summability theory has been originated from the attempts made by the mathematicians to give limits to the divergent sequences and series. In particular, theory of sequence spaces and matrix transformations is a significant area of research in summability theory as a part of functional analysis, and so many mathematicians have done a lot of work in this field. In fact, the most important methods of summability are given by infinite matrices and matrix transformations. So, our concern is with those infinite matrices that map a sequence space into another one. Such matrices arise naturally from the infinite-dimensions of sequence spaces [10].

The historical roots of interest in matrix transformations was stimulated by special results in the summability theory which were obtained by E. Cesàro, L. Euler, N. Nörlund, F. Riesz and others. The earliest idea of summability were perhaps contained in a letter written by Leibnitz to C. Wolf in 1713, the sum of the oscillatory series $1-1+1-\cdots$ as given by Leibnitz was in 1880. After that, Frobenius introduced the method of summability by arithmetic mean which has later been generalized by Cesàro in 1890 as the (C,α) method of summability [67]. With the emergence of functional analysis, sequence spaces were studied with greater insight and motivation and the earliest applications of functional analysis to summability was made by S. Banach, H. Hahn, S. Mazur, G. Köthe and O. Toeplitz. In 1911, the celebrated mathematician Toeplitz determined the necessary and sufficient conditions for an infinite matrix to be regular, that is, he characterized those conservative matrices that preserve the limits invariant. In fact, Toeplitz was the first person who studies the summability methods as a class of operators defined on sequences by infinite matrices [35]. It was followed by the works done by I. Schur, W. Orlicz, K. Knopp, G. Petersen, H. Nakano, S. Simons, G. Lorentz, G. Hardy, A. Wilansky, I. Maddox, W. Sargent, C. Lascarides, S. Nanda, D. Rath, G. Das, Z. Ahmed, B. Kuttner and many others like Russell and Rhoade.

Many years after the Toeplitz's work, exactly in 1950, Robinson initiated the study of summability by infinite matrices of linear operators on normed linear spaces which enabled the workers on summability to extend their results. Also, in 1951, the famous mathematician K. Zeller introduced the concept of BK spaces* which has proved its useful in summability theory, especially in the characterizations of matrix transformations between sequence spaces, and the most important result is that matrix operators between BK spaces are continuous [36, 38].

But, why we should study matrix operators and transformations between sequence spaces; why not study the general linear operators? The reason is that, in many important cases, the most general linear operators acting between sequence spaces are actually determined by infinite matrices. So, there is no loss of generality in such study. Moreover, there is often a gain in that specific conditions on the entries of an infinite matrix which may be easy to verify [16].

The sequence spaces were motivated by problems in Fourier series, power series and systems of equations with infinitely many variables, and the theory of sequence spaces and infinite matrices occupies a very prominent position in several branches of analysis and plays an important role in various fields of Mathematics as a powerful and pervading tool in almost all these branches with several important applications. For example, in the structural theory of topological vector spaces, Schauder basis

*The letters **B** and **K** stand for **B**anach and the German word **K**oordinate which means 'coordinate' as in the Zeller's terminology.

theory and theory of integral and differential equations and special functions (e.g., see [10, 39, 59, 61, 71]).

Recently, the sequence spaces have been generalized in several directions by many mathematicians and some of them introduce new sequence spaces and study their various properties. At the present time, a lot of work have been done by many researchers around the world, like Boos, Rakočević, Malkowsky, Savaş, Başar, Altay, Mursaleen, Noman, Kiriçi, Kara and many others, only a few was named (e.g., see [3, 6, 12, 26, 28, 31, 38, 48, 55, 69, 74]). In particular, the difference sequence spaces have been studied by many researchers, like Kızmaz, Et, Başar, Karakaya, Polat, Meng and others (e.g., see [2, 7, 13, 19, 22, 32, 33, 40, 41, 62, 65, 68]).

Finally, the λ-sequence spaces have been introduced and studied by Mursaleen and Noman in 2010 [44, 45, 46, 47] and these spaces are very interesting which have been taken away by researchers and authors upto so far limits. For instance, they have introduced the concepts of statistical λ-convergence and strong λ-convergence by using the generalized de la Vallée-Poussin mean in 2011 (see Mursaleen and Alotaibi [42]) and other authors have studied some general difference forms of the λ-matrix and λ-sequence spaces (e.g., Sönmez with others in 2012 [62] and Bişgin and others in 2014 [14]). Also, some of them have introduced the concept of almost convergence of double sequences in λ-sequence spaces (e.g., Ahmad and Ganie in 2013 [1] and Raj with others in 2015 [57]) and others have introduced the spaces of almost lambda null, almost lambda convergent and almost lambda bounded sequences (e.g, Yeşilkayagil and Başar in 2015 [73] and Ercan in 2020 [20]). Besides, the λ-sequence spaces have proved their useful in some subjects with various applications in the operator theory, spectral theory and measure of non-compactness (e.g., see [10, 49, 52, 71, 72]).

1.1.2 Notions and Notations*

Here, we give a short survey on the basic definitions, concepts and notions which are the elementary tools in the theory of sequence spaces and matrix transformations. Also, we will define the common notations which are usually used by all authors and researchers in this area. Thus, our terminologies, as given here, will have the same meanings throughout this thesis (unless stated otherwise).

1.1.2.1 Scalars and Sequences: Let $\mathbb{K}$ be the scalar field (consisting of real or complex numbers), that is $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and so our scalars are either real or complex numbers (according to the case of our spaces). Also, we will use the symbols k and n to be positive integers while $p, q \geq 1$ are real numbers.

By a "sequence", we mean an infinite sequence of real or complex terms, and if $x = (x_1, x_2, x_3, \cdots)$ is a real or complex sequence; then we denote it by $x = (x_k)_{k=1}^{\infty}$ or simply $x = (x_k)$, where x_k is called the k-th term of x. Further, we shall use the following conventions: the first is that any term with a non-positive subscript is assumed to be nothing (e.g., the terms x_0 and x_{-1} have no meaning and can be considered to be not exist). Next, we will frequently use the sequences $e = (1, 1, 1, \ldots)$ and e_k for each $k \geq 1$, where e_k is the sequence whose only one non-zero term which is 1 in the k-th place for each $k \geq 1$, that is $e_1 = (1, 0, 0, \cdots),\ e_2 = (0, 1, 0, 0, \cdots), \cdots$ etc. Also, the absolute value of a sequence x and its positive power are defined by means of their meanings for its scalar-terms, that is $|x| = (|x_k|)$ and $|x|^r = (|x_k|^r)$ for any real number $r > 0$. The last conventions are concerning with some algebraic operations defined on sequences, namely the coordinate-wise addition, scalar multiplication, product and division. More precisely, if x and y are sequences and α is a scalar; then $x \pm y = (x_k \pm y_k)$, $\alpha x = (\alpha x_k)$,

*We refer the reader to [10, 16, 35, 37, 67] for the elementary concepts.

$xy = (x_k y_k)$ and if $y_k \neq 0$ for all k; then $x/y = (x_k/y_k)$ and $1/y = (1/y_k)$.

Together with any sequence $x = (x_k)$, there always exist two sequences, namely the difference sequence $\Delta(x)$ and the sum sequence $\sigma(x)$, where

$$\Delta(x) = (x_1, x_2 - x_1, x_3 - x_2, \cdots) \text{ and } \sigma(x) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, \cdots).$$

That is $\Delta(x) = (\Delta(x_k))_{k=1}^{\infty} = (x_k - x_{k-1})_{k=1}^{\infty}$ and $\sigma(x) = (\sigma_k(x))_{k=1}^{\infty} = (\sum_{j=1}^{k} x_j)_{k=1}^{\infty}$ which leads us to write their terms as follows:

$$\sigma_k(x) = \sum_{j=1}^{k} x_j \text{ and } \Delta(x_k) = x_k - x_{k-1} \text{ with } \Delta(x_1) = x_1 \qquad (k \geq 1). \qquad (1.1.1)$$

Also, for any two sequences $x, y \in w$, the difference sequence of their product is the sequence $\Delta(xy)$ with terms $\Delta(x_k y_k) = x_k\, y_k - x_{k-1} y_{k-1}$ for all k which can be obtained by the following formula:

$$\Delta(x_k y_k) = x_k \Delta(y_k) + y_{k-1} \Delta(x_k) \qquad (k \geq 1). \qquad (1.1.2)$$

1.1.2.2 Boundedness and Convergence: A sequence $x = (x_k)$ is said to be *bounded* if there exists a positive real number $M > 0$ such that $|x_k| \leq M$ for all $k \geq 1$, that is x is bounded if and only if $\sup_k |x_k| < \infty$, where the supremum of $|x_k|$ is taken over all positive integers k. Also, the sequence x is said to be *convergent* if its limit $\lim_{k\to\infty} x_k$ exists (in $\mathbb{K}$). That is, there must exist $a \in \mathbb{K}$ such that $\lim_{k\to\infty} x_k = a$ which can be written as $x_k \to a$ as $k \to \infty$ (or briefly $x_k \to a$). In particular, by a *null sequence*, we mean a convergent sequence which converges to zero, i.e. $\lim_{k\to\infty} x_k = 0$. On other side, if a sequence $x = (x_k)$ is not convergent; then we say that x is *divergent* which means that its limit does not exist (in $\mathbb{K}$) [35].

In connection with some kinds of divergence, the real sequences have a special importance in this study. For instance, if $z = (z_k)$ is a real sequence such that

$\lim_{k\to\infty} z_k = \infty$ (or $\lim_{k\to\infty} z_k = -\infty$); then we say that z diverges to ∞ (or $-\infty$) which can be denoted by $z_k \to \infty$ (or $z_k \to -\infty$) as $k \to \infty$. Further, a divergent real sequence $z = (z_k)$ is said to be *oscillated* if neither $z_k \to \infty$ nor $z_k \to -\infty$ (as $k \to \infty$). That is, if $\lim_{k\to\infty} z_k$ exists, $\lim_{k\to\infty} z_k = \infty$ or $\lim_{k\to\infty} z_k = -\infty$; then z is not oscillated. In other words, by an oscillated sequence, we mean a divergent real sequence which has no a unique limit for all its subsequences (including the limits $\pm\infty$). For example, the real sequences $((-1)^k)$, $((-2)^k)$ and $(k + k(-1)^k)$ are oscillated while the sequences $((-1)^k/k)$, $((-2)^k - 3^k)$ and $(k^2 + k(-1)^k)$ are not oscillated [52].

1.1.2.3 Series: Every sequence $x = (x_k) \in w$ is associated with a series $\sum_{k=1}^{\infty} x_k$ whose terms are exactly those of x and so it has the same sequence of partial sum which is $\sigma(x)$. Thus, it seems to be quite natural to similarly say that $\sum_{k=1}^{\infty} x_k$ is a null, convergent or bounded series if its sequence of partial sum $\sigma(x)$ is a null, convergent or bounded sequence, respectively. That is, the series $\sum_{k=1}^{\infty} x_k$ is bounded if $\sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty$, and it is convergent if $\lim_{n\to\infty} \sum_{k=1}^{n} x_k$ exists (in $\mathbb{K}$). Also, by a null series, we mean a convergent series to zero, i.e. $\lim_{n\to\infty} \sum_{k=1}^{n} x_k = \sum_{k=1}^{\infty} x_k = 0$.

A series $\sum_{k=1}^{\infty} x_k$ is said to be *absolutely convergent* if the series $\sum_{k=1}^{\infty} |x_k|$ converges and we denote it by $\sum_{k=1}^{\infty} |x_k| < \infty$ (it is well-known that absolute convergence of series implies their convergent, but the converse is not). In general, for any real number $p \geq 1$, the series $\sum_{k=1}^{\infty} x_k$ is said to be *p-absolutely convergent* if $\sum_{k=1}^{\infty} |x_k|^p < \infty$.

A sequence x is said to be of *bounded variation* if $\sum_{k=1}^{\infty} |x_k - x_{k-1}| < \infty$ or equivalently $\sum_{k=1}^{\infty} |\Delta(x_k)| < \infty$. More generally, for any real number $p \geq 1$, we say that x is of *p-bounded variation* if $\sum_{k=1}^{\infty} |x_k - x_{k-1}|^p < \infty$ or $\sum_{k=1}^{\infty} |\Delta(x_k)|^p < \infty$ [10].

1.1.2.4 Sequence Spaces: By w, we denote the linear space of all (real or complex) sequences over the scalar field $\mathbb{K}$ (with coordinate-wise addition and scalar multiplication) and any vector subspace of w is called a *sequence space*. Throughout, we shall write ℓ_∞, c and c_0 for the sequence spaces of bounded, convergent and null sequences, respectively. Also, for each real number $1 \leq p < \infty$, the sequence space ℓ_p is consisting of all sequences associated with p-absolutely convergent series. These sequence spaces are known as the *classical sequence spaces*. Further, we write bs, cs and cs_0 for the spaces of all sequences associated with bounded, convergent and null series, respectively. Moreover, by $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, we stand for the difference spaces of all sequences with bounded, convergent and null difference sequences, respectively. Furthermore, for each real number $1 \leq p < \infty$, we denote the space of all sequences of p-bounded variation by bv_p. That is, we have the following sequence spaces:

$$c_0 = \big\{x = (x_k) \in w : \lim_{k\to\infty} x_k = 0\big\},$$

$$c = \big\{x = (x_k) \in w : \lim_{k\to\infty} x_k \text{ exists}\big\},$$

$$\ell_\infty = \big\{x = (x_k) \in w : \sup_k |x_k| < \infty\big\},$$

$$\ell_p = \big\{x = (x_k) \in w : \textstyle\sum_{k=1}^\infty |x_k|^p < \infty\big\} \quad (1 \leq p < \infty),$$

$$cs_0 = \big\{x = (x_k) \in w : \lim_{n\to\infty} \textstyle\sum_{k=1}^n x_k = 0\big\},$$

$$cs = \big\{x = (x_k) \in w : \lim_{n\to\infty} \textstyle\sum_{k=1}^n x_k \text{ exists}\big\},$$

$$bs = \big\{x = (x_k) \in w : \sup_n \big|\textstyle\sum_{k=1}^n x_k\big| < \infty\big\},$$

$$c_0(\Delta) = \big\{x = (x_k) \in w : \lim_{k\to\infty}(x_k - x_{k-1}) = 0\big\},$$

$$c(\Delta) = \big\{x = (x_k) \in w : \lim_{k\to\infty}(x_k - x_{k-1}) \text{ exists}\big\},$$

$$\ell_\infty(\Delta) = \big\{x = (x_k) \in w : \sup_k |x_k - x_{k-1}| < \infty\big\},$$

$$bv_p = \big\{x = (x_k) \in w : \textstyle\sum_{k=1}^\infty |x_k - x_{k-1}|^p < \infty\big\} \quad (1 \leq p < \infty),$$

and we define the sequence space bv_0 by $bv_0 = c_0 \cap bv_1$ [37].

1.1.2.5 Normed Sequence Spaces: A *normed sequence space* is of course a sequence space X equipped with a *norm* $\|\cdot\|$ defined on X as a mapping $\|\cdot\| : X \to \mathbb{R}$ such that $\|x\| \geq 0$, $x = 0$ whenever $\|x\| = 0$, $\|\alpha x\| = |\alpha|\|x\|$ and $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ and every $\alpha \in \mathbb{K}$. In addition, a normed sequence space X is called a *Banach sequence space* if it is complete with the topology generated by its norm.

Further, if X and Y are normed sequence spaces; then we say that X is *isometrically linear-isomorphic to* Y, denoted by $X \cong Y$, if there exists a linear isomorphism $L : X \to Y$ which preserves the norms, i.e. $\|L(x)\|_Y = \|x\|_X$ for all $x \in X$, where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms on X and Y, respectively. That is, the operator $L : X \to Y$ is linear, bijective and norm-preserving (or isometry).

Furthermore, if X is a normed sequence space; then for each positive integer k, there exists a mapping $\pi_k : X \to \mathbb{K}$ defined by $x \mapsto \pi_k(x) = x_k$ for all $x \in X$, these mappings π_k's (for all k) are called the *coordinate-maps* of X or the *coordinates* of X, where $\mathbb{K}$ is the scalar field of X [16].

1.1.2.6 Schauder Basis: As in the general case of arbitrary normed spaces, if a normed sequence space X contains a sequence $(b_k)_{k=1}^{\infty}$ with the property that for every $x \in X$ there exists a unique sequence $(\alpha_k)_{k=1}^{\infty}$ of scalars such that

$$\lim_{n\to\infty} \|x - (\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n)\| = 0;$$

then the sequence $(b_k)_{k=1}^{\infty}$ is called a *Schauder basis* for X (or briefly a basis for X) and the series $\sum_{k=1}^{\infty} \alpha_k b_k$ which has the sum x is then called the expansion of x, with respect to the given basis, and we then say that x has uniquely been represented in the form $x = \sum_{k=1}^{\infty} \alpha_k b_k$. Further, a normed sequence space X is said to be *separable* if it contains a countable dense subset, and it is well-known that every normed space with Schauder basis must be separable [35].

1.1.2.7 BK Spaces: A normed sequence space X is called a BK *space* if it is complete and all its coordinate-maps are continuous. In other words, by a BK space, we mean a Banach sequence space with continuous coordinates. It is well-known that the above mentioned sequence spaces are all BK spaces with their natural norms. More precisely, the spaces ℓ_∞, c and c_0 are BK spaces with the sup-norm $\|\cdot\|_\infty$ given by $\|x\|_\infty = \sup_k |x_k|$. Also, for $1 \leq p < \infty$, the spaces ℓ_p are BK spaces with the p-norm $\|\cdot\|_p$ defined by $\|x\|_p = \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p}$ and the spaces bv_p are BK spaces with their norm $\|\cdot\|_{bv_p}$ given by $\|x\|_{bv_p} = \left(\sum_{k=1}^{\infty}|x_k - x_{k-1}|^p\right)^{1/p}$. Moreover, the spaces bs, cs and cs_0 are BK spaces with the series-norm $\|\cdot\|_s$ defined by $\|x\|_s = \sup_n \left|\sum_{k=1}^{n} x_k\right|$. Besides, the difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are BK spaces with the Δ-norm $\|\cdot\|_\Delta$ given by $\|x\|_\Delta = \sup_k |x_k - x_{k-1}|$ [16].

1.1.2.8 Dual Spaces: For any sequence space X, the concept of *Köthe-Toeplitz duality* of X, so-called as the *α-, β- and γ-duals* of X can simply be given by means of the spaces $\langle\alpha\rangle = \ell_1$, $\langle\beta\rangle = cs$ and $\langle\gamma\rangle = bs$. For this, let θ be any of the duality symbols α, β or γ, that is $\theta := \alpha$, β or γ. Then, the θ-dual of X is a sequence space denoted by X^θ which can be defined as follows:

$$X^\theta = \{a \in w : \; ax \in \langle\theta\rangle \; \text{ for all } x \in X\} \qquad (\theta = \alpha, \beta \text{ or } \gamma), \tag{1.1.3}$$

where $\langle\alpha\rangle = \ell_1$, $\langle\beta\rangle = cs$ and $\langle\gamma\rangle = bs$. In other words, the α-, β- and γ-duals of X are respectively denoted by X^α, X^β and X^γ which are sequence spaces defined as follows:

$$X^\alpha = \Big\{a = (a_k) \in w : \; ax = (a_k x_k) \in \ell_1 \; \text{ for all } \; x = (x_k) \in X\Big\},$$

$$X^\beta = \Big\{a = (a_k) \in w : \; ax = (a_k x_k) \in cs \; \text{ for all } \; x = (x_k) \in X\Big\},$$

$$X^\gamma = \Big\{a = (a_k) \in w : \; ax = (a_k x_k) \in bs \; \text{ for all } \; x = (x_k) \in X\Big\}.$$

Further, it is well-known that $X^\alpha \subset X^\beta \subset X^\gamma$, the inclusion $X \subset Y$ implies that $Y^\theta \subset X^\theta$, and we have $c_0^\theta = c^\theta = \ell_\infty^\theta = \ell_1$, $\ell_1^\theta = \ell_\infty$ and $\ell_p^\theta = \ell_q$ for $p > 1$ with $q = p/(p-1)$, where X and Y are sequence spaces. The basic properties of dual spaces can be found in [37, 67].

1.1.2.9 Matrix Transformations: Due to the infinite dimensions of sequence spaces in the general case, the notion of *matrix transformations* between sequence spaces has been arisen to study the linear operators between such spaces which can be given by infinite matrices. For an infinite matrix A with real or complex entries a_{nk} $(n, k \geq 1)$, we write $A = [a_{nk}]_{n,k=1}^\infty$ or simply $A = [a_{nk}]$, and we will write A_n for the n-th row sequence in A, that is $A_n = (a_{nk})_{k=1}^\infty$ for each $n \geq 1$. Also, for any sequence $x \in w$, the A-*transform* of x, denoted by $A(x)$, is defined to be the sequence $A(x) = (A_n(x))_{n=1}^\infty$ whose terms given by

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk}\, x_k \qquad (n \geq 1) \tag{1.1.4}$$

provided the convergence of series for each $n \geq 1$ and we then say that $A(x)$ exists. Further, for any two sequence spaces X and Y, we say that A acts from X into Y if $A(x)$ exists and $A(x) \in Y$ for every $x \in X$ [35]. Furthermore, the *matrix class* (X, Y) is defined to be the collection of all infinite matrices acting from X into Y. In particular, an infinite matrix A is said to be *conservative* if $A \in (c, c)$ and a conservative matrix A is said to be *regular* if $\lim_{n\to\infty} A_n(x) = \lim_{n\to\infty} x_n$ for all $x \in c$. In fact, there may exists an infinite matrix A such that $A \notin (X, Y)$ and so the infinite matrices in the class (X, Y) must be characterized from those matrices which are not in (X, Y). That is, there must exist a list of necessary and sufficient conditions on the entries of a given infinite matrix A to be in the class (X, Y), where $A \in (X, Y)$ if and only if $A(x)$ exists as well as $A(x) \in Y$ for every $x \in X$. In other words, $A \in (X, Y)$ if and only if $A_n \in X^\beta$

for every $n \geq 1$ and $A(x) \in Y$ for all $x \in X$, where each A_n is the n-th row sequence in A, and so the β-duality is an important tool for characterizing matrix classes [16].

1.1.2.10 Matrix Operators: If $A \in (X, Y)$; then A defines a linear operator $A : X \to Y$ by $x \mapsto A(x)$, and we may call it as a *matrix operator* (matrix mapping) and the same for every linear operator from X into Y which can be given by an infinite matrix. That is, a linear operator between sequence spaces $L : X \to Y$ is called a matrix operator if there exists an infinite matrix $A \in (X, Y)$ such that $L(x) = A(x)$ for all $x \in X$ and we then say that L is given by an infinite matrix, viz A. Moreover, it is worth mentioning that the most general forms of linear operators between sequence spaces can be given by infinite matrices [37]. This fact gives a special importance for the notion of matrix transformations between sequence spaces, which has been studied by several authors in many research papers (e.g., see [12, 32, 36, 38, 48, 64]) and has recently been used to introduce new sequence spaces and characterize their matrix classes by means of the idea of matrix domains (e.g., see [3, 25, 27, 30, 37, 53, 70]).

1.1.2.11 Matrix Domains and Triangles: For an infinite matrix A and a sequence space X, the *matrix domain* of A in X is a sequence space denoted by X_A which can be defined as follows:

$$X_A = \Big\{x \in w : \ A(x) \in X\Big\}. \tag{1.1.5}$$

The most useful cases of matrix domains are those obtained from special types of infinite matrices called as triangles, where an infinite matrix $T = [t_{nk}]_{k,n=1}^{\infty}$ is called a *triangle* if $t_{nn} \neq 0$ for every $n \geq 1$ and $t_{nk} = 0$ for all $k > n$ $(n, k \geq 1)$. That is, the infinite matrix T has non-zero diagonal entries while all its entries on the upper side are zeros. For example, by using (1.1.1) and (1.1.4), it can easily be seen that the sum-matrix σ and the band-matrix Δ are infinite matrices which are triangles defining the partial sum and the difference operator, respectively. To see that, consider the following triangles:

$$\sigma = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

with their transforms as $\sigma(x) = (\sigma_n(x))$ and $\Delta(x) = (\Delta_n(x))$ which can be obtained by using (1.1.4) to get $\sigma_n(x) = \sum_{k=1}^{n} x_k$ and $\Delta_n(x) = x_n - x_{n-1}$ for all $n \geq 1$ with $\Delta_1(x) = x_1$ $(x \in w)$ which is the same result as given in (1.1.1) (with noting that the term $\Delta(x_n)$ has been used instead of $\Delta_n(x)$ for every $n \geq 1$, and this convention will only be used for the Δ-transform). That is, for every $x \in w$, the sum sequence $\sigma(x)$ and the difference sequence $\Delta(x)$ are the σ-transform and Δ-transform of x, respectively. This fact, together with (1.1.5), leads us to use the concept of matrix domain in order to redefine some sequence spaces (given in Section 1.1.2.4, p. 8) as follows:

$$cs_0 = (c_0)_\sigma = \{x \in w : \sigma(x) \in c_0\},$$

$$cs = (c)_\sigma = \{x \in w : \sigma(x) \in c\},$$

$$bs = (\ell_\infty)_\sigma = \{x \in w : \sigma(x) \in \ell_\infty\},$$

$$c_0(\Delta) = (c_0)_\Delta = \{x \in w : \Delta(x) \in c_0\},$$

$$c(\Delta) = (c)_\Delta = \{x \in w : \Delta(x) \in c\},$$

$$\ell_\infty(\Delta) = (\ell_\infty)_\Delta = \{x \in w : \Delta(x) \in \ell_\infty\},$$

$$bv_p = (\ell_p)_\Delta = \{x \in w : \Delta(x) \in \ell_p\} \qquad (1 \leq p < \infty)$$

which means that these spaces are the matrix domains of the triangles σ and Δ in the classical sequence spaces. This idea has been applied by many authors in several interesting studies as presented in the next section.

1.1.3 Review of Literature

The approach constructing a new sequence space by means of the matrix domain of a particular infinite matrix has been employed by Maddox, Wang, Ng, Lee, Kızmaz, Rakočević, Malkowsky, Savaş, Başar, Altay, Mursaleen, Noman, Karakaya, Kiriçi, Kara, Polat, Aydın, Bektaş and many others (e.g., see [15, 18, 27, 28, 30, 32, 36, 38, 44, 47, 48, 51, 60, 70]). More recently, the idea of introducing a new sequence space by means of the matrix domain of a particular triangle has largely been used by several authors in many research studies with different manners. For instance, we display here the following previous studies:

1.1.3.1 The Nörlund sequence spaces have been introduced by Wang in 1978 [66] as domains of the Nörlund matrix N^q in the spaces c_0, c and ℓ_∞, that is

$$n_0^q = \{x \in w : N^q(x) \in c_0\}, \quad n_c^q = \{x \in w : N^q(x) \in c\}, \quad n_\infty^q = \{x \in w : N^q(x) \in \ell_\infty\}$$

which are BK spaces with $\|x\|_{N^q} = \|N^q(x)\|_\infty$, where $q = (q_k)$ is a sequence of non-negative real numbers such that $q_1 \neq 0$ and

$$N_n^q(x) = \frac{1}{Q_n}\sum\nolimits_{k=1}^{n} q_{n-k+1}\, x_k \text{ with } Q_n = \sum\nolimits_{k=1}^{n} q_k \quad (n \geq 1).$$

1.1.3.2 The difference sequence spaces have been defined by Kızmaz in 1981 [32, 33] as domains of the band matrix Δ in the spaces c_0, c and ℓ_∞, that is

$$\mu(\Delta) = \{x \in w : \Delta(x) \in \mu\} \qquad (\mu = c_0, c \text{ or } \ell_\infty)$$

which are BK spaces with $\|x\|_\Delta = \|\Delta(x)\|_\infty$, where $\Delta(x) = (x_n - x_{n-1})$.

1.1.3.3 The Riesz sequence spaces have been constructed by Malkowsky in 1997 [36] as domains of the Riesz matrix R^t in the spaces c_0, c and ℓ_∞, that is

$$r_0^t = \{x \in w : R^t(x) \in c_0\}, \quad r_c^t = \{x \in w : R^t(x) \in c\}, \quad r_\infty^t = \{x \in w : R^t(x) \in \ell_\infty\}$$

which are BK spaces with $\|x\|_{R^t} = \|R^t(x)\|_\infty$, where $t = (t_k)$ is a sequence of positive real numbers and

$$R_n^t(x) = \frac{1}{T_n}\sum_{k=1}^{n} t_k\, x_k \quad \text{with} \quad T_n = \sum_{k=1}^{n} t_k \qquad (n \geq 1).$$

1.1.3.4 The sequence spaces of p-bounded variation have been introduced by Başar and Altay in 2003 [11] as domains of the band matrix Δ in the spaces ℓ_p, where $1 \leq p \leq \infty$. That is $bv_p = \{x \in w : \Delta(x) \in \ell_p\}$ which are BK spaces with $\|x\|_{bv_p} = \|\Delta(x)\|_p$ for all $x \in bv_p$ $(1 \leq p \leq \infty)$.

1.1.3.5 The Euler sequence spaces have been constructed by Altay and Başar in 2005 [4] (and together with Mursaleen, 2006 [43, 56]) as domains of the Euler matrix E^r in the spaces c_0, c, ℓ_∞ and ℓ_p for $1 < p < \infty$, that is

$$e_0^r = \{x \in w : E^r(x) \in c_0\}, \quad e_c^r = \{x \in w : E^r(x) \in c\},$$

$$e_\infty^r = \{x \in w : E^r(x) \in \ell_\infty\}, \quad e_p^r = \{x \in w : E^r(x) \in \ell_p\}.$$

Also e_0^r, e_c^r and e_∞^r are BK spaces with $\|x\|_{E^r} = \|E^r(x)\|_\infty$ and all e_p^r are BK spaces with $\|x\|_{E_p^r} = \|E^r(x)\|_p$ $(1 < p < \infty)$, where $0 < r < 1$ and

$$E_n^r(x) = \sum_{k=1}^{n} \binom{n-1}{k-1}(1-r)^{n-k}\, r^{k-1}\, x_k \qquad (n \geq 1).$$

1.1.3.6 The sequence spaces of weighted means have been defined by Malkowsky and Savaş in 2008 [38] as domains of the matrix W_s^t of weighted means in the spaces μ, where $\mu = c_0,\ c$ or ℓ_∞, that is $w_s^t(\mu) = \{x \in w : W_s^t(x) \in \mu\}$ which are BK spaces with $\|x\|_{w_s^t} = \|W_s^t(x)\|_\infty$, where s and t are sequences of non-zero scalars and

$$(W_s^t)_n(x) = \frac{1}{s_n}\sum_{k=1}^{n} t_k x_k \qquad (n \geq 1).$$

1.1.3.7 The generalized Nörlund sequence spaces have been studied by Stadtmüller and Tali in 2009 [63] as domains of the generalized Nörlund matrix (N, s, t) in the spaces $\mu = c_0$, c or ℓ_∞, that is $\mu(N,s,t) = \{x \in w : (N,s,t)(x) \in \mu\}$ which are BK spaces with $\|x\|_{\mu(N,s,t)} = \|(N,s,t)(x)\|_\infty$, where s and t are sequences of scalars such that $\sum_{k=1}^{n} s_{n-k+1}t_k \neq 0$ and

$$(N,s,t)_n(x) = \frac{1}{r_n}\sum_{k=1}^{n} s_{n-k+1}t_k\, x_k \quad \text{with} \quad r_n = \sum_{k=1}^{n} s_{n-k+1}t_k \qquad (n \geq 1).$$

1.1.3.8 The sequence spaces of generalized means have been defined in 2011 by Mursaleen and Noman[48] as domains of the matrix $A(r,s,t)$ of generalized means in the spaces μ, where $\mu = c_0,\ c,\ \ell_\infty$ or ℓ_p $(1 \leq p < \infty)$, that is $\mu(r,s,t) = \{x \in w : A(r,s,t)(x) \in \mu\}$ which are BK spaces with $\|x\|_{\mu(r,s,t)} = \|A(r,s,t)(x)\|_\mu$, where r and t are sequences of non-zero scalars, s is a sequence of scalars with $s_1 \neq 0$ and

$$A(r,s,t)_n(x) = \frac{1}{r_n}\sum_{k=1}^{n} s_{n-k+1}t_k\, x_k \qquad (n \geq 1).$$

1.1.3.9 The $\boldsymbol{\lambda}$-sequence spaces μ^λ have been introduced by Mursaleen and Noman in 2010 - 2011 [44, 46, 47] as domains of the λ-matrix Λ in the spaces μ, where μ is any of the spaces c_0, c, ℓ_∞ or ℓ_p for $1 \leq p < \infty$, that is $\mu^\lambda = \{x \in w : \Lambda(x) \in \mu\}$ which are BK spaces with $\|x\|_{\mu^\lambda} = \|\Lambda(x)\|_\mu$ and the triangle Λ is given by

$$\Lambda = \begin{bmatrix} \Delta(\lambda_1)/\lambda_1 & 0 & 0 & \cdots \\ \Delta(\lambda_1)/\lambda_2 & \Delta(\lambda_2)/\lambda_2 & 0 & \cdots \\ \Delta(\lambda_1)/\lambda_3 & \Delta(\lambda_2)/\lambda_3 & \Delta(\lambda_3)/\lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix},$$

where $\lambda = (\lambda_k)$ is a strictly increasing sequence of positive real numbers and

$$\Lambda_n(x) = \frac{1}{\lambda_n}\sum_{k=1}^{n} \Delta(\lambda_k)\, x_k \qquad (n \geq 1).$$

• It is worth mentioning that the notions of λ-matrix and λ-sequence spaces have proved their useful in some subjects with various applications in the operator theory, spectral theory and measure of non-compactness (e.g., see [10, 49, 71, 72]). In fact, the idea of

λ-sequence spaces is very interesting and so it has been taken away by researchers and authors upto so far limits. For instance, they have introduced the concepts of statistical λ-convergence and strong λ-convergence by using the generalized de la Vallée-Poussin mean in 2011 (see Mursaleen and Alotaibi [42]) and other authors have studied some general difference forms of the λ-matrix and λ-sequence spaces (e.g., Sönmez with others in 2012 [62] and Bişgin and others in 2014 [14]). Also, some of them have introduced the concept of almost convergence of double sequences in λ-sequence spaces (e.g., Ahmad and Ganie in 2013 [1] and Raj with others in 2015 [57]) and others have introduced the spaces of almost lambda null, almost lambda convergent and almost lambda bounded sequences (e.g, Yeşilkayagil and Başar in 2015 [73] and Ercan in 2020 [20]). Further, we have selected the next six studies, based on the λ-sequence spaces, to be included in our list of previous studies.

1.1.3.10 The difference $\boldsymbol{\lambda}$-sequence spaces have been introduced by Mursaleen and Noman in 2010 [45] as domains of the matrix $\bar{\Lambda}$ in the spaces c_0, c and ℓ_∞, that is

$$\mu^\lambda(\Delta) = \{x \in w : \bar{\Lambda}(x) \in \mu\} \qquad (\mu = c_0, c \text{ or } \ell_\infty)$$

which are BK spaces with $\|x\|_{\bar{\Lambda}} = \|\bar{\Lambda}(x)\|_\infty$, where

$$\bar{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum\nolimits_{k=1}^{n} \Delta(\lambda_k)\,(x_k - x_{k-1}) \qquad (n \geq 1).$$

1.1.3.11 The paranormed $\boldsymbol{\lambda}$-sequence spaces have been introduced by Karakaya, Noman and Polat in 2011 [29] as domains of the λ-matrix Λ in the paranormed sequence spaces $\mu(p)$ of Maddox [35], that is

$$\mu(\lambda, p) = \{x \in w : \Lambda(x) \in \mu(p)\} \qquad (\mu = c_0, c \text{ or } \ell_\infty)$$

which are paranormed spaces with their paranorm $\|x\|_{\mu(\lambda,p)} = \|\Lambda(x)\|_{\mu(p)}$, where $p =$

(p_k) is a bounded sequence of positive real numbers, $\|\cdot\|_{\mu(p)}$ is the paranorm on the Maddox's spaces $\mu(p)$ and μ is any of the spaces c_0, c or ℓ_∞.

1.1.3.12 The A_λ-sequence spaces have been defined by Braha and Başar in 2013 [15] as domains of the matrix A_λ in the spaces c_0, c and ℓ_∞, that is

$$A_\lambda(\mu) = \{x \in w : A_\lambda(x) \in \mu\} \qquad (\mu = c_0, c \text{ or } \ell_\infty)$$

which are BK spaces with $\|x\|_{A_\lambda} = \|A_\lambda(x)\|_\infty$, where A_λ is the same matrix Λ with the sequence $\Delta(\lambda)$ instead of λ provided that $\Delta(\lambda)$ is increasing, that is

$$(A_\lambda)_n(x) = \frac{1}{\Delta(\lambda_n)}\sum_{k=1}^{n} (\Delta(\lambda_k) - \Delta(\lambda_{k-1}))\, x_k \qquad (n \geq 1).$$

1.1.3.13 The Δ_u^λ-sequence spaces have been studied by Ganie and Sheikh in 2013 [23] as domains of the matrix Δ_u^λ in the spaces c_0, c and ℓ_∞, that is

$$\mu(\Delta_u^\lambda) = \{x \in w : \Delta_u^\lambda(x) \in \mu\} \qquad (\mu = c_0, c \text{ or } \ell_\infty)$$

which are BK spaces with $\|x\|_{\Delta_u^\lambda} = \|\Delta_u^\lambda(x)\|_\infty$, where $u = (u_k)$ is a real or complex sequence of non-zero terms and

$$(\Delta_u^\lambda)_n(x) = \frac{1}{\lambda_n}\sum_{k=1}^{n} u_k(\lambda_k - \lambda_{k-1})\,(x_k - x_{k-1}) \qquad (n \geq 1).$$

1.1.3.14 The Δ_v^λ-sequence spaces have been defined by Ercan and Bektaş in 2014 [21] as domains of the matrix Δ_v^λ in the spaces c_0, c and ℓ_∞, that is

$$\mu^\lambda(\Delta_v) = \{x \in w : \Delta_v^\lambda(x) \in \mu\} \qquad (\mu = c_0, c \text{ or } \ell_\infty)$$

which are BK spaces with $\|x\|_{\Delta_v^\lambda} = \|\Delta_v^\lambda(x)\|_\infty$, where $v = (v_k)$ is a real or complex sequence of non-zero terms and

$$(\Delta_v^\lambda)_n(x) = \frac{1}{\lambda_n}\sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})\,(v_k x_k - v_{k-1} x_{k-1}) \qquad (n \geq 1).$$

1.1.3.15 The U^λ-sequence spaces have been studied by Zeren and Bektaş in 2014 [75] as domains of the matrix U^λ in the spaces c_0, c and ℓ_∞, that is

$$\mu^\lambda(u) = \{x \in w : U^\lambda(x) \in \mu\} \qquad (\mu = c_0, c \text{ or } \ell_\infty)$$

which are BK spaces with $\|x\|_{U^\lambda} = \|U^\lambda(x)\|_\infty$, where $u = (u_k)$ is a real or complex sequence of non-zero terms and

$$U_n^\lambda(x) = \frac{u_n}{\lambda_n}\sum\nolimits_{k=1}^{n}(\lambda_k - \lambda_{k-1})\, x_k \qquad (n \geq 1).$$

1.1.3.16 The binomial sequence spaces have been introduced by Bişgin in 2016 [13] as domains of the binomial matrix $B^{r,s}$ in the spaces ℓ_p for $1 \leq p \leq \infty$, that is

$$b_p^{r,s} = \{x \in w : B^{r,s}(x) \in \ell_p\} \qquad (1 \leq p \leq \infty)$$

which are BK spaces with $\|x\|_{B_p^{r,s}} = \|B^{r,s}(x)\|_p$ $(1 \leq p \leq \infty)$, where r and s are non-zero real numbers such that $r + s \neq 0$ and

$$B_n^{r,s}(x) = \frac{1}{(r+s)^{n-1}}\sum\nolimits_{k=1}^{n}\binom{n-1}{k-1} s^{n-k}\, r^{k-1}\, x_k \qquad (n \geq 1).$$

1.1.3.17 The Pascal sequence spaces have been constructed by Aydin and Polat in 2018 [8] as domains of the Pascal matrix P in the spaces c_0, c and ℓ_∞, that is

$$P_0 = \{x \in w : P(x) \in c_0\}, \quad P_c = \{x \in w : P(x) \in c\}, \quad P_\infty = \{x \in w : P(x) \in \ell_\infty\}$$

which are BK spaces with $\|x\|_P = \|P(x)\|_\infty$, where

$$P_n(x) = \sum\nolimits_{k=1}^{n}\binom{n-1}{n-k}\, x_k \qquad (n \geq 1).$$

1.1.3.18 The Pascal difference spaces have been introduced by Aydin and Polat in 2019 [53] as domains of the generalized band matrix $\Delta^{(m)}$ of order m in the Pascal sequence spaces P_0, P_c and P_∞, that is

$$\eta(\Delta^{(m)}) = \{x \in w : \Delta^{(m)}(x) \in \eta\} \qquad (\eta = P_0, P_c \text{ or } P_\infty)$$

which are BK spaces with $\|x\|_{P(\Delta^{(m)})} = \|P(\Delta^{(m)}(x))\|_\infty$, where P is Pascal matrix and

$$\Delta_n^{(m)}(x) = \sum\nolimits_{k=\max\{1,\, n-m\}}^{n} (-1)^{n-k} \binom{m}{n-k} x_k \qquad (n \geq 1).$$

Lastly, we refer the reader to [3, 24, 26, 37, 55, 68, 69, 70, 74] for additional studies constructing new sequence spaces as matrix domains of infinite matrices.

1.2 Research Methodology

In this section, we display the research methodology used in our investigation.

1.2.1 Research Problem

It is obvious, from the previous studies mentioned in above section, that there are many new sequence spaces of λ-type which have been introduced and studied including the difference λ-sequence spaces (see previous studies, from Section 1.1.3.10 to Section 1.1.3.15, pp. 17–19) but the λ-difference spaces defined by the λ-difference sequences have not, which forms a problem in the literature. So, we are going to introduce and study the λ-difference spaces of bounded, convergent and null λ-difference sequences.

1.2.2 Research Objectives

In this study, our aim is to add the following contributions:

- Introducing some new λ-difference spaces by means of λ-difference sequences.
- Study some isomorphic, algebraic and topological properties of these new spaces.
- Constructing the Schauder bases for the new λ-difference spaces.
- Establishing some new inclusion relations between these new spaces.
- Concluding the Köthe-Toeplitz duals of the new λ-difference spaces.
- Characterizing some new classes of matrix operators on and into these spaces.

1.2.3 Research Tools

In the present thesis, our study and investigation will be based on the usual mathematical tools as the proof and conclusion, and the usual mathematical methodology as the mathematical induction, conclusion and investigation. Also, many mathematical concepts will be used as main tools in our thesis, and the most important tools among them are sequence, series, matrix and space.

1.3 Preliminaries

In this section, we give a list of the preliminary results which are already known in the literature of the theory of sequence spaces and matrix transformations.

Lemma 1.3.1 (Boos [16]) *If $p < p'$ $(1 \leq p < p' < \infty)$; then the inclusions $\ell_p \subset \ell_{p'}$ and $bv_p \subset bv_{p'}$ are strictly satisfied. Further, we have the following strict inclusions:*

$$c_0 \subset c \subset \ell_\infty, \quad \ell_p \subset c_0, \quad \ell_p \subset bv_p \subset c_0(\Delta), \quad cs_0 \subset cs \subset bs,$$

$$c_0(\Delta) \subset c(\Delta) \subset \ell_\infty(\Delta), \quad bs \subset \ell_\infty \subset \ell_\infty(\Delta), \quad bv_1 \subset c \subset c_0(\Delta), \quad \ell_1 \subset cs \subset c_0.$$

Lemma 1.3.2 (Malkowsky [36]) *We have the following facts:*

(1) *The spaces ℓ_∞, c and c_0 are BK spaces with the sup-norm $\|\cdot\|_\infty$ given by $\|x\|_\infty = \sup_k |x_k|$.*

(2) *The spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are BK spaces with the Δ-norm $\|\cdot\|_\Delta$ defined by $\|x\|_\Delta = \sup_n |\Delta(x_n)| = \sup_n |x_n - x_{n-1}|$.*

Lemma 1.3.3 (Malkowsky [37]) *We have the following facts:*

(1) *The sequence $(e_1, e_2, e_3, \cdots)$ is a Schauder basis for the space c_0 and every $x \in c_0$ has the unique representation $x = \sum_{k=1}^{\infty} x_k e_k$.*

(2) *The sequence* $(e, e_1, e_2, \cdots)$ *is a Schauder basis for the space* c *and every* $x \in c$ *has the unique representation* $x = Le + \sum_{k=1}^{\infty}(x_k - L)\, e_k$, *where* $L = \lim_{k\to\infty} x_k$.

(3) *The spaces* c_0 *and* c *are separable while* ℓ_∞ *is not separable and has no Schauder basis* (*in general, every normed sequence space with Schauder basis is separable*).

Lemma 1.3.4 (Kızmaz [32, 33]) *For every sequence* $x = (x_k)$, *we have the following:*

(1) *If* $x \in \ell_\infty(\Delta)$; *then* $(x_k/k) \in \ell_\infty$.

(2) *If* $x \in c(\Delta)$; *then* $(x_k/k) \in c$ *and* $\lim_{k\to\infty} \Delta(x_k) = \lim_{k\to\infty} x_k/k$ (*in particular, if* $x \in c_0(\Delta)$; *then* $(x_k/k) \in c_0$).

(3) *If* x *is a real sequence such that* $\lim_{k\to\infty} \Delta(x_k) = \infty$; *then* $\lim_{k\to\infty} x_k/k = \infty$.

Lemma 1.3.5 (Wilansky [67]) *Let* X *and* Y *be sequence spaces. Then, we have:*

(1) $X^\alpha \subset X^\beta \subset X^\gamma$.

(2) *If* $X \subset Y$; *then* $Y^\theta \subset X^\theta$, *where* $\theta = \alpha$, β *or* γ.

(3) $c_0^\theta = c^\theta = \ell_\infty^\theta = \ell_1$, $\ell_1^\theta = \ell_\infty$ *and* $\ell_p^\theta = \ell_q$ *for* $p > 1$ *with* $q = p/(p-1)$.

Lemma 1.3.6 (Maddox [35]) *Let* X, Y *and* Z *be sequence spaces, and* A *an infinite matrix. Then, we have the following facts:*

(1) $A \in (X, Y) \Longleftrightarrow A_n \in X^\beta$ *for every* $n \geq 1$ *and* $A(x) \in Y$ *for all* $x \in X$.

(2) *If* $X \subset Y$; *then* $(Y, Z) \subset (X, Z)$.

(3) *If* $Y \subset Z$; *then* $(X, Z) \subset (X, Y)$.

Lemma 1.3.7 (Malkowsky [38]) *Let* X *and* Y *be sequence spaces,* A *an infinite matrix and* T *a triangle. Then, we have the following facts:*

(1) $T \in (X, Y) \Longleftrightarrow T(x) \in Y$ *for all* $x \in X$ (*note that:* $T(x)$ *exists for all* $x \in w$).

(2) *If X is a BK space with a norm $\|\cdot\|$; then X_T is a BK space with the norm $\|\cdot\|_T$ defined by $\|x\|_T = \|T(x)\|$ for all $x \in X_T$.*

(3) $A \in (X, Y_T) \Longleftrightarrow TA \in (X, Y)$.

Further, it seems to be quite natural, in view of the fact that matrix operators between BK spaces are continuous, to find necessary and sufficient conditions for the entries of an infinite matrix to define a linear operator between BK spaces which means the characterization of matrix classes of sequence spaces. The following familiar results can be found in [64, pp. 2–9] and will be needed to prove our main results in the next chapters. In the following, we will use the symbol μ to be any of the spaces c_0, c or ℓ_∞, and $\mathcal{K}$ stands for the collection of all non-empty finite subsets of positive integers.

Lemma 1.3.8 *Let $1 \leq p < \infty$. Then, we have $(c_0, \ell_p) = (c, \ell_p) = (\ell_\infty, \ell_p)$, and $A \in (\mu, \ell_p)$ if and only if the following condition holds:*

$$\sup_{K \in \mathcal{K}} \sum_{n=1}^{\infty} \left| \sum_{k \in K} a_{nk} \right|^p < \infty \,,$$

where $\mathcal{K}$ stands for the collection of all non-empty finite subsets of positive integers.

Lemma 1.3.9 *We have $(c_0, \ell_\infty) = (c, \ell_\infty) = (\ell_\infty, \ell_\infty)$, and $A \in (\mu, \ell_\infty)$ if and only if the following condition holds:*

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty \,. \tag{1.3.1}$$

Lemma 1.3.10 *We have the following:*

(1) $A \in (c_0, c)$ *if and only if* (1.3.1) *and the following condition hold:*

$$\lim_{n\to\infty} a_{nk} = a_k \ \text{exists for every}\ k \geq 1 \,. \tag{1.3.2}$$

Further, if $A \in (c_0, c)$; then $\lim_{n\to\infty} A_n(x) = \sum_{k=1}^{\infty} a_k x_k$ for all $x \in c_0$.

(2) $A \in (c, c)$ *if and only if* (1.3.1), (1.3.2) *and the following condition hold:*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = a \ \text{ exists.}$$

Further, if $A \in (c, c)$*; then* $\lim_{n\to\infty} A_n(x) = L(a - \sum_{k=1}^{\infty} a_k) + \sum_{k=1}^{\infty} a_k x_k$ *for all* $x \in c$*, where* $L = \lim_{k\to\infty} x_k$.

(3) $A \in (\ell_\infty, c)$ *if and only if* (1.3.1), (1.3.2) *and the following condition hold:*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk} - a_k| = 0\,.$$

Further, if $A \in (\ell_\infty, c)$*; then* $\lim_{n\to\infty} A_n(x) = \sum_{k=1}^{\infty} a_k x_k$ *for all* $x \in \ell_\infty$.

Lemma 1.3.11 *We have the following:*

(1) $A \in (\ell_\infty, c_0)$ *if and only if the following condition holds:*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = 0.$$

(2) $A \in (c, c_0)$ *if and only if* (1.3.1) *and the following conditions hold:*

$$\lim_{n\to\infty} a_{nk} = 0 \ \text{ for every } \ k \geq 1, \tag{1.3.3}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 0.$$

(3) $A \in (c_0, c_0)$ *if and only if* (1.3.1) *and* (1.3.3) *hold.*

Lemma 1.3.12 *We have the following:*

(1) $A \in (\ell_1, \ell_\infty)$ *if and only if the following condition holds:*

$$\sup_{n,\,k} |a_{nk}| < \infty. \tag{1.3.4}$$

(2) $A \in (\ell_1, c)$ *if and only if* (1.3.2) *and* (1.3.4) *hold.*

(3) $A \in (\ell_1, c_0)$ *if and only if* (1.3.3) *and* (1.3.4) *hold.*

Lemma 1.3.13 *Let $1 < p < \infty$ and $q = p/(p-1)$. Then, we have the following:*

(1) *$A \in (\ell_p, \ell_\infty)$ if and only if the following condition holds:*

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}|^q < \infty. \tag{1.3.5}$$

(2) *$A \in (\ell_p, c)$ if and only if (1.3.2) and (1.3.5) hold.*

(3) *$A \in (\ell_p, c_0)$ if and only if (1.3.3) and (1.3.5) hold.*

Lastly, to prove the main results in this study, we need the following two lemmas concerning the λ-matrix and the λ-sequence spaces (see Section 1.1.3.9, p. 16).

Lemma 1.3.14 (Noman [52]) *We have the following:*

(1) *The matrix Λ is regular* if and only if $\lambda_k \to \infty$ as $k \to \infty$ (or equivalently $1/\lambda \in c_0$, where $1/\lambda = (1/\lambda_k)_{k=1}^{\infty}$).*

(2) *For every sequence $x \in w$, we have the following satisfied equality:*

$$x_n - \Lambda_n(x) = \frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} [\Lambda_n(x) - \Lambda_{n-1}(x)] \qquad (n \geq 1).$$

Lemma 1.3.15 (Mursaleen [44, 46]) *Let $1 \leq p < \infty$. Then, we have the following:*

(1) *The inclusions $\ell_p^\lambda \subset c_0^\lambda \subset c^\lambda \subset \ell_\infty^\lambda$ strictly hold.*

(2) *The inclusions $c \subset c^\lambda$ and $\ell_\infty \subset \ell_\infty^\lambda$ hold.*

(3) *The inclusion $c_0 \subset c_0^\lambda$ holds if and only if $1/\lambda \in c_0$, where $1/\lambda = (1/\lambda_k)_{k=1}^{\infty}$.*

*That is $\lim_{k\to\infty} \Lambda_k(x) = \lim_{k\to\infty} x_k$ for every $x \in c$.

Chapter 2

NEW λ-DIFFERENCE SPACES

2 NEW λ-DIFFERENCE SPACES

The idea of constructing a new sequence space by means of the matrix domain of a particular limitation triangle has recently been employed by several authors in many research papers (see for example [3, 25, 27, 53, 70]). In this chapter, we first study some additional properties of the well-known difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ of bounded, convergent and null difference sequences, respectively. After that, we will introduce the new λ-difference spaces $\ell_\infty(\Delta^\lambda)$, $c(\Delta^\lambda)$ and $c_0(\Delta^\lambda)$ of bounded, convergent and null λ-difference sequences, respectively. Further, we study some isomorphic, algebraic and topological properties of our new spaces. Finally, we construct the Schauder bases for the spaces $c(\Delta^\lambda)$ and $c_0(\Delta^\lambda)$ with concluding their separability. This chapter is divided into three sections, the first is devoted to study the usual difference spaces, the second is for introducing our new spaces with study their properties and the last is to construct their Schauder bases. The materials of this chapter are part of our research paper* which has been published in the Albaydha Univ. J., and presented in the 2nd Conference of Albaydha University on 2021.

2.1 Difference Sequence Spaces

In this section, we study some additional properties of the usual difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ of all sequences having bounded, convergent and null difference sequences, respectively. These spaces have been introduced by Kızmaz in 1981 [32] which have been defined in [33] as the domains of the triangle Δ, so-called as the band-

*A.K. Noman and O.H. Al-Sabri, *On the new λ-difference spaces of convergent and bounded sequences*, Albaydha Univ. J., **3**(2) (2021), 18–30.

matrix, in the spaces ℓ_∞, c and c_0, respectively. That is $\ell_\infty(\Delta) = (\ell_\infty)_\Delta$, $c(\Delta) = (c)_\Delta$ and $c_0(\Delta) = (c_0)_\Delta$ which can be written as $\ell_\infty(\Delta) = \{x \in w : \Delta(x) \in \ell_\infty\}$, $c(\Delta) = \{x \in w : \Delta(x) \in c\}$ and $c_0(\Delta) = \{x \in w : \Delta(x) \in c_0\}$, where $\Delta(x) = (\Delta(x_n))$ and $\Delta(x_n) = x_n - x_{n-1}$ for all $n \geq 1$ with $\Delta(x_1) = x_1$. This yields the following:

$$c_0(\Delta) = \{x = (x_n) \in w : \lim_{n\to\infty} \Delta(x_n) = 0\},$$

$$c(\Delta) = \{x = (x_n) \in w : \lim_{n\to\infty} \Delta(x_n) \text{ exists}\},$$

$$\ell_\infty(\Delta) = \big\{x = (x_n) \in w : \sup_n \big|\Delta(x_n)\big| < \infty\big\}.$$

Further, since ℓ_∞, c and c_0 are BK spaces with $\|\cdot\|_\infty$ and Δ is a triangle; it follows that $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are BK spaces with the norm $\|\cdot\|_\Delta$ given by $\|x\|_\Delta = \sup_n |\Delta(x_n)|$ (see (2) of Lemma 1.3.2). Furthermore, we may begin with proving the following three well-known results which are necessary to be proved here in order to understand the basic idea and tools in this field.

Lemma 2.1.1 *The difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are isometrically linear-isomorphic to the spaces ℓ_∞, c and c_0, respectively. That is $\ell_\infty(\Delta) \cong \ell_\infty$, $c(\Delta) \cong c$ and $c_0(\Delta) \cong c_0$.*

Proof. Let μ be standing for any one of the spaces ℓ_∞, c or c_0, and let $\mu(\Delta)$ be the respective one of the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. Then, it follows by definition that the spaces $\mu(\Delta)$ are the domains of the band-matrix Δ in the spaces μ, that is $\mu(\Delta) = \mu_\Delta$ and so we have the linear operator $\Delta : \mu(\Delta) \to \mu$. Also, since Δ is a triangle and so invertible with $\Delta^{-1} = \sigma$ [37]; we deduce that Δ is a linear bijection preserving the norm, where $\|\Delta(x)\|_\infty = \|x\|_\Delta$ for all $x \in \mu(\Delta)$. Hence, the space $\mu(\Delta)$ is isometrically linear-isomorphic to the corresponding space μ, that is $\mu(\Delta) \cong \mu$ and this completes the proof. $\square$

Lemma 2.1.2 *Suppose that*

$$\bar{e}_1 = (1, 1, 1, 1, \cdots), \quad \bar{e}_2 = (0, 1, 1, 1, \cdots), \quad \bar{e}_3 = (0, 0, 1, 1, \cdots), \cdots \; etc.$$

Then, we have the following facts:

(1) *The sequence* $(\bar{e}_1, \bar{e}_2, \; \bar{e}_3 \cdots)$ *is a Schauder basis for the space* $c_0(\Delta)$ *and every* $x \in c_0(\Delta)$ *has the unique representation* $x = \sum_{k=1}^{\infty} \Delta_k(x)\, \bar{e}_k$.

(2) *The sequence* $(\bar{e}, \bar{e}_1, \bar{e}_2, \bar{e}_3 \cdots)$ *is a Schauder basis for the space* $c(\Delta)$ *and every* $x \in c(\Delta)$ *has the unique representation* $x = L\bar{e} + \sum_{k=1}^{\infty}(\Delta_k(x) - L)\, \bar{e}_k$, *where* $\bar{e} = (k) = (1, 2, 3, \cdots)$ *and* $L = \lim_{n\to\infty} \Delta_n(x)$.

Proof. Since each of the spaces $c_0(\Delta)$ and $c(\Delta)$ is isometrically linear-isomorphic to the corresponding space c_0 or c (by Lemma 2.1.1); parts (1) and (2) are respectively immediate by (1) and (2) of Lemma 1.3.3, as $\Delta(\bar{e}) = e$ and $\Delta(\bar{e}_k) = e_k$ for all $k \geq 1$, where $(e_1, e_2, e_3 \cdots)$ and $(e, e_1, e_2, e_3 \cdots)$ are Schauder bases for c_0 and c (respt.). □

Lemma 2.1.3 *The spaces* $c_0(\Delta)$ *and* $c(\Delta)$ *are separable while the space* $\ell_\infty(\Delta)$ *is not separable and has no Schauder basis.*

Proof. This result follows from (3) of Lemma 1.3.3, since $c_0(\Delta)$ and $c(\Delta)$ are BK spaces and so normed spaces having Schauder bases (by Lemma 2.1.2) while the space ℓ_∞ (and so $\ell_\infty(\Delta)$) is not separable and has no Schauder basis. □

Moreover, some inclusion relations concerning the difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ can be found in Lemma 1.3.1. For example, we have the strict inclusions $c_0(\Delta) \subset c(\Delta) \subset \ell_\infty(\Delta)$, $c \subset c_0(\Delta)$, $\ell_\infty \subset \ell_\infty(\Delta)$ and $bv_p \subset c_0(\Delta)$ for $1 \leq p < \infty$.

Furthermore, we may now add some new idea and prove additional properties of these difference sequence spaces. So, we have the following results:

Lemma 2.1.4 *We have the following relations:*

(1) *The inclusion* $\ell_\infty \cap c(\Delta) \subset c_0(\Delta)$ *strictly holds.*

(2) *The equality* $\ell_\infty \cap c(\Delta) = \ell_\infty \cap c_0(\Delta)$ *holds.*

(3) *The inclusion* $c \subset \ell_\infty \cap c_0(\Delta)$ *strictly holds.*

Proof. For (1), take any $x \in \ell_\infty \cap c(\Delta)$. Then $x \in \ell_\infty$ as well as $x \in c(\Delta)$ and so $\Delta(x) \in c$. Also, it follows by (2) of Lemma 1.3.4 that $(x_k/k) \in c$ such that $\lim_{k\to\infty} \Delta(x_k) = \lim_{k\to\infty} x_k/k$. But $x \in \ell_\infty$ and so $\lim_{k\to\infty} x_k/k = 0$ which implies $\lim_{k\to\infty} \Delta(x_k) = 0$ (as $\lim_{k\to\infty} \Delta(x_k) = \lim_{k\to\infty} x_k/k$). Thus $x \in c_0(\Delta)$ and hence $\ell_\infty \cap c(\Delta) \subset c_0(\Delta)$. Also, to show that this inclusion is strict, consider the unbounded sequence $y = (\sqrt{k}\,)$. Then $\Delta(y) = (\sqrt{k} - \sqrt{k-1}\,)$ and so $\lim_{k\to\infty} \Delta(y_k) = \lim_{k\to\infty} 1/(\sqrt{k} + \sqrt{k-1}\,) = 0$ which means that $\Delta(y) \in c_0$ and hence $y \in c_0(\Delta)$. Thus $y \in c_0(\Delta)$ while $y \notin \ell_\infty$ and so $y \notin \ell_\infty \cap c(\Delta)$. Therefore, the inclusion $\ell_\infty \cap c(\Delta) \subset c_0(\Delta)$ is strict.

To prove (2), we have $c_0(\Delta) \subset c(\Delta)$ and so $\ell_\infty \cap c_0(\Delta) \subset \ell_\infty \cap c(\Delta)$. Also, for the converse inclusion, it is clear that $\ell_\infty \cap c(\Delta) \subset \ell_\infty$ and we have $\ell_\infty \cap c(\Delta) \subset c_0(\Delta)$ by part (1) which together imply that $\ell_\infty \cap c(\Delta) \subset \ell_\infty \cap c_0(\Delta)$. Consequently, we deduce the equality $\ell_\infty \cap c(\Delta) = \ell_\infty \cap c_0(\Delta)$.

For the last part (3), it is obvious that the inclusion $c \subset \ell_\infty \cap c_0(\Delta)$ holds by Lemma 1.3.1. To show that this inclusion is strict, consider the sequence $z = (z_k)$ with terms z_k's given, via k and any positive integer n, by

$$z_k = \frac{1}{2} + (-1)^n \left(\frac{2k - (n+1)^2}{2(n+1)} \right) \text{ for all } \frac{n(n+1)}{2} \leq k \leq \frac{(n+1)(n+2)}{2} \qquad (n \geq 1).$$

That is, our z is the following sequence

$$z = \left(1,\ \frac{1}{2},\ 0,\ \frac{1}{3},\ \frac{2}{3},\ 1,\ \frac{3}{4},\ \frac{2}{4},\ \frac{1}{4},\ 0, \cdots\right).$$

Then, it can easily be seen that $-1/2 \leq [2k-(n+1)^2]/[2(n+1)] \leq 1/2$ whenever $n(n+1)/2 \leq k \leq (n+1)(n+2)/2$ and so $0 \leq z_k \leq 1$ for all $k \geq 1$ which means that $z \in \ell_\infty$. But z is oscillated between 0 and 1 (as $z_{n(n+1)/2} = (1-(-1)^n)/2$ for all n) which means that $z \notin c$. On other side, we find that

$$\Delta(z) = \Big(1, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \cdots\Big)$$

which can be written as $\Delta(z_k) = (-1)^{n-1}/n$ for all $(n^2-n)/2 < k \leq (n^2+n)/2$, where $n \geq 1$. Thus, it is clear that $\Delta(z) \in c_0$ and so $z \in c_0(\Delta)$. Therefore, we have shown that $z \in \ell_\infty \cap c_0(\Delta)$ while $z \notin c$. Hence, the inclusion $c \subset \ell_\infty \cap c_0(\Delta)$ is strict. □

Remark 2.1.5 We may note the follwing:

(1) Although the spaces c_0 and c are strictly included in $c_0(\Delta)$, the space ℓ_∞ is never included in $c(\Delta)$ or $c_0(\Delta)$. To see that, the alternating sequence $x = ((-1)^k) = (-1, 1, -1, 1, \cdots)$ is bounded and so $x \in \ell_\infty$, but its difference sequence is not convergent, where $\Delta(x) = (-1, 2, -2, 2, \cdots) \notin c$ and so $x \notin c(\Delta)$.

(2) Although the spaces $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ overlap with ℓ_∞, none of them is included in ℓ_∞. To see that, we have $(\sqrt{k}\,) \in c_0(\Delta) \setminus \ell_\infty$ (note also that $\Delta(k) = 1$ for all k and so $(k) \in c(\Delta) \setminus \ell_\infty$).

Example 2.1.6 We will show here that the converse implications of those given in Lemma 1.3.4 are not true in the general case. For this, let μ be standing for any of the spaces c_0, c or ℓ_∞, and so $\mu(\Delta)$ is the respective one of the spaces $c_0(\Delta)$, $c(\Delta)$ or $\ell_\infty(\Delta)$. Then, we have the inclusions $c_0 \subset \mu \subset \ell_\infty$ as well as $c_0(\Delta) \subset \mu(\Delta) \subset \ell_\infty(\Delta)$. Besides, it is known, by Lemma 1.3.4, that $x \in \mu(\Delta)$ implies $(x_k/k) \in \mu$, but the converse need not be true. For example, let $x = ((-1)^k\sqrt{k}\,)$. Then, it is obvious that $(x_k/k) \in c_0$ and so $(x_k/k) \in \mu$, but $\Delta(x) = ((-1)^k(\sqrt{k} + \sqrt{k-1})\,) \notin \ell_\infty$ and so

$\Delta(x) \notin \mu$ which means that $x \notin \mu(\Delta)$. That is $(x_k/k) \in \mu$ while $x \notin \mu(\Delta)$. Similarly, let $x = (1, 1, 9, 9, 25, 25, \cdots)$. That is $x_k = k^2$ when k is odd and $x_k = (k-1)^2$ when k is even. Then $x_k/k \to \infty$ while $\Delta(x_k) \not\to \infty$ (as $k \to \infty$), since $\Delta(x_{2k}) \to 0$.

So, because of the above example, we have to deduce the precise characterization of sequences in the difference spaces $\mu(\Delta)$, which will be given in the next theorem.

Theorem 2.1.7 *For any sequence $x = (x_k)$, we have the following equivalences:*

(1) $x \in \ell_\infty(\Delta)$ *if and only if* $(x_k/k) \in \ell_\infty$ *and* $(\, k\Delta(x_k/k)\,) \in \ell_\infty$.

(2) $x \in c(\Delta)$ *if and only if* $(x_k/k) \in c$ *and* $(\, k\Delta(x_k/k)\,) \in c_0$.

(3) $x \in c_0(\Delta)$ *if and only if* $(x_k/k) \in c_0$ *and* $(\, k\Delta(x_k/k)\,) \in c_0$.

Proof. Let $x = (x_k)$ be given. Then, we can use (1.1.2) to derive the following

$$k\,\Delta\left(\frac{x_k}{k}\right) = \Delta(x_k) - \frac{x_{k-1}}{k-1} \qquad (k > 1), \tag{2.1.1}$$

$$\Delta(x_k) = k\,\Delta\left(\frac{x_k}{k}\right) + \frac{x_{k-1}}{k-1} \qquad (k > 1). \tag{2.1.2}$$

To prove (1), suppose that $x \in \ell_\infty(\Delta)$. Then $(x_k/k) \in \ell_\infty$ (by (1) of Lemma 1.3.4). Also, since $x \in \ell_\infty(\Delta)$; we get $\Delta(x) \in \ell_\infty$ (by definition). Thus, it follows by (2.1.1) that $(\, k\Delta(x_k/k)\,) \in \ell_\infty$. Conversely, if $(x_k/k) \in \ell_\infty$ and $(\, k\Delta(x_k/k)\,) \in \ell_\infty$; we deduce from (2.1.2) that $\Delta(x) \in \ell_\infty$ and so $x \in \ell_\infty(\Delta)$ which proves part (1).

Similarly, parts (2) and (3) can be proved by using (2) of Lemma 1.3.4 with noting that $\lim_{k\to\infty} \Delta(x_k) = \lim_{k\to\infty} x_k/k = \lim_{k\to\infty} x_{k-1}/(k-1)$ for every $x \in c(\Delta)$. □

Finally, the last result in this section is analogous to that famous result in calculus (the H'Lôpital's rule).

Corollary 2.1.8 *If $x, y \in c(\Delta)$, $y_k \neq 0$ for all k and $\lim_{k\to\infty} \Delta(x_k)/\Delta(y_k)$ exists; then $x/y \in c$ and we have*

$$\lim_{k\to\infty} \frac{x_k}{y_k} = \lim_{k\to\infty} \frac{\Delta(x_k)}{\Delta(y_k)} \cdot$$

Proof. Suppose that $x, y \in c(\Delta)$, $y_k \neq 0$ for all $k \geq 1$ and $\lim_{\kappa\to\infty} \Delta(x_k)/\Delta(y_k)$ exists. Then, since $x, y \in c(\Delta)$; we have $\lim_{k\to\infty} \Delta(x_k) = \lim_{k\to\infty} x_k/k$ as well as $\lim_{k\to\infty} \Delta(y_k) = \lim_{k\to\infty} y_k/k$ (by (2) of Theorem 2.1.7 or (2) of Lemma 1.3.4). Also, since $y_k \neq 0$ for all k; the the quotient sequence x/y is well-defined. Besides, since $\lim_{\kappa\to\infty} \Delta(x_k)/\Delta(y_k)$ exists; we dedeuce that

$$\lim_{k\to\infty} \frac{\Delta(x_k)}{\Delta(y_k)} = \lim_{k\to\infty} \frac{x_k/k}{y_k/k} = \lim_{k\to\infty} \frac{x_k}{y_k}$$

which means that $\lim_{k\to\infty} x_k/y_k$ exists. Thus $x/y \in c$ and the given formula is true. □

Remark 2.1.9 It must be noted that Corollary 2.1.8 can be generalized as follows:

If $x, y \in c(\Delta^{(m)})$, $y_k \neq 0$ for all k and $\lim_{k\to\infty} \Delta^{(m)}(x_k)/\Delta^{(m)}(y_k)$ exists; then $x/y \in c$ and we have

$$\lim_{k\to\infty} \frac{x_k}{y_k} = \lim_{k\to\infty} \frac{\Delta^{(m)}(x_k)}{\Delta^{(m)}(y_k)},$$

where $\Delta^{(m)}$ is the generalized band-matrix defined for every positive integer m by $\Delta^{(m)}(x) = \Delta(\Delta^{(m-1)}(x))$ with $\Delta^{(0)}(x) = x$ and $c(\Delta^{(m)}) = \{x \in w : \Delta^{(m)}(x) \in c\}$.

2.2 λ-Difference Spaces

In this section, we will present the idea of λ-sequence spaces and introduce the new λ-difference spaces $\ell_\infty(\Delta^\lambda)$, $c(\Delta^\lambda)$ and $c_0(\Delta^\lambda)$ of bounded, convergent and null difference sequences of λ-type, respectively. Also, we will show that our new spaces are BK spaces and conclude their isomorphic relations to the spaces ℓ_∞, c and c_0, and to the usual difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, respectively.

Throughout this study, we assume that $\lambda = (\lambda_k)_{k=1}^\infty$ is a strictly increasing sequence of positive real numbers, that is $0 < \lambda_1 < \lambda_2 < \cdots$ and so $\Delta(\lambda_k) > 0$ for all $k \geq 1$.

Also, the λ-*matrix* Λ is a triangle $\Lambda = [\lambda_{nk}]_{n,k=1}^{\infty}$ defined for every $n, k \geq 1$ by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n}\,; & (1 \leq k \leq n), \\ 0\,; & (k > n \geq 1). \end{cases} \tag{2.2.1}$$

Then, with help of (1.1.4), the Λ-transform of any sequence $x \in w$ is the sequence $\Lambda(x) = (\Lambda_n(x))_{n=1}^{\infty}$ given by

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})\, x_k \qquad (n \geq 1). \tag{2.2.2}$$

The λ-*sequence spaces* c_0^λ, c^λ, ℓ_∞^λ and ℓ_p^λ $(1 \leq p < \infty)$ have been introduced by Mursaleen and Noman in 2010 [44, 46, 47] as the matrix domains of Λ in the spaces c_0, c, ℓ_∞ and ℓ_p, respectively. That is

$$c_0^\lambda = (c_0)_\Lambda = \{x \in w : \ \Lambda(x) \in c_0\},$$

$$c^\lambda = (c)_\Lambda = \{x \in w : \ \Lambda(x) \in c\},$$

$$\ell_\infty^\lambda = (\ell_\infty)_\Lambda = \{x \in w : \ \Lambda(x) \in \ell_\infty\},$$

$$\ell_p^\lambda = (\ell_p)_\Lambda = \{x \in w : \ \Lambda(x) \in \ell_p\} \qquad (1 \leq p < \infty).$$

Further, it has been shown that the spaces c_0^λ, c^λ and ℓ_∞^λ are BK spaces with the norm $\|x\|_{\Lambda\infty} = \|\Lambda(x)\|_\infty = \sup_n |\Lambda_n(x)|$. Also, for each real number $1 \leq p < \infty$, the space ℓ_p^λ is BK space with the norm $\|x\|_{\Lambda p} = \|\Lambda(x)\|_p = \left(\sum_{n=1}^{\infty} |\Lambda_n(x)|^p \right)^{1/p}$.

Moreover, the *difference* λ-*sequence spaces* $c_0^\lambda(\Delta)$, $c^\lambda(\Delta)$ and $\ell_\infty^\lambda(\Delta)$ have been studied in [45] as the domains of the band-matrix Δ in the λ-sequence spaces c_0^λ, c^λ and ℓ_∞^λ, respectively. That is, the difference λ-sequence spaces are defined as follows:

$$c_0^\lambda(\Delta) = \left(c_0^\lambda\right)_\Delta, \quad c^\lambda(\Delta) = \left(c^\lambda\right)_\Delta \quad \text{and} \quad \ell_\infty^\lambda(\Delta) = \left(\ell_\infty^\lambda\right)_\Delta. \tag{2.2.3}$$

We refer the reader to [44, 45, 46] and [47] for additional knowledge concerning the λ-sequence spaces and the difference λ-sequence spaces.

Now, as a natural continuation of above work, we go away from the technique used in [45] and introduce the λ-*difference sequence spaces* in the following definition:

Definition 2.2.1 *The λ-difference spaces $c_0(\Delta^\lambda)$, $c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$ are defined as the matrix domains of the triangle Λ in the difference spaces $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$, respectively. That is*

$$c_0(\Delta^\lambda) = \{c_0(\Delta)\}_\Lambda = \big\{x \in w : \Lambda(x) \in c_0(\Delta)\big\},$$

$$c(\Delta^\lambda) = \{c(\Delta)\}_\Lambda = \big\{x \in w : \Lambda(x) \in c(\Delta)\big\},$$

$$\ell_\infty(\Delta^\lambda) = \{\ell_\infty(\Delta)\}_\Lambda = \big\{x \in w : \Lambda(x) \in \ell_\infty(\Delta)\big\}.$$

From above definition, the new contribution of this study can be given as follows:

$$c_0(\Delta^\lambda) = \left\{x \in w : \lim_{n\to\infty} \big(\Lambda_n(x) - \Lambda_{n-1}(x)\big) = 0\right\},$$

$$c(\Delta^\lambda) = \left\{x \in w : \lim_{n\to\infty} \big(\Lambda_n(x) - \Lambda_{n-1}(x)\big) \text{ exists}\right\},$$

$$\ell_\infty(\Delta^\lambda) = \left\{x \in w : \sup_n \big|\Lambda_n(x) - \Lambda_{n-1}(x)\big| < \infty\right\}.$$

In other words, we introduce the $\tilde{\lambda}$-*matrix* $\tilde{\Lambda}$ to be a triangle $\tilde{\Lambda} = [\tilde{\lambda}_{nk}]_{n,k=1}^\infty$ defined for all $n, k \geq 1$ as follows:

$$\tilde{\lambda}_{nk} = \begin{cases} \dfrac{\lambda_n - \lambda_{n-1}}{\lambda_n}\,; & (n = k), \\ (\lambda_k - \lambda_{k-1})\left(\dfrac{1}{\lambda_n} - \dfrac{1}{\lambda_{n-1}}\right); & (n > k), \\ 0\,; & (n < k). \end{cases} \tag{2.2.4}$$

Then, for every sequence $x \in w$, we have

$$\tilde{\Lambda}_n(x) = \frac{\Delta(\lambda_n)}{\lambda_n}\, x_n + \Delta\left(\frac{1}{\lambda_n}\right) \sum_{k=1}^{n-1} \Delta(\lambda_k) x_k \qquad (n > 1), \tag{2.2.5}$$

which is also valid for $n = 1$ by noting that $\Delta(1/\lambda_1) = 1/\lambda_1$ and the sum on the right-hand side is equal to zero when $n = 1$. That is $\tilde{\Lambda}_1(x) = \Lambda_1(x) = x_1$. Thus, the equality (2.2.5) can be rewritten as

$$\tilde{\Lambda}_n(x) = \frac{\Delta(\lambda_n)}{\lambda_n}\, x_n + \Delta\left(\frac{1}{\lambda_n}\right) \sum_{j=0}^{n-1} \Delta(\lambda_j) x_j \qquad (n \geq 1) \tag{2.2.6}$$

to be more suitable with the case $n = 1$. Besides, by using (2.2.2) and (2.2.5), it can easily be shown that

$$\tilde{\Lambda}_n(x) = \Lambda_n(x) - \Lambda_{n-1}(x) \qquad (n \geq 1) \tag{2.2.7}$$

and so $\tilde{\Lambda}(x) = (\,\Lambda_n(x) - \Lambda_{n-1}(x)\,)_{n=1}^{\infty}$. That is $\tilde{\Lambda}(x) = \Delta(\Lambda(x))$ for all $x \in w$ which means that $\tilde{\Lambda} = \Delta\Lambda$. Thus, the new spaces $c_0(\Delta^\lambda), c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$ can equivalently be redefined as in the following equivalent definition:

Definition 2.2.2 *The λ-difference spaces $c_0(\Delta^\lambda), c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$ are defined as the matrix domains of the triangle $\tilde{\Lambda}$ in the spaces c_0, c and ℓ_∞, respectively. That is*

$$c_0\left(\Delta^\lambda\right) = (c_0)_{\tilde{\Lambda}}, \quad c\left(\Delta^\lambda\right) = (c)_{\tilde{\Lambda}} \;\text{ and }\; \ell_\infty\left(\Delta^\lambda\right) = (\ell_\infty)_{\tilde{\Lambda}}. \tag{2.2.8}$$

From Definition 2.2.2, it follows that $\mu(\Delta^\lambda) = \left\{x \in w : \tilde{\Lambda}(x) \in \mu\right\}$, where μ is any of the spaces c_0, c or ℓ_∞. More precisely, we have the following:

$$c_0\left(\Delta^\lambda\right) = \left\{x \in w : \lim_{n\to\infty} \tilde{\Lambda}_n(x) = 0\right\},$$

$$c\left(\Delta^\lambda\right) = \left\{x \in w : \lim_{n\to\infty} \tilde{\Lambda}_n(x) \;\text{ exists}\right\},$$

$$\ell_\infty\left(\Delta^\lambda\right) = \left\{x \in w : \sup_n \left|\tilde{\Lambda}_n(x)\right| < \infty\right\}$$

and by using the help of (2.2.7) we obviously observe that

$$x \in \mu(\Delta^\lambda) \Longleftrightarrow \tilde{\Lambda}(x) \in \mu \Longleftrightarrow \Lambda(x) \in \mu(\Delta)$$

and this shows that Definitions 2.2.1 and 2.2.2 are equivalent which can also be noted by the following example:

Example 2.2.3 Consider the sequence $\lambda = (\lambda_k)$ given by $\lambda_k = k$ for all positive integers k. Then $\Delta(\lambda_k) = 1$ for all k. Also, let $x \in w$ be given. Then, by using (2.2.2) and (2.2.5), we respectively obtain that

$$\Lambda_n(x) = \frac{\sigma_n(x)}{n} \quad (n \geq 1) \quad \text{and} \quad \tilde{\Lambda}_n(x) = \frac{x_n}{n} - \frac{\sigma_{n-1}(x)}{n(n-1)} \quad (n > 1),$$

where $\sigma_n(x) = \sum_{k=1}^{n} x_k$ for all n. Besides, let μ be standing for any of the spaces c_0, c or ℓ_∞. Then, by using Definitions 2.2.1 and 2.2.2, we obtain two formulae for the space $\mu(\Delta^\lambda)$, which can respectively be given as follows:

$$\mu(\Delta^\lambda) = \left\{ x \in w : \left(\frac{\sigma_n(x)}{n} \right)_{n=1}^{\infty} \in \mu(\Delta) \right\},$$

$$\mu(\Delta^\lambda) = \left\{ x \in w : \left(\frac{x_n}{n} - \frac{\sigma_{n-1}(x)}{n(n-1)} \right)_{n=2}^{\infty} \in \mu \right\}.$$

In addition, by using (1.1.2), we get

$$\Delta\left(\frac{\sigma_n(x)}{n} \right) = \frac{1}{n}\Delta(\sigma_n(x)) + \Delta\left(\frac{1}{n} \right)\sigma_{n-1}(x) = \frac{x_n}{n} - \frac{\sigma_{n-1}(x)}{n(n-1)} \quad (n > 1)$$

which shows that both above two formulae of $\mu(\Delta^\lambda)$ are equivalent to each others. Further, let's take the sequence $x = (2k) \in c(\Delta)$. Then $\Delta(x_k) = 2$ for all k and so $\lim_{k\to\infty} \Delta(x_k) = 2$. On other side, we have $\sigma_n(x) = n(n+1)$ and so $\Lambda_n(x) = n+1$ for all n. Thus, we get $\tilde{\Lambda}_n(x) = 1$ for all $n > 1$ and hence $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = 1$ which means that $x \in c(\Delta^\lambda)$ (we may note that: although $x \in c(\Delta)$ and $x \in c(\Delta^\lambda)$, but $\lim_{n\to\infty} \Delta(x_n) \neq \lim_{n\to\infty} \tilde{\Lambda}_n(x)$).

From Example 2.2.3, it must be noted that the matrix $\tilde{\Lambda}$ does not preserve the limits of convergent difference sequences. This leads us to introduce the following new concept* of *regularity of* $\tilde{\Lambda}$ *over* $c(\Delta)$ as in the following definition:

*This concept can be generalized for two triangles (or two infinite matrices).

Definition 2.2.4 *We say that the matrix* $\tilde{\Lambda}$ *is regular over* $c(\Delta)$ *if* $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \lim_{n\to\infty} \Delta(x_n)$ *for every* $x \in c(\Delta)$.

Obviously, we note that $\tilde{\Lambda}$, given in Example 2.2.3, is not regular over $c(\Delta)$ (also, some other examples will be given in next chapter and it will be shown that the regularity of $\tilde{\Lambda}$ over $c(\Delta)$ is depending on its own λ). Now, we may begin with proving some main results which are essential in the text.

Theorem 2.2.5 *The* λ*-difference spaces* $c_0(\Delta^\lambda), c(\Delta^\lambda)$ *and* $\ell_\infty(\Delta^\lambda)$ *are* BK *spaces with the norm* $\|\cdot\|_{\Delta^\lambda}$ *defined, for every sequence* x *in these spaces, by*

$$\|x\|_{\Delta^\lambda} = \|\tilde{\Lambda}(x)\|_\infty = \sup_n \left|\tilde{\Lambda}_n(x)\right| = \sup_n \left|\Lambda_n(x) - \Lambda_{n-1}(x)\right|.$$

Proof. Since c_0, c and ℓ_∞ are BK spaces with their natural norm $\|\cdot\|_\infty$ (by (1) of Lemma 1.3.2) and $\tilde{\Lambda}$ is a triangle; it follows by (2) of Lemma 1.3.7 that the matrix domains of $\tilde{\Lambda}$ in the spaces c_0, c and ℓ_∞ are BK spaces with the norm $\|\cdot\|_{\tilde{\Lambda}}$ given by $\|x\|_{\tilde{\Lambda}} = \|\tilde{\Lambda}(x)\|_\infty = \|x\|_{\Delta^\lambda}$. Thus, from (2.2.8) of Definition 2.2.2, we deduce that $c_0(\Delta^\lambda), c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$ are BK spaces with the norm $\|\cdot\|_{\Delta^\lambda}$. □

Remark 2.2.6 It maybe noted that Theorem 2.2.5 can be proved by using Definition 2.2.1, where $c_0(\Delta^\lambda), c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$ are respectively the matrix domains of the triangle Λ in the BK spaces $c_0(\Delta), c(\Delta)$ and $\ell_\infty(\Delta)$ with their norm $\|\cdot\|_\Delta$ (by (2) of Lemma 1.3.2). Hence $c_0(\Delta^\lambda), c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$ are BK spaces with the norm $\|\cdot\|_\Lambda$ given by $\|x\|_\Lambda = \|\Lambda(x)\|_\Delta = \|\Delta(\Lambda(x))\|_\infty = \|\tilde{\Lambda}(x)\|_\infty = \|x\|_{\Delta^\lambda}$ (as $\Delta\Lambda = \tilde{\Lambda}$ by (2.2.7)). Therefore $\|x\|_{\Delta^\lambda} = \|\tilde{\Lambda}(x)\|_\infty = \|\Lambda(x)\|_\Delta$ for every x in the λ-difference spaces.

Theorem 2.2.7 *The* λ*-difference spaces* $c_0(\Delta^\lambda), c(\Delta^\lambda)$ *and* $\ell_\infty(\Delta^\lambda)$ *are isometrically linear-isomorphic to the spaces* c_0, c *and* ℓ_∞, *respectively. That is* $c_0(\Delta^\lambda) \cong c_0$, $c(\Delta^\lambda) \cong c$ *and* $\ell_\infty(\Delta^\lambda) \cong \ell_\infty$.

Proof. To show that $\mu(\Delta^{\lambda}) \cong \mu$, we will prove the existence of a linear operator between $\mu(\Delta^{\lambda})$ and μ which is bijective and norm-preserving, where $\mu = c_0$, c or ℓ_∞. For this, we define the mapping $\tilde{\Lambda} : \mu(\Delta^{\lambda}) \to \mu$ by $x \mapsto \tilde{\Lambda}(x)$ for all $x \in \mu(\Delta^{\lambda})$. Then, this mapping is well-defined by (2.2.8) which is clearly a linear operator. Also, it is easy to see that $\tilde{\Lambda}(x) = 0$ implies $x = 0$ which means that $\tilde{\Lambda}$ is injective. Further, to show that $\tilde{\Lambda}$ is surjective, let $y \in \mu$ and define the sequence $x = (x_k)$ by

$$x_k = \frac{\Delta(\lambda_k\, \sigma_k(y))}{\Delta(\lambda_k)} = \frac{\lambda_k\, \sigma_k(y) - \lambda_{k-1}\, \sigma_{k-1}(y)}{\lambda_k - \lambda_{k-1}} \qquad (k \geq 1). \tag{2.2.9}$$

That is
$$x_k = \frac{1}{\lambda_k - \lambda_{k-1}} \left(\lambda_k \sum_{j=1}^{k} y_j - \lambda_{k-1} \sum_{j=1}^{k-1} y_j \right) \qquad (k \geq 1),$$

where $x_1 = y_1$ (since $\lambda_0 = y_0 = 0$). Then, for every $n \geq 1$, it follows by (2.2.2) that

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) x_k = \frac{1}{\lambda_n} \sum_{k=1}^{n} \big(\lambda_k\, \sigma_k(y) - \lambda_{k-1}\, \sigma_{k-1}(y) \big) = \sigma_n(y)$$

and by using (2.2.7) we get $\tilde{\Lambda}_n(x) = \Lambda_n(x) - \Lambda_{n-1}(x) = \sigma_k(y) - \sigma_{n-1}(y) = y_n$ for every $n \geq 1$ which implies $\tilde{\Lambda}(x) = y \in \mu$ (as $y \in \mu$) and hence $x \in \mu(\Delta^{\lambda})$. Thus $x \in \mu(\Delta^{\lambda})$ such that $\tilde{\Lambda}(x) = y$. This shows that $\tilde{\Lambda}$ is surjective and hence $\tilde{\Lambda}$ is a linear isomorphism. Finally, from Theorem 2.2.5, we have $\|\tilde{\Lambda}(x)\|_\infty = \|x\|_{\Delta^{\lambda}}$ for all $x \in \mu(\Delta^{\lambda})$ which means that $\tilde{\Lambda}$ is norm-preserving, and so $\tilde{\Lambda}$ is a linear bijection which preserves the norm. Therefore, we deduce that $\mu(\Delta^{\lambda}) \cong \mu$ and we have done. □

Corollary 2.2.8 *The λ-difference sequence spaces $c_0(\Delta^{\lambda})$, $c(\Delta^{\lambda})$ and $\ell_\infty(\Delta^{\lambda})$ are isometrically linear-isomorphic to the usual difference spaces $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$, respectively. That is $c_0(\Delta^{\lambda}) \cong c_0(\Delta)$, $c(\Delta^{\lambda}) \cong c(\Delta)$ and $\ell_\infty(\Delta^{\lambda}) \cong \ell_\infty(\Delta)$.*

Proof. It is immediate by combining Lemma 2.1.1 with Theorem 2.2.7. □

Remark 2.2.9 The isomorphic relations $\mu(\Delta^{\lambda}) \cong \mu$ have very important consequences and it will frequently be used throughout. For instance, we may note the following:

(1) The matrix operator $\tilde{\Lambda} : \mu(\Delta^{\lambda}) \to \mu$ is an isometry linear isomorphism, and this implies the continuity of the matrix operator $\tilde{\Lambda}$ which will be used in the next section.

(2) For every $x \in \mu(\Delta^{\lambda})$, there exists a unique sequence $y \in \mu$ connected with x by $y = \tilde{\Lambda}(x)$ and so $x = \Delta(\lambda\sigma(y))/\Delta(\lambda)$. Conversely, for every $y \in \mu$, there exists a unique sequence $x \in \mu(\Delta^{\lambda})$ given by $x = \Delta(\lambda\sigma(y))/\Delta(\lambda)$ and so $y = \tilde{\Lambda}(x)$.

Furthermore, we have the following results characterizing the sequences in $\mu(\Delta^{\lambda})$.

Corollary 2.2.10 *We have the following:*

(1) *If $x \in \ell_\infty(\Delta^{\lambda})$; then $(\Lambda_n(x)/n) \in \ell_\infty$.*

(2) *If $x \in c(\Delta^{\lambda})$; then $(\Lambda_n(x)/n) \in c$ and $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \lim_{n\to\infty} \Lambda_n(x)/n$.*

(3) *In particular, if $x \in c_0(\Delta^{\lambda})$; then $(\Lambda_n(x)/n) \in c_0$.*

Proof. This fact follows from (1) and (2) of Lemma 1.3.4 (since $x \in \mu(\Delta^{\lambda})$ implies that $\Lambda(x) \in \mu(\Delta)$). □

Corollary 2.2.11 *We have the following facts:*

(1) *$x \in \ell_\infty(\Delta^{\lambda})$ if and only if $(\Lambda_n(x)/n) \in \ell_\infty$ and $(\, n\,\Delta(\Lambda_n(x)/n)\,) \in \ell_\infty$.*

(2) *$x \in c(\Delta^{\lambda})$ if and only if $(\Lambda_n(x)/n) \in c$ and $(\, n\,\Delta(\Lambda_n(x)/n)\,) \in c_0$.*

(3) *$x \in c_0(\Delta^{\lambda})$ if and only if $(\Lambda_n(x)/n) \in c_0$ and $(\, n\,\Delta(\Lambda_n(x)/n)\,) \in c_0$.*

Proof. This result is an immediate consequence of Theorem 2.1.7 (since $x \in \mu(\Delta^{\lambda})$ if and only if $\Lambda(x) \in \mu(\Delta)$). □

At the end of this section, we give an example to show that the new λ-difference spaces (in some particular cases of the sequence λ) are totally different from the related classical sequence spaces and other λ-sequence spaces. That is, each one of the spaces $c_0(\Delta^{\lambda})$, $c(\Delta^{\lambda})$ or $\ell_\infty(\Delta^{\lambda})$ is totally different from all the spaces c_0, c, ℓ_∞, c_0^{λ}, c^{λ}, ℓ_∞^{λ},

$c_0(\Delta)$, $c(\Delta)$, $\ell_\infty(\Delta)$, $c_0^\lambda(\Delta)$, $c^\lambda(\Delta)$ and $\ell_\infty^\lambda(\Delta)$. For this, we shall use our terminologies used Example 2.1.6. That is, we will use the symbol μ to denote any of the spaces c_0, c or ℓ_∞ and so $\mu(\Delta)$ stands for the respective one of the spaces $c_0(\Delta)$, $c(\Delta)$ or $\ell_\infty(\Delta)$ while μ^λ is the corresponding space of c_0^λ, c^λ or ℓ_∞^λ and hence $\mu(\Delta^\lambda)$ is the respective one of the spaces $c_0(\Delta^\lambda)$, $c(\Delta^\lambda)$ or $\ell_\infty(\Delta^\lambda)$, respectively. Also, we assume that $\bar{\mu}$ has the same meaning of μ, that is $\bar{\mu}$ is any of the spaces c_0, c or ℓ_∞ (but the equality $\mu = \bar{\mu}$ need not be valid). Then, our aim in the following example is to show that some particular cases of our spaces $\mu(\Delta^\lambda)$ are different from all $\bar{\mu}$, $\bar{\mu}(\Delta)$, $\bar{\mu}^\lambda$ and $\bar{\mu}^\lambda(\Delta)$.

Example 2.2.12 Consider the spaces $\mu(\Delta^\lambda)$ in the particular case of the sequence $\lambda = (\lambda_k)$ given by $\lambda_k = (2^{2k} - 1)/2^{2k-1}$ $(k \geq 1)$ which is a strictly increasing sequence of positive real numbers. Then $\Delta(\lambda_k) = 3/2^{2k-1}$ $(k \geq 1)$ and for any $x \in w$ we have

$$\tilde{\Lambda}_n(x) = \Lambda_n(x) - \Lambda_{n-1}(x) \quad \text{and} \quad \Lambda_n(x) = \frac{2^{2n-1}}{2^{2n}-1} \sum_{k=1}^{n} \frac{3\,x_k}{2^{2k-1}} \qquad (n \geq 1).$$

Also, consider the unbounded sequence $x = (x_k)$ given by $x_k = 2^{2k-1}(\sqrt{k} - \sqrt{k-1}\,)$ for all $k \geq 1$. Then, it can easily be shown that

$$\Lambda_n(x) = \frac{3\sqrt{n}}{2}\Big(1 + \frac{1}{2^{2n}-1}\Big) \qquad (n \geq 1)$$

$$\tilde{\Lambda}_n(x) = \frac{3}{2}\Bigg(\sqrt{n} - \sqrt{n-1} + \frac{\sqrt{n}}{2^{2n}-1} - \frac{\sqrt{n-1}}{2^{2n-2}-1}\Bigg) \qquad (n > 1)$$

which shows that $\tilde{\Lambda}(x) \in c_0$. Thus $x \in c_0(\Delta^\lambda)$ and hence $x \in \mu(\Delta^\lambda)$ (since $c_0(\Delta^\lambda) \subset c(\Delta^\lambda) \subset \ell_\infty(\Delta^\lambda)$). But, it is clear that $x \notin \ell_\infty$ and so $x \notin \bar{\mu}$ (as $\bar{\mu} \subset \ell_\infty$ by Lemma 1.3.1). Thus, we have $x \in \mu(\Delta^\lambda)$ while $x \notin \bar{\mu}$ and so $\mu(\Delta^\lambda) \neq \bar{\mu}$.

On other side, it is obvious that $\Lambda(x) \notin \ell_\infty$ and hence $x \notin \ell_\infty^\lambda$ which means that $x \notin \bar{\mu}^\lambda$ (as $\bar{\mu}^\lambda \subset \ell_\infty^\lambda$ by (1) of Lemma 1.3.15). Thus $\mu(\Delta^\lambda) \neq \bar{\mu}^\lambda$.

Further, for every $k \geq 1$, it can easily be seen that $\sqrt{k}+\sqrt{k-1} \geq (\sqrt{k+1}+\sqrt{k}\,)/2$ and hence $\sqrt{k}-\sqrt{k-1} \leq 2(\sqrt{k+1}-\sqrt{k}\,)$ which implies $\Delta(x_{k+1}) \geq 2^{2k}(\sqrt{k+1}-\sqrt{k}\,)$

and so $\Delta(x_k) \geq x_k/2 \to \infty$ (as $k \to \infty$) as well as $\Lambda_n(\,\Delta(x)\,) \geq \Lambda_n(x)/2 \to \infty$ (as $n \to \infty$). Thus $\Delta(x) \notin \ell_\infty$ and $\Delta(x) \notin \ell_\infty^\lambda$. Hence, we deduce that $x \notin \ell_\infty(\Delta)$ as well as $x \notin \ell_\infty^\lambda(\Delta)$ (by using (2.2.3)). This implies that $x \notin \bar{\mu}(\Delta)$ and $x \notin \bar{\mu}^\lambda(\Delta)$ which means that $\mu(\Delta^\lambda) \neq \bar{\mu}(\Delta)$ as well as $\mu(\Delta^\lambda) \neq \bar{\mu}^\lambda(\Delta)$.

Therefore, we conclude that each of the spaces $\mu(\Delta^\lambda)$ is totally different from all the spaces $\bar{\mu}$, $\bar{\mu}(\Delta)$, $\bar{\mu}^\lambda$ and $\bar{\mu}^\lambda(\Delta)$.

2.3 Schauder Basis

In the last section, we will construct the Schauder bases for the λ-difference spaces $c_0(\Delta^\lambda)$ and $c(\Delta^\lambda)$ and conclude their separability while $\ell_\infty(\Delta^\lambda)$ is not separable and has no Schauder basis.

If a normed sequence space X (or an arbitrary normed spaces X) contains a sequence $(b_k)_{k=1}^\infty$ with the property that for every $x \in X$ there exists a unique sequence $(\alpha_k)_{k=1}^\infty$ of scalars such that $\lim_{n\to\infty} \|x - (\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n)\| = 0$; then the sequence $(b_k)_{k=1}^\infty$ is called a Schauder basis for X (or briefly a basis for X) and the series $\sum_{k=1}^\infty \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to the given basis, and we then say that x has the unique representation $x = \sum_{k=1}^\infty \alpha_k b_k$. For example, Lemma 1.3.3 tells us that the sequence $(e_1, e_2, e_3, \cdots)$ is the Schauder basis for c_0 while $(e, e_1, e_2, \cdots)$ is the Schauder basis for c (see also Lemma 2.1.2 for the Schauder bases of $c_0(\Delta)$ and $c(\Delta)$).

Now, we may begin this section with obtaining the Schauder basis for the space $c_0(\Delta^\lambda)$ of λ-null difference sequences. That is, we shall construct a sequence of points in the space $c_0(\Delta^\lambda)$ (every term in this sequence is a sequence in $c_0(\Delta^\lambda)$) to be a Schauder basis for $c_0(\Delta^\lambda)$ such that every sequence in $c_0(\Delta^\lambda)$ can uniquely be written as an infinite linear combination of the sequences in this basis.

Theorem 2.3.1 *For each $k \geq 1$, define the sequence $e_k^\lambda = (e_{nk}^\lambda)_{n=1}^\infty$ by*

$$e_{nk}^\lambda = \begin{cases} 0\,; & (n < k), \\ \dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}}\,; & (n = k), \qquad (n \geq 1) \\ 1\,; & (n > k). \end{cases}$$

Then, the sequence $\left(e_k^\lambda\right)_{k=1}^\infty$ is a Schauder basis for the space $c_0(\Delta^\lambda)$ and every $x \in c_0(\Delta^\lambda)$ has a unique representation in the following form:

$$x = \sum_{k=1}^\infty \tilde{\Lambda}_k(x)\, e_k^\lambda\,. \tag{2.3.1}$$

Proof. For each $k \geq 1$, it is clear that $\Lambda_n(e_k^\lambda) = 0$ for $1 \leq n < k$ and $\Lambda_n(e_k^\lambda) = 1$ for all $n \geq k$ and so $\tilde{\Lambda}_n(e_k^\lambda) = \delta_{nk}$ for all $n \geq 1$. Thus $\tilde{\Lambda}(e_k^\lambda) = e_k \in c_0$ and hence $e_k^\lambda \in c_0(\Delta^\lambda)$ for every $k \geq 1$. This means that $\left(e_k^\lambda\right)_{k=1}^\infty$ is a sequence in $c_0(\Delta^\lambda)$. Further, let $x \in c_0(\Delta^\lambda)$ be given and for every positive integer m, we put

$$x^{(m)} = \sum_{k=1}^m \tilde{\Lambda}_k(x)\, e_k^\lambda\,.$$

Then, we find that

$$\tilde{\Lambda}(x^{(m)}) = \sum_{k=1}^m \tilde{\Lambda}_k(x)\, \tilde{\Lambda}(e_k^\lambda) = \sum_{k=1}^m \tilde{\Lambda}_k(x)\, e_k$$

and hence

$$\tilde{\Lambda}_n(x - x^{(m)}) = \begin{cases} 0\,; & (1 \leq n \leq m), \\ \tilde{\Lambda}_n(x)\,; & (n > m). \end{cases}$$

Now, for any positive real $\epsilon > 0$, there is a positive integer m_0 such that $|\tilde{\Lambda}_m(x)| < \epsilon$ for all $m \geq m_0$. Thus, for every $m \geq m_0$, we find that

$$\|x - x^{(m)}\|_{\Delta^\lambda} = \sup_{n>m} |\tilde{\Lambda}_n(x)| \leq \sup_{n>m_0} |\tilde{\Lambda}_n(x)| \leq \epsilon\,.$$

Therefore, it follows that $\lim_{m\to\infty} \|x - x^{(m)}\|_{\Delta^\lambda} = 0$ which means that x is represented as in (2.3.1). Thus, it is remaining to show the uniqueness of the representation (2.3.1) of x. For this, suppose that $x = \sum_{k=1}^{\infty} \alpha_k \, e_k^{\lambda}$. Then, we have to show that $\alpha_n = \tilde{\Lambda}_n(x)$ for all n, which is immediate by operating $\tilde{\Lambda}_n$ on both sides of (2.3.1) for each $n \geq 1$, where the continuity of $\tilde{\Lambda}$ (see (1) of Remark 2.2.9) allows us to obtain that

$$\tilde{\Lambda}_n(x) = \sum_{k=1}^{\infty} \alpha_k \, \tilde{\Lambda}_n(e_k^{\lambda}) = \sum_{k=1}^{\infty} \alpha_k \, \delta_{nk} = \alpha_n$$

for all $n \geq 1$ and hence the representation (2.3.1) of x is unique. □

Moreover, we have the following result constructing the Schauder basis for the sequence space $c(\Delta^{\lambda})$ of λ-convergent difference sequences.

Theorem 2.3.2 *The sequence $\left(e^{\lambda}, e_1^{\lambda}, e_2^{\lambda}, \cdots\right)$ is a Schauder basis for the space $c(\Delta^{\lambda})$ and every $x \in c(\Delta^{\lambda})$ has a unique representation in the following form:*

$$x = L \, e^{\lambda} + \sum_{k=1}^{\infty} \left(\tilde{\Lambda}_k(x) - L\right) e_k^{\lambda} \,, \tag{2.3.2}$$

where $L = \lim_{n\to\infty} \tilde{\Lambda}_n(x)$, the sequence $\left(e_k^{\lambda}\right)_{k=1}^{\infty}$ is as in Theorem 2.3.1 and e^{λ} is the following sequence:

$$e^{\lambda} = \left(\frac{n\lambda_n - (n-1)\,\lambda_{n-1}}{\lambda_n - \lambda_{n-1}}\right)_{n=1}^{\infty}.$$

Proof. By using (2.2.2), we get $\Lambda_n(e^{\lambda}) = n$ and so $\tilde{\Lambda}_n(e^{\lambda}) = 1$ for all n which yields that $\tilde{\Lambda}(e^{\lambda}) = e \in c$ and hence $e^{\lambda} \in c(\Delta^{\lambda})$. This, together with $e_k^{\lambda} \in c_0(\Delta^{\lambda}) \subset c(\Delta^{\lambda})$ for all k, implies that that $\left(e^{\lambda}, e_1^{\lambda}, e_2^{\lambda}, \cdots\right)$ is a sequence in $c(\Delta^{\lambda})$. Also, let $x \in c(\Delta^{\lambda})$ be given. Then $\tilde{\Lambda}(x) \in c$ which means the convergence of the sequence $\tilde{\Lambda}(x)$ to a unique limit, say $L = \lim_{n\to\infty} \tilde{\Lambda}_n(x)$. Thus, by taking $y = x - L\, e^{\lambda}$, we get $\tilde{\Lambda}(y) = \tilde{\Lambda}(x) - L\, e \in c_0$ and so $y \in c_0(\Delta^{\lambda})$. Hence, it follows by Theorem 2.3.1 that y can be uniquely represented in the following form:

$$y = \sum_{k=1}^{\infty} \tilde{\Lambda}_k(y) \, e_k^{\lambda} = \sum_{k=1}^{\infty} \left(\tilde{\Lambda}_k(x) - L \, \tilde{\Lambda}_k(e^{\lambda})\right) e_k^{\lambda} = \sum_{k=1}^{\infty} \left(\tilde{\Lambda}_k(x) - L\right) e_k^{\lambda}.$$

Consequently, our x can also be uniquely written as

$$x = L\,e^{\lambda} + y = L\,e^{\lambda} + \sum_{k=1}^{\infty}\left(\tilde{\Lambda}_k(x) - L\right) e_k^{\lambda}$$

which proves the unique representation (2.3.2) of x and completes the proof. □

Further, the topological property of separability will be discussed in the next result:

Corollary 2.3.3 *We have the following facts:*

(1) *The spaces $c_0(\Delta^{\lambda})$ and $c(\Delta^{\lambda})$ are separable BK spaces.*

(2) *The space $\ell_{\infty}(\Delta^{\lambda})$ is a non-separable BK space and has no Schauder basis.*

Proof. Since $c_0(\Delta^{\lambda})$ and $c(\Delta^{\lambda})$ are BK spaces and so normed spaces having Schauder bases; this result is immediate by (3) of Lemma 1.3.3. □

Finally, we end this chapter with the following example to apply our results in Theorems 2.3.1 and 2.3.2 to Schauder bases for the spaces $c_0(\Delta^{\lambda})$ and $c(\Delta^{\lambda})$.

Example 2.3.4 Here, we give an example of the unique representation of a single sequence in a particular case of our spaces $c_0(\Delta^{\lambda})$ and $c(\Delta^{\lambda})$. For this, consider the sequence $\lambda = (\lambda_k)$ given by $\lambda_k = k(k+1)$ for $k \geq 1$. Then, for every $x \in w$, we have

$$\tilde{\Lambda}_n(x) = \Lambda_n(x) - \Lambda_{n-1}(x) \quad \text{and} \quad \Lambda_n(x) = \frac{2}{n(n+1)} \sum_{k=1}^{n} k\,x_k \qquad (n \geq 1).$$

Thus, by using Definition 2.2.1, we obtain the following particular cases of the general spaces of λ-difference sequences:

$$c_0(\Delta^{\lambda}) = \left\{ x = (x_k) \,:\, \left(\frac{2}{n(n+1)} \sum_{k=1}^{n} k\,x_k \right)_{n=1}^{\infty} \in c_0(\Delta) \right\},$$

$$c(\Delta^{\lambda}) = \left\{ x = (x_k) \,:\, \left(\frac{2}{n(n+1)} \sum_{k=1}^{n} k\,x_k \right)_{n=1}^{\infty} \in c(\Delta) \right\}.$$

Further, with help of Theorems 2.3.1 and 2.3.2, the Schauder bases for these two spaces are respectively the two sequences $\left(e_1^\lambda, e_2^\lambda, e_3^\lambda, \cdots\right)$ and $\left(e^\lambda, e_1^\lambda, e_2^\lambda, \cdots\right)$, where

$$e_1^\lambda = (1,\, 1,\, 1,\, \cdots)\,,\;\; e_2^\lambda = (0,\, 3/2,\, 1,\, 1, \cdots)\,,$$

$$\mathrm{e}_3^\lambda = (0,\, 0,\, 2,\, 1,\, 1, \cdots)\,,\;\; e_4^\lambda = (0,\, 0,\, 0,\, 5/2,\, \cdots)\,,\, \ldots \text{etc.}$$

$$\text{and }\; e^\lambda = \big(\,(3k-1)/2\,\big)_{k=1}^{\infty} = (1,\, 5/2,\, 4,\, 11/2,\, \cdots)\,.$$

Now, consider the sequence $y = (y_k) \in c_0(\Delta^\lambda)$ defined by $y_k = (k+1)\sqrt{k} - (k-1)\sqrt{k-1}$ for all $k \geq 1$. Then, we have

$$\Lambda_n(y) = 2\sqrt{n} \quad \text{and} \quad \tilde{\Lambda}_n(y) = 2(\sqrt{n} - \sqrt{n-1}\,) \qquad (n \geq 1).$$

Thus, our sequence y has the unique representation

$$y = 2\sum_{n=1}^{\infty}(\sqrt{n} - \sqrt{n-1}\,)\, e_n^\lambda$$

with respect to Schauder basis (e_n^λ) of the space $c_0(\Delta^\lambda)$.

In addition, if we define $x = (x_k)$ by $x_k = 1 - 3k + y_k$ for $k \geq 1$. Then, we find that $\tilde{\Lambda}_n(x) = -2 + \tilde{\Lambda}_n(y)$ for all $n \geq 1$. Thus $x \in c(\Delta^\lambda)$ such that $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = -2$. Hence, by applying Theorem 2.3.2, the sequence x has the unique representation

$$x = -2\, e^\lambda + 2\sum_{n=1}^{\infty}(\sqrt{n} - \sqrt{n-1} + 1\,)\, e_n^\lambda$$

with respect to the Schauder basis $\left(e^\lambda, e_1^\lambda, e_2^\lambda, \cdots\right)$ of the space $c(\Delta^\lambda)$.

Chapter 3

SOME INCLUSION RELATIONS

3 SOME INCLUSION RELATIONS

In the present chapter, we will deduce some inclusion relations between the new λ-difference spaces and derive some other inclusion relations between these spaces and the classical sequence spaces. This chapter is divided into three sections, the first is devoted to derive some basic inclusion relations, the second is for proving some preliminary results to be used in deriving our main results in last section. The materials of this chapter are part of our research paper* which has been published in Albaydha Univ. J., and presented in the 2nd Conference of Albaydha University on 2021.

3.1 Basic Results

In this section, we establish some basic inclusion relations concerning with the new λ-difference spaces $c_0(\Delta^\lambda)$, $c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$.

Lemma 3.1.1 *The inclusions $c_0(\Delta^\lambda) \subset c(\Delta^\lambda) \subset \ell_\infty(\Delta^\lambda)$ strictly hold.*

Proof. These inclusions are immediate from the inclusions $c_0 \subset c \subset \ell_\infty$ (see Lemma 1.3.1). To see that, let $x \in c_0(\Delta^\lambda)$. Then $\tilde{\Lambda}(x) \in c_0$ and so $\tilde{\Lambda}(x) \in c$ which means that $x \in c(\Delta^\lambda)$. Thus $c_0(\Delta^\lambda) \subset c(\Delta^\lambda)$ and we can similarly prove that $c(\Delta^\lambda) \subset \ell_\infty(\Delta^\lambda)$. Also, to show that these inclusions are strict, we define two sequences x and y by

$$x_k = \frac{k\lambda_k - (k-1)\,\lambda_{k-1}}{\lambda_k - \lambda_{k-1}} \quad \text{and} \quad y_k = \frac{(-1)^k}{2}\left(\frac{\lambda_k + \lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\right) \qquad (k \geq 1)\,.$$

*A.K. Noman and O.H. Al-Sabri, *On the new λ-difference spaces of convergent and bounded sequences*, Albaydha Univ. J., **3**(2) (2021), 18–30.

Then, for every $n \geq 1$, it can be easily seen that

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} (k\lambda_k - (k-1)\lambda_{k-1}) = n\,,$$

$$\Lambda_n\,(y) = \frac{1}{2\lambda_n} \sum_{k=1}^{n} (-1)^k (\lambda_k + \lambda_{k-1}) = \frac{(-1)^n}{2}$$

and so $\tilde{\Lambda}_n\,(x) = \Lambda_n\,(x) - \Lambda_{n-1}\,(x) = 1$ and $\tilde{\Lambda}_n\,(y) = \Lambda_n\,(y) - \Lambda_{n-1}\,(y) = (-1)^n\ (n > 1)$. Thus $\tilde{\Lambda}_n\,(x) = e \in c \setminus c_0$ and $\tilde{\Lambda}_n(y) \in \ell_\infty \setminus c$ which means that $x \in c(\Delta^\lambda) \setminus c_0(\Delta^\lambda)$ and $y \in \ell_\infty(\Delta^\lambda) \setminus c(\Delta^\lambda)$. This ends the proof. □

Lemma 3.1.2 *The inclusions $c \subset c_0(\Delta^\lambda)$ and $c^\lambda \subset c_0(\Delta^\lambda)$ strictly hold.*

Proof. First, we show that $c^\lambda \subset c_0(\Delta^\lambda)$. For this, take any $x \in c^\lambda$. Then, we have $\Lambda(x) \in c$ and so $\tilde{\Lambda}\,(x) = (\Lambda_n\,(x) - \Lambda_{n-1}\,(x)) \in c_0$ which means that $x \in c_0(\Delta^\lambda)$ and hence $c^\lambda \subset c_0(\Delta^\lambda)$. Also, to show that this inclusion is strict, define $x = (x_k)$ by

$$x_k = \frac{\lambda_k \sqrt{k} - \lambda_{k-1}\sqrt{k-1}}{\lambda_k - \lambda_{k-1}} \qquad (k \geq 1).$$

Then $\Lambda_n\,(x) = \sqrt{n}$ and so $\Lambda\,(x) = (\sqrt{n}) \notin c$ which means that $x \notin c^\lambda$, but $\tilde{\Lambda}\,(x) = \left(\sqrt{n} - \sqrt{n-1}\right) \in c_0$ and hence $x \in c_0(\Delta^\lambda)$. Thus $x \in c_0(\Delta^\lambda) \setminus c^\lambda$ and so the inclusion $c^\lambda \subset c_0(\Delta^\lambda)$ is strict. Besides, since $c \subset c^\lambda$ (by (2) of Lemma 1.3.15); we deduce the other strict inclusion and this completes the proof. □

Corollary 3.1.3 *We have the following facts:*

(1) *The inclusions $c_0^\lambda \subset c_0(\Delta^\lambda)\,,\ c^\lambda \subset c(\Delta^\lambda)$ and $\ell_\infty^\lambda \subset \ell_\infty(\Delta^\lambda)$ strictly hold.*

(2) *The inclusions $c_0 \subset c_0(\Delta^\lambda)\,,\ c \subset c(\Delta^\lambda)$ and $\ell_\infty \subset \ell_\infty(\Delta^\lambda)$ strictly hold.*

Proof. (1) The first two inclusions follow from Lemmas 1.3.15 and 3.1.2, and the third inclusion can be proved by the same way used in the proof of Lemma 3.1.2.

(2) The first inclusion follows from Lemma 3.1.2 (as $c_0 \subset c$) and the second two inclusions follow from part (1) (as $c \subset c^\lambda$ and $\ell_\infty \subset \ell_\infty^\lambda$ by Lemma 1.3.15). □

Corollary 3.1.4 *Let $1 \leq p < \infty$. Then, we have the following:*

(1) *All the spaces cs_0, cs, ℓ_p, ℓ_p^λ and bv_p are strictly included in $c_0(\Delta^\lambda)$.*

(2) *The inclusion $bs \subset \ell_\infty(\Delta^\lambda)$ strictly holds.*

Proof. This result can similarly be proved as Corollary 3.1.3 by using the help of Lemmas 1.3.1, 1.3.15 and 3.1.2. □

Remark 3.1.5 The spaces ℓ_∞ and $c_0(\Delta^\lambda)$ overlap (since $c \subset \ell_\infty \cap c_0(\Delta^\lambda)$), but $c_0(\Delta^\lambda)$ cannot be included in ℓ_∞. To see that, we have already obtained an unbounded sequence $x \in c_0(\Delta) \setminus \ell_\infty$ (see the proof of Lemma 3.1.2 and note that $x_k \geq \sqrt{k}$ for all $k \geq 1$). On other side, although the space ℓ_∞ cannot be included in $c_0(\Delta)$ (see (1) of Remark 2.1.5), but it can be strictly included in $c_0(\Delta^\lambda)$ for some particular λ (as shown in the next theorem).

Theorem 3.1.6 *The inclusion $\ell_\infty \subset c_0(\Delta^\lambda)$ strictly holds* $\Longleftrightarrow \lim_{n\to\infty} \lambda_{n-1}/\lambda_n = 1$ $\Longleftrightarrow \lim_{n\to\infty} \lambda_n/\Delta(\lambda_n) = \infty$.

Proof. Since $\tilde{\Lambda}$ is a triangle; we can use (1) of Lemma 1.3.7 to deduce that

$$\begin{aligned} \ell_\infty \subset c_0(\Delta^\lambda) &\Longleftrightarrow x \in c_0(\Delta^\lambda) \text{ for all } x \in \ell_\infty \\ &\Longleftrightarrow \tilde{\Lambda}(x) \in c_0 \text{ for all } x \in \ell_\infty \\ &\Longleftrightarrow \tilde{\Lambda} \in (\ell_\infty, c_0) \quad (\tilde{\Lambda} \text{ is a triangle}). \end{aligned}$$

That is, the inclusion $\ell_\infty \subset c_0(\Delta^\lambda)$ holds if and only if $\tilde{\Lambda} \in (\ell_\infty, c_0)$. This together with (1) of Lemma 1.3.11 lead us to obtain that $\ell_\infty \subset c_0(\Delta^\lambda) \Longleftrightarrow \lim_{n\to\infty} \sum_{k=1}^\infty |\tilde{\lambda}_{nk}| = 0$. On other side, for every $n > 1$, it follows by (2.2.4) that

$$\sum_{k=1}^\infty \left|\tilde{\lambda}_{nk}\right| = \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right) \sum_{k=1}^{n-1} (\lambda_k - \lambda_{k-1}) + \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} = 2\left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right).$$

Thus, we find that $\lim_{n\to\infty} \sum_{k=1}^\infty |\tilde{\lambda}_{nk}| = 0$ if and only if $\lim_{n\to\infty} \lambda_{n-1}/\lambda_n = 1$. Therefore, the inclusion $\ell_\infty \subset c_0(\Delta^\lambda)$ holds if and only if $\lim_{n\to\infty} \lambda_{n-1}/\lambda_n = 1$. Also, this inclusion must be strict, since the equality cannot be satisfied by Remark 3.1.5. Finally, to conclude the proof, it must be noted that

$$\lim_{n\to\infty} \lambda_{n-1}/\lambda_n = 1 \Longleftrightarrow \lim_{n\to\infty} \Delta(\lambda_n)/\lambda_n = 0 \Longleftrightarrow \lim_{n\to\infty} \lambda_n/\Delta(\lambda_n) = \infty. \quad \square$$

Theorem 3.1.7 *We have the following:*

(1) *The inclusion* $\ell_\infty^\lambda \cap c(\Delta^\lambda) \subset c_0(\Delta^\lambda)$ *strictly holds.*

(2) *The equality* $\ell_\infty^\lambda \cap c(\Delta^\lambda) = \ell_\infty^\lambda \cap c_0(\Delta^\lambda)$ *holds.*

(3) *The inclusion* $c^\lambda \subset \ell_\infty^\lambda \cap c_0(\Delta^\lambda)$ *strictly holds.*

Proof. This result is immediate by Lemma 2.1.4. To see that, we have the following:

For (1), take any $x \in \ell_\infty^\lambda \cap c(\Delta^\lambda)$. Then $x \in \ell_\infty^\lambda$ as well as $x \in c(\Delta^\lambda)$. Thus $\Lambda(x) \in \ell_\infty$ and $\Lambda(x) \in c(\Delta)$. This implies that $\Lambda(x) \in \ell_\infty \cap c(\Delta)$ and so $\Lambda(x) \in c_0(\Delta)$ (by (1) of Lemma 2.1.4). Thus $x \in c_0(\Delta^\lambda)$ and so $\ell_\infty^\lambda \cap c(\Delta^\lambda) \subset c_0(\Delta^\lambda)$ which is a strict inclusion by the example given in the proof of Lemma 3.1.2 ($x \in c_0(\Delta^\lambda) \setminus \ell_\infty^\lambda$).

Similarly, the equality in part (2) can be proved by using the corresponding equality given in (2) of Lemma 2.1.4.

To prove (3), we have $c^\lambda \subset \ell_\infty^\lambda$ (by (1) of Lemma 1.3.15) as well as $c^\lambda \subset c_0(\Delta^\lambda)$ (by Lemma 3.1.2). Thus, we deduce the inclusion $c^\lambda \subset \ell_\infty^\lambda \cap c_0(\Delta^\lambda)$. To show that this inclusion is strict, there must exist a sequence $z \in \ell_\infty \cap c_0(\Delta)$ such that $z \notin c$ (since the inclusion $c \subset \ell_\infty \cap c_0(\Delta)$ is strict by (3) of Lemma 2.1.4). This implies that $z \in \ell_\infty \setminus c$ and $\Delta(z) \in c_0$. Now, define a sequence x in terms of z by $x_k = \Delta(\lambda_k z_k)/\Delta(\lambda_k)$ for all k. Then, by using (2.2.2), we find that $\Lambda(x) = z \in \ell_\infty \setminus c$ and so $\tilde{\Lambda}(x) = \Delta(z) \in c_0$. Thus, it follows that $x \in \ell_\infty^\lambda \setminus c^\lambda$ and $x \in c_0(\Delta^\lambda)$. Therefore $x \in \ell_\infty^\lambda \cap c_0(\Delta^\lambda)$ while $x \notin c^\lambda$ which shows that the inclusion $c^\lambda \subset \ell_\infty^\lambda \cap c_0(\Delta^\lambda)$ is strict and this stage completes the proof of our result. $\square$

Corollary 3.1.8 *We have the following:*

(1) *The inclusion* $\ell_\infty \cap c(\Delta^\lambda) \subset c_0(\Delta^\lambda)$ *strictly holds.*

(2) *The equality* $\ell_\infty \cap c(\Delta^\lambda) = \ell_\infty \cap c_0(\Delta^\lambda)$ *holds.*

(3) $\ell_\infty \subset c(\Delta^\lambda) \iff \ell_\infty \subset c_0(\Delta^\lambda) \iff \lim_{n\to\infty} \lambda_n/\Delta(\lambda_n) = \infty$.

Proof. (1) Since $\ell_\infty \subset \ell_\infty^\lambda$ (by (2) of Lemma 1.3.15); we have $\ell_\infty \cap c(\Delta^\lambda) \subset \ell_\infty^\lambda \cap c(\Delta^\lambda)$ and so we deduce the strict inclusion $\ell_\infty \cap c(\Delta^\lambda) \subset c_0(\Delta^\lambda)$ (by (1) of Theorem 3.1.7).

(2) Since $\ell_\infty \cap \ell_\infty^\lambda = \ell_\infty$; the given equality can immediately be obtained by taking the intersection of ℓ_∞ with both sides of the equality in (2) of Theorem 3.1.7.

(3) By using part (2) with Theorem 3.1.6, we find that

$$\begin{aligned} \ell_\infty \subset c(\Delta^\lambda) &\iff \ell_\infty \cap c(\Delta^\lambda) = \ell_\infty \\ &\iff \ell_\infty \cap c_0(\Delta^\lambda) = \ell_\infty \\ &\iff \ell_\infty \subset c_0(\Delta^\lambda) \\ &\iff \lim_{n\to\infty} \lambda_n/\Delta(\lambda_n) = \infty \end{aligned}$$

which ends the proof of this result. □

3.2 Preliminary Results

In this section, we will derive some preliminaries and define some terminologies which will be used in proving the main results in the next section.

We may begin with the following result which gives another formula for the $\tilde{\Lambda}$-transform of any sequence $x \in w$.

Lemma 3.2.1 *For every sequence* $x \in w$*, we have the following equality:*

$$\tilde{\Lambda}_n(x) = \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right) \sum_{k=2}^{n} \lambda_{k-1} \Delta(x_k) \qquad (n \geq 2).$$

Proof. Let $x \in w$. Then, for every $n \geq 2$, we have $\tilde{\Lambda}_n(x) = \Lambda_n(x) - \Lambda_{n-1}(x)$ and by using (2.2.2) we find that

$$\tilde{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} \Delta(\lambda_k)\, x_k - \frac{1}{\lambda_{n-1}} \sum_{k=1}^{n-1} \Delta(\lambda_k)\, x_k$$

which leads us to derive the following:

$$\begin{aligned}
\tilde{\Lambda}_n(x) &= \frac{1}{\lambda_n} \sum_{k=1}^{n} \Delta(\lambda_k)\, x_k - \frac{1}{\lambda_{n-1}} \sum_{k=1}^{n} \Delta(\lambda_k)\, x_k + \frac{\Delta(\lambda_n)}{\lambda_{n-1}}\, x_n \\
&= \frac{\Delta(\lambda_n)}{\lambda_{n-1}}\, x_n - \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n} \right) \sum_{k=1}^{n} \Delta(\lambda_k)\, x_k \\
&= \frac{\Delta(\lambda_n)}{\lambda_n \lambda_{n-1}} \sum_{k=1}^{n} \Delta(\lambda_k x_k) - \frac{\Delta(\lambda_n)}{\lambda_n \lambda_{n-1}} \sum_{k=1}^{n} \Delta(\lambda_k)\, x_k \\
&= \frac{\Delta(\lambda_n)}{\lambda_n \lambda_{n-1}} \sum_{k=1}^{n} \lambda_{k-1}(x_k - x_{k-1}) \\
&= \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \right) \sum_{k=2}^{n} \lambda_{k-1}\, \Delta\left(x_k\right) \\
&= \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n} \right) \sum_{k=2}^{n} \lambda_{k-1}\, \Delta\left(x_k\right)
\end{aligned}$$

which proves the given relation and concludes the proof. □

As we have seen in the proof of above lemma, we are in need for more simplicity in notations. That is, we need to quoting some additional conventions and terminologies to be used in the sequel. So, we define the real sequences $u = (u_k)$ and $v = (v_k)$ by

$$u_k = \frac{\lambda_k}{\Delta(\lambda_k)} = \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \quad \text{and} \quad v_k = \frac{\lambda_{k-1}}{\Delta(\lambda_k)} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}} \qquad (k \geq 1). \qquad (3.2.1)$$

Then, it is obvious that $u_1 = 1$ and $u_k > 1$ for all $k > 1$ while $v_1 = 0$ and $v_k > 0$ for all $k > 1$. Thus, by using the equality given in Lemma 3.2.1, the $\tilde{\Lambda}$-transform of any sequence $x \in w$ is given by

$$\tilde{\Lambda}_n(x) = \frac{1}{\lambda_n v_n} \sum_{k=2}^{n} \lambda_{k-1}\, \Delta(x_k) = \frac{1}{\lambda_{n-1} u_n} \sum_{k=2}^{n} \lambda_{k-1}\, \Delta(x_k) \qquad (n \geq 2)\,. \qquad (3.2.2)$$

Remark 3.2.2 The three real sequences $(k)_{k=1}^{\infty}$, $u = (u_k)_{k=1}^{\infty}$ and $v = (v_k)_{k=1}^{\infty}$ will play important roles in the remaining part of this study and will frequently be used throughout. Besides, by using (3.2.2), the $\tilde{\Lambda}$-transform of the sequence (k) is given by

$$\tilde{\Lambda}_n(k) = \frac{1}{\lambda_n v_n} \sum_{k=2}^{n} \lambda_{k-1} = \frac{1}{\lambda_{n-1} u_n} \sum_{k=2}^{n} \lambda_{k-1} \qquad (n \geq 2)\,. \tag{3.2.3}$$

Further, by taking $y = (1, 1, 1, \cdots) \in c$ in (2.2.9), we note by (2) of Remark 2.2.9 that $(\,\Delta(k\lambda_k)/\Delta(\lambda_k)\,)_{k=1}^{\infty} \in c(\Delta^{\lambda})$ and we have

$$\tilde{\Lambda}_n \left(\frac{\Delta(k\lambda_k)}{\Delta(\lambda_k)} \right) = 1 \qquad (n \geq 1), \tag{3.2.4}$$

$$\frac{\Delta(k\lambda_k)}{\Delta(\lambda_k)} = k + v_k \qquad (k \geq 1). \tag{3.2.5}$$

Now, we may prove the following preliminaries:

Lemma 3.2.3 *We have the following equalities:*

(1) $u_k = 1 + v_k \qquad (k \geq 1)\,,$

(2) $\Delta(u_k) = \Delta(v_k) \qquad (k \geq 2),$

(3) $\tilde{\Lambda}_n(u) = \tilde{\Lambda}_n(v) \qquad (n \geq 2)\,,$

(4) $\tilde{\Lambda}_n(k + v_k) = 1 \qquad (n \geq 1)\,,$

(5) $\tilde{\Lambda}_n(k) + \tilde{\Lambda}_n(v) = 1 \qquad (n \geq 1)\,.$

Proof. For (1), we may note that $\lambda_k = \Delta(\lambda_k) + \lambda_{k-1}$ for all k and then by using (3.2.1) we get the equality in (1). Also, part (2) is immediate from (1) by operating Δ on both sides and noting that $\Delta(1, 1, 1, \cdots) = (1, 0, 0, \cdots)$, that is $\Delta(e) = e_1$. Besides, the equality in part (3) follows from (3.2.2) with help of (2). Further, part (4) is obtained by combining (3.2.4) and (3.2.5). Finally, part (5) is immediate from (4) by the linear property of the matrix operator $\tilde{\Lambda}$. □

Lemma 3.2.4 *We have the following inequalities:*

(1) $\lambda_{k-1}\, \Delta(v_k) \leq\, \Delta(\lambda_k v_k) \qquad (k \geq 1)\,,$

(2) $\left|\tilde{\Lambda}_n(v)\right| \leq\, 1 \qquad (n \geq 1)\,,$

(3) $0\, \leq\, \tilde{\Lambda}_n\,(k)\, \leq\, 2 \qquad (n \geq 1)\,.$

Proof. To prove (1), let $k \geq 1$. Then, by using (1.1.2) and then (3.2.1), we find that

$$\Delta(\lambda_k v_k) =\, v_k\, \Delta(\lambda_k) + \lambda_{k-1}\Delta(v_k) =\, \lambda_{k-1} + \lambda_{k-1}\Delta(v_k) \geq\, \lambda_{k-1}\Delta(v_k).$$

For (2), we have $\tilde{\Lambda}_1(v) = v_1 = 0$ and so the given inequality is true for $n = 1$. Also, for $n \geq 2$, we can use (3.2.2) and part (1) to obtain that

$$\begin{aligned}\left|\tilde{\Lambda}_n(v)\right| &= \frac{1}{\lambda_n v_n}\left|\sum\nolimits_{k=2}^{n} \lambda_{k-1}\, \Delta(v_k)\right| \\ &\leq \frac{1}{\lambda_n v_n}\left|\sum\nolimits_{k=2}^{n} \Delta(\lambda_k\, v_k)\right| \\ &= \frac{1}{\lambda_n v_n}\sum\nolimits_{k=2}^{n} \Delta(\lambda_k\, v_k) \\ &= \frac{1}{\lambda_n v_n}\,(\lambda_n\, v_n) \qquad (\text{as } v_1 = 0) \\ &= 1.\end{aligned}$$

Finally, to prove (3), we have $\tilde{\Lambda}_1\,(k) = 1$ and this together with (3.2.3) lead us to deduce that $\tilde{\Lambda}_n\,(k) \geq 0$ for all $n \geq 1$ and so the given inequality is immediate by (5) of Lemma 3.2.3 with help of part (2), where

$$\tilde{\Lambda}_n\,(k) = \left|\tilde{\Lambda}_n\,(k)\right| = \left|1 - \tilde{\Lambda}_n(v)\right|\, \leq\, 1 + \left|\tilde{\Lambda}_n(v)\right| \leq\, 2. \qquad \square$$

Lemma 3.2.5 *We have the following facts:*

(1) $(k) \in \ell_\infty(\Delta^\lambda),\; v \in \ell_\infty(\Delta^\lambda)$ *and* $u \in \ell_\infty(\Delta^\lambda)$.

(2) $\tilde{\Lambda}(k + v_k) = e \in c$ *and so* $(k + v_k) \in c(\Delta^\lambda)$.

(3) $(k) \in c(\Delta^\lambda) \Longleftrightarrow v \in c(\Delta^\lambda)$.

Proof. Part (1) is obtained from (2) and (3) of Lemma 3.2.4 with help of part (3) of Lemma 3.2.3. Also, part (2) is immediate by (4) of Lemma 3.2.3, where $\tilde{\Lambda}_n(k+v_k) = 1$ for all n which means that $\tilde{\Lambda}(k+v_k) = e \in c$ and so $(k+v_k) \in c(\Delta^\lambda)$.

Finally, part (3) is an immediate consequence of part (2). To see that, we have $(k) + v = (k+v_k) \in c(\Delta^\lambda)$. Thus, we deduce the given equivalence (in part (3)) from the properties of any linear space. In other words, from (5) of Lemma 3.2.3, we find that $\tilde{\Lambda}(k) + \tilde{\Lambda}(v) = e \in c$. Thus $\tilde{\Lambda}(k) \in c \Longleftrightarrow \tilde{\Lambda}(v) \in c$ which can equivalently be written as $(k) \in c(\Delta^\lambda) \Longleftrightarrow v \in c(\Delta^\lambda)$ (note that: we will show, in next section, that both sequences $\tilde{\Lambda}(k)$ and $\tilde{\Lambda}(v)$ can together be not in c while their summation $\tilde{\Lambda}(k) + \tilde{\Lambda}(v)$ always belongs to c). □

Lemma 3.2.6 *We have the following equivalences:*

(1) $u \in \ell_\infty \Longleftrightarrow v \in \ell_\infty$ *(the same is true for c instead of ℓ_∞).*

(2) $(u_k/k) \in c \Longleftrightarrow (v_k/k) \in c$ *(in such case:* $\lim_{k\to\infty} u_k/k = \lim_{k\to\infty} v_k/k \geq 0$*).*

(3) $\lim_{k\to\infty} u_k/k = \infty \iff \lim_{k\to\infty} v_k/k = \infty$.

(4) $u \in c(\Delta) \Longleftrightarrow v \in c(\Delta)$ *(in such case:* $\lim_{k\to\infty} \Delta(u_k) = \lim_{k\to\infty} \Delta(v_k) \geq 0$*).*

Proof. The given equivalences in parts (1), (2) and (3) are immediate by (1) of Lemma 3.2.3 while the equivalence in part (4) follows from (2) of Lemma 3.2.3.

On other side, in part (2), if $(u_k/k) \in c$ and so $(v_k/k) \in c$; then their limits must be equal to each others. To see that, from (1) of Lemma 3.2.3, we have $u_k/k = 1/k + v_k/k$ and $(1/k) \in c_0$. Also, since $u_k \geq 0$ and $v_k \geq 0$ for all k; it follows that $\lim_{k\to\infty} u_k/k \geq 0$ as well as $\lim_{k\to\infty} v_k/k \geq 0$.

Similarly, in part (4), if $u \in c(\Delta)$ and so $v \in c(\Delta)$; then $\lim_{k\to\infty} \Delta(u_k) = \lim_{k\to\infty} u_k/k$ as well as $\lim_{k\to\infty} \Delta(v_k) = \lim_{k\to\infty} v_k/k$ (by (2) of Lemma 1.3.4). Thus, we deduce that $\lim_{k\to\infty} \Delta(u_k) \geq 0$ and $\lim_{k\to\infty} \Delta(v_k) \geq 0$ (by part (2)) and these two limits must be equal to each others (by (2) of Lemma 3.2.3). $\square$

Theorem 3.2.7 *We have the following facts:*

(1) $u \in c(\Delta^\lambda) \Longleftrightarrow v \in c(\Delta^\lambda)$ (*in such case:* $\lim_{n\to\infty} \tilde{\Lambda}_n(u) = \lim_{n\to\infty} \tilde{\Lambda}_n(v)$).

(2) *If* $v \in c(\Delta^\lambda)$*; then* $0 \leq \lim_{n\to\infty} \tilde{\Lambda}_n(v) \leq 1$ *and we have*

$$\lim_{n\to\infty} \tilde{\Lambda}_n(k) = 1 - \lim_{n\to\infty} \tilde{\Lambda}_n(v) \quad \text{(the same is true for } u \text{ instead of } v\text{).}$$

(3) *If* $(k) \in c(\Delta^\lambda)$*; then* $0 \leq \lim_{n\to\infty} \tilde{\Lambda}_n(k) \leq 1$ *and we have*

$$\lim_{n\to\infty} \tilde{\Lambda}_n(v) = 1 - \lim_{n\to\infty} \tilde{\Lambda}_n(k) \quad \text{(the same is true for } u \text{ instead of } v\text{).}$$

(4) $\tilde{\Lambda}(u) \in c \iff \tilde{\Lambda}(v) \in c \iff \tilde{\Lambda}(k) \in c$.

Proof. (1) Since $\tilde{\Lambda}_n(u) = \tilde{\Lambda}_n(v)$ for all $n \geq 2$ (by (3) of Lemma 3.2.3); we deduce that $\tilde{\Lambda}(u) \in c \Longleftrightarrow \tilde{\Lambda}(v) \in c$, and their limits must be equal to each others, i.e. $\lim_{n\to\infty} \tilde{\Lambda}_n(u) = \lim_{n\to\infty} \tilde{\Lambda}_n(v)$. Hence, part (1) is proved.

(2) Suppose that $v \in c(\Delta^\lambda)$, i.e. $\tilde{\Lambda}(v) \in c$. Then $\lim_{n\to\infty} \tilde{\Lambda}_n(v) = \lim_{n\to\infty} \Lambda_n(v)/n$ (by (2) of Corollary 2.2.10). But $v_k \geq 0$ for all k and it follows by (2.2.2) that $\Lambda_n(v) \geq 0$ for all n and hence $\lim_{n\to\infty} \Lambda_n(v)/n \geq 0$ which implies that $\lim_{n\to\infty} \tilde{\Lambda}_n(v) \geq 0$. Besides, by using (2) of Lemma 3.2.4, we deduce that $\lim_{n\to\infty} \tilde{\Lambda}_n(v) \leq \lim_{n\to\infty} |\tilde{\Lambda}_n(v)| \leq 1$. Thus, it follows that $0 \leq \lim_{n\to\infty} \tilde{\Lambda}_n(v) \leq 1$. Also, since $v \in c(\Delta^\lambda)$; we get $(k) \in c(\Delta^\lambda)$ (by (3) of Lemma 3.2.5) such that $\lim\limits_{n\to\infty} \tilde{\Lambda}_n(k) = 1 - \lim\limits_{n\to\infty} \tilde{\Lambda}_n(v)$ (by (5) of Lemma 3.2.3).

(3) Assume that $(k) \in c(\Delta^\lambda)$. Then, it follows, by (3) of Lemma 3.2.5, that $v \in c(\Delta^\lambda)$. Therefore, by using part (2), we deduce that $0 \leq \lim_{n\to\infty} \tilde{\Lambda}_n(k) \leq 1$ and $\lim_{n\to\infty} \tilde{\Lambda}_n(v) = 1 - \lim_{n\to\infty} \tilde{\Lambda}_n(k)$ (in parts (2) and (3), v can be replaced by u).

(4) By combining (3) of Lemma 3.2.5 with part (1), we conclude that $u \in c(\Delta^\lambda) \Longleftrightarrow v \in c(\Delta^\lambda) \Longleftrightarrow (k) \in c(\Delta^\lambda)$ which can equivalently be written as $\tilde{\Lambda}(u) \in c \iff \tilde{\Lambda}(v) \in c \iff \tilde{\Lambda}(k) \in c$ and this completes the proof. $\square$

Remark 3.2.8 From previous results, we may note that there exists a similarity or an equivalence between the sequences u and v with respect to their belong to any of the spaces c, ℓ_∞, $c(\Delta)$ or $c(\Delta^\lambda)$. So, it is useful to use a common symbol for these two sequences. That is, we will use the sequence $t = (t_k)$ to stand for any of the sequences u or v (we may note that $v_k \leq t_k \leq u_k$ for all k). Thus, we can replace v (or u) by t in Theorem 3.2.7.

Moreover, for any sequence $x = (x_n)$, we have the following equality:

$$x_n - \Lambda_n(x) = v_n \tilde{\Lambda}_n(x) \qquad (n \geq 1) \tag{3.2.6}$$

which can be obtained from that equality given in part (2) of Lemma 1.3.14 by using (2.2.7) and (3.2.1). Besides, by using (2.2.2) and (3.2.2), we find (for every $n \geq 2$) that

$$\Lambda_n(v\,\Delta(x)) = \frac{1}{\lambda_n}\sum_{k=1}^{n} \lambda_{k-1}\Delta\,(x_k) = \frac{1}{\lambda_n}\sum_{k=2}^{n} \lambda_{k-1}\Delta\,(x_k) = v_n\,\tilde{\Lambda}_n(x).$$

That is $\Lambda_n(v\,\Delta(x)) = v_n\,\tilde{\Lambda}_n(x)$ $(n \geq 2)$ which is true for $n = 1$ (as $v_1 = 0$) and by combining this equality with (3.2.6) we obtain that

$$x_n - \Lambda_n(x) = v_n \tilde{\Lambda}_n(x) = \Lambda_n(v\,\Delta(x)) \qquad (n \geq 1)$$

and by operating Δ on both sides, we get

$$\Delta(x_n) - \tilde{\Lambda}_n(x) = \Delta(v_n \tilde{\Lambda}_n(x)) = \tilde{\Lambda}_n(v\,\Delta(x)) \qquad (n \geq 1). \tag{3.2.7}$$

On other side, by noting that $u_n = 1 + v_n$ for all n (by (1) of Lemma 3.2.3), the equality (3.2.6) can be rewritten, for any sequence $x = (x_n)$, as follows

$$x_n = v_n \tilde{\Lambda}_n(x) + \Lambda_n(x) \quad \text{or} \quad x_n = u_n \tilde{\Lambda}_n(x) + \Lambda_{n-1}(x)$$

from which we can respectively derive the following useful identities:

$$\frac{x_n}{n+v_n} = \tilde{\Lambda}_n(x) - \frac{n}{n+v_n}\left(\tilde{\Lambda}_n(x) - \frac{\Lambda_n(x)}{n}\right) \qquad (n \geq 1), \tag{3.2.8}$$

$$\frac{x_n}{n+u_n} = \tilde{\Lambda}_n(x) - \frac{n}{n+u_n}\left(\tilde{\Lambda}_n(x) - \frac{\Lambda_{n-1}(x)}{n}\right) \qquad (n \geq 1). \tag{3.2.9}$$

Now, by using the convention mentioned in Remark 3.2.8, we prove the following theorem which is analogous to Lemma 1.3.4.

Theorem 3.2.9 *Let $t = (t_n)$ be any of the sequences $u = (u_n)$ or $v = (v_n)$. Then, for any sequence $x = (x_n)$, we have the following facts:*

(1) *If $x \in \ell_\infty(\Delta^\lambda)$; then $(x_n/(n+t_n)) \in \ell_\infty$.*

(2) *If $x \in c(\Delta^\lambda)$; then $(x_n/(n+t_n)) \in c$ and $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \lim_{n\to\infty} x_n/(n+t_n)$.*

(3) *In particular, if $x \in c_0(\Delta^\lambda)$; then $(x_n/(n+t_n)) \in c_0$.*

Proof. At the beginning, we may note that the sequences $(n/(n+v_n))$ and $(n/(n+u_n))$ are bounded (i.e. $(n/(n+v_n)) \in \ell_\infty$ and $(n/(n+u_n)) \in \ell_\infty$). Also, it is obvious that $((n-1)/n) \in c \setminus c_0$, where $\lim_{n\to\infty}(n-1)/n = 1$. Thus, we have the following:

To prove (1), let $x \in \ell_\infty(\Delta^\lambda)$ which means that $\tilde{\Lambda}(x) \in \ell_\infty$ and so $(\Lambda_n(x)/n) \in \ell_\infty$ (by (1) of Corollary 2.2.10). Then, it follows by (3.2.8) that $(x_n/(n+v_n)) \in \ell_\infty$. Also, since $(\Lambda_{n-1}(x)/n) \in \ell_\infty$; it follows by (3.2.9) that $(x_n/(n+u_n)) \in \ell_\infty$. This proves (1), as t is either u or v (see Remark 3.2.8).

For (2), let $x \in c(\Delta^\lambda)$ which means that $\tilde{\Lambda}(x) \in c$ and so $(\Lambda_n(x)/n) \in c$ such that $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \lim_{n\to\infty} \Lambda_n(x)/n$ (by (2) of Corollary 2.2.10). Then, we get $(\tilde{\Lambda}_n(x) - \Lambda_n(x)/n) \in c_0$ and since $(n/(n+v_n)) \in \ell_\infty$; we immediately deduce that $\lim_{n\to\infty}(n/(n+v_n))(\tilde{\Lambda}_n(x) - \Lambda_n(x)/n) = 0$. Therefore, by passing to the limits in both sides of (3.2.8) as $n \to \infty$, we get $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \lim_{n\to\infty} x_n/(n+v_n)$. Also, we have

$$\lim_{n\to\infty}\frac{\Lambda_{n-1}(x)}{n}=\lim_{n\to\infty}\left(\frac{n-1}{n}\right)\frac{\Lambda_{n-1}(x)}{n-1}=\lim_{n\to\infty}\frac{\Lambda_{n-1}(x)}{n-1}=\lim_{n\to\infty}\tilde{\Lambda}_n(x).$$

Therefore, by using (3.2.9), we similarly get $\lim_{n\to\infty}\tilde{\Lambda}_n(x)=\lim_{n\to\infty}x_n/(n+u_n)$ which proves (2). Finally, part (3) is immediate by (2) and we have done. □

In addition, in the light of Theorem 3.2.9, we have the following consequences as important particular cases concerning with the sequences u, v and (k).

Corollary 3.2.10 *Let t be either u or v. Then, we have the following facts:*

(1) *If $t\in c(\Delta^\lambda)$; then $(t_n/(n+t_n))\in c$ and $\lim_{n\to\infty}\tilde{\Lambda}_n(t)=\lim_{n\to\infty}t_n/(n+t_n)$.*

(2) *If $(k)\in c(\Delta^\lambda)$; then $(n/(n+t_n))\in c$ and $\lim_{n\to\infty}\tilde{\Lambda}_n(k)=\lim_{n\to\infty}n/(n+t_n)$.*

Proof. It is immediate by taking $x=t$ and $x=(k)$ in Theorem 3.2.9. □

Corollary 3.2.11 *Assume that $t=(t_n)$ is either $u=(u_n)$ or $v=(v_n)$. If $\tilde{\Lambda}(t)\in c$; then the real sequence (t_n/n) is not oscillated (i.e., either $(t_n/n)\in c$ or $t_n/n\to\infty$ as $n\to\infty$). Further, in such case, we have*

$$\lim_{n\to\infty}\tilde{\Lambda}_n(t)=\lim_{n\to\infty}\frac{t_n/n}{1+t_n/n}\quad\text{and}\quad\lim_{n\to\infty}\tilde{\Lambda}_n(k)=\lim_{n\to\infty}\frac{1}{1+t_n/n}\cdot\tag{3.2.10}$$

Proof. Suppose that $\tilde{\Lambda}(t)\in c$. Then, it follows by Theorem 3.2.7 that $\tilde{\Lambda}(k)\in c$ such that $0\le\lim_{n\to\infty}\tilde{\Lambda}_n(t)\le 1$ and $0\le\lim_{n\to\infty}\tilde{\Lambda}_n(k)\le 1$. Besides, the limits given in (3.2.10) are immediate by Corollary 3.2.10. Also, since these limits exist; we deduce that either $(t_n/n)\in c$ or $t_n/n\to\infty$ as $n\to\infty$, that is (t_n/n) cannot be oscillated. □

Corollary 3.2.12 *Let t be either u or v. Then, we have the following facts:*

(1) $\tilde{\Lambda}(k)\in c_0 \iff \lim_{n\to\infty}t_n/n=\infty$.

(2) $\tilde{\Lambda}(t)\in c_0 \implies \lim_{n\to\infty}t_n/n=0$.

(3) $\Delta(t)\in c_0 \implies \tilde{\Lambda}(t)\in c_0$.

Proof. (1) If $\tilde{\Lambda}(k) \in c_0$ and so $\tilde{\Lambda}(t) \in c$; then it follows by (3.2.10) of Corollary 3.2.11 that $(1/(1+t_n/n)) \in c_0$ which implies that $t_n/n \to \infty$ as $n \to \infty$. Conversely, suppose that $\lim_{n\to\infty} t_n/n = \infty$. Then $\lim_{n\to\infty} n/t_n = 0$, that is $(n/t_n)_{n=2}^{\infty} \in c_0$. Besides, since λ is increasing; we get $\sum_{k=2}^{n} \lambda_{k-1} = \sum_{k=1}^{n-1} \lambda_k \le (n-1)\lambda_{n-1} \le n\lambda_{n-1}$ for all $n \ge 2$. So, by using (3.2.3), we obtain that $\tilde{\Lambda}_n(k) \le n/u_n \le n/t_n$ and so $0 \le \tilde{\Lambda}_n(k) \le n/t_n$ for all $n \ge 2$. Therefore, by passing to the limits as $n \to \infty$, we get $\lim_{n\to\infty} \tilde{\Lambda}_n(k) = 0$ which means that $\tilde{\Lambda}(k) \in c_0$.

For (2), suppose that $\tilde{\Lambda}(t) \in c_0$ and so $\lim_{n\to\infty} \tilde{\Lambda}_n(k) = 1$ by (3) of Theorem 3.2.7. Then, from (3.2.10) of Corollary 3.2.11 we deduce that $\lim_{n\to\infty} 1/(1+t_n/n) = 1$ which implies that $\lim_{n\to\infty} t_n/n = 0$.

Lastly, to prove (3), assume that $\Delta(t) \in c_0$. Then, for every positive real number $\epsilon > 0$ there is an integer $k_1 \ge 1$ such that $|\Delta(t_k)| < \epsilon/4$ for all $k > k_1$. Also, let $M = \sum_{k=2}^{k_1} \lambda_{k-1}|\Delta(t_k)|$. Then, for every $n > k_1$, we can use (3.2.2) and (3.2.3) to get

$$\begin{aligned}
\left|\tilde{\Lambda}_n(t)\right| &\le \frac{1}{\lambda_n v_n} \sum_{k=2}^{n} \lambda_{k-1}|\Delta(t_k)| \\
&< \frac{1}{\lambda_n v_n} \left(M + \frac{\epsilon}{4} \sum_{k=k_1+1}^{n} \lambda_{k-1} \right) \\
&\le \frac{1}{\lambda_n v_n} \left(M + \frac{\epsilon}{4} \sum_{k=2}^{n} \lambda_{k-1} \right) \\
&= \tilde{\Lambda}_n(k) \left(M_n + \frac{\epsilon}{4} \right) \\
&\le 2 \left(M_n + \frac{\epsilon}{4} \right),
\end{aligned}$$

where $M_n = M / \sum_{k=2}^{n} \lambda_{k-1}$ for all n and $\tilde{\Lambda}_n(k) \le 2$ for evey $n \ge 1$ (by (3) of Lemma 3.2.4). On other side, since $\lim_{n\to\infty} \sum_{k=2}^{n} \lambda_{k-1} = \infty$; we have $(M_n) \in c_0$. Thus, there must exist an integer $k_2 \ge 1$ such that $M_n < \epsilon/4$ for all $n > k_2$. Finally, let $k_0 = \max\{k_1, k_2\}$. Then, for every $n > k_0$, we find that $\left|\tilde{\Lambda}_n(t)\right| < 2(\epsilon/4 + \epsilon/4) = \epsilon$. That is $\left|\tilde{\Lambda}_n(t)\right| < \epsilon$ for all $n > k_0$ and so $\tilde{\Lambda}(t) \in c_0$. $\square$

Corollary 3.2.13 *Assume that $t = (t_n)$ is either $u = (u_n)$ or $v = (v_n)$. If the real sequence $\Delta(t)$ is not oscillated (i.e., either $\Delta(t) \in c$ or $\Delta(t_n) \to \infty$ as $n \to \infty$); then $\tilde{\Lambda}(t) \in c$. Further, in such case, we have*

$$\lim_{n\to\infty} \tilde{\Lambda}_n(t) = \lim_{n\to\infty} \frac{\Delta(t_n)}{1+\Delta(t_n)} \quad \textit{and} \quad \lim_{n\to\infty} \tilde{\Lambda}_n(k) = \lim_{n\to\infty} \frac{1}{1+\Delta(t_n)} \cdot \tag{3.2.11}$$

Proof. Suppose that $\Delta(t)$ is not oscillated. Then $\lim_{k\to\infty} \Delta(t_k) = \lim_{k\to\infty} t_k/k \geq 0$ and so we have two distinct cases: either $\Delta(t) \in c$ or $\Delta(t_k) \to \infty$ as $k \to \infty$.

First, let's consider the case $\Delta(t) \in c$. Then $\Delta(v) \in c$ by (4) of Lemma 3.2.6. Also, let $L = \lim_{k\to\infty} \Delta(v_k)$, where $L \geq 0$. Then, we have $(\Delta(v_k) - L) \in c_0$ and so $\lim_{k\to\infty} \Delta(v_k - Lk) = 0$. Thus, by following the same technique used in the proof of pqart (3) of Corollary 3.2.12, we can similarly show that $\lim_{n\to\infty} \tilde{\Lambda}_n(v_k - Lk) = 0$ and hence $\lim_{n\to\infty}(\,\tilde{\Lambda}_n(v) - L\tilde{\Lambda}_n(k)\,) = 0$ which can equivalently be rewritten as $\lim_{n\to\infty}(\,(1+L)\tilde{\Lambda}_n(v) - L\,) = 0$ (as $\tilde{\Lambda}_n(k) = 1 - \tilde{\Lambda}_n(v)$ for all n by (5) of Lemma 3.2.3). Therefore, we deduce that $\tilde{\Lambda}(v) \in c$ and so $\tilde{\Lambda}(k) \in c$ with limits given by

$$\lim_{n\to\infty} \tilde{\Lambda}_n(v) = \frac{L}{1+L} = \lim_{n\to\infty} \frac{\Delta(v_n)}{1+\Delta(v_n)}\,,$$

$$\lim_{n\to\infty} \tilde{\Lambda}_n(k) = 1 - \lim_{n\to\infty} \tilde{\Lambda}_n(v) = \frac{1}{1+L} = \lim_{n\to\infty} \frac{1}{1+\Delta(v_n)}$$

and since $\Delta(v_n) = \Delta(t_n)$, $\tilde{\Lambda}_n(v) = \tilde{\Lambda}_n(t)$ for $n > 1$; we get $\tilde{\Lambda}(t) \in c$ and obtain (3.2.11).

Next, if $\lim_{k\to\infty} \Delta(t_k) = \infty$; then $\lim_{k\to\infty} t_k/k = \infty$ (by (3) of Lemma 1.3.4). Thus $\tilde{\Lambda}(k) \in c_0$ (by (1) of Corollary 3.2.12) and so $\lim_{n\to\infty} \tilde{\Lambda}_n(t) = 1$ which can be obtained from (3.2.11) an this completes the proof. $\square$

Lastly, combining Corollaries 3.2.11 and 3.2.13 yields the following implications:

$$\Delta(t) \text{ is not oscillated} \implies \tilde{\Lambda}(t) \in c \implies (t_k/k) \text{ is not oscillated},$$

$$(t_k/k) \text{ is oscillated} \implies \tilde{\Lambda}(t) \notin c \implies \Delta(t) \text{ is oscillated}$$

but the converse of each implication is not true as will be shown in the next example.

Example 3.2.14 Consider the particular sequence $\lambda = (3, 6, 18, 36, 108, \cdots)$, that is $\lambda = (\lambda_k)$ is the sequence defined for all $k \geq 1$ by

$$\lambda_k = \begin{cases} \dfrac{6^{(k+1)/2}}{2}\,; & (k \text{ is odd}), \\ 6^{k/2}\,; & (k \text{ is even}). \end{cases}$$

Then, for every $k > 2$, we have

$$u_k = \begin{cases} 3/2\,; & (k \text{ is odd}), \\ 2\,; & (k \text{ is even}). \end{cases} \qquad \Delta(u_k) = \begin{cases} -1/2\,; & (k \text{ is odd}), \\ 1/2\,; & (k \text{ is even}). \end{cases}$$

Also, for every $n \geq 1$ it can easily be shown that

$$\tilde{\Lambda}_n(u) = \begin{cases} -\dfrac{1}{5}\Big(1 - \dfrac{1}{6^{(n-3)/2}}\Big)\,; & (n \text{ is odd}), \\ \dfrac{1}{5}\Big(1 + \dfrac{9}{6^{n/2}}\Big)\,; & (n \text{ is even}) \end{cases}$$

which means that $\tilde{\Lambda}(u) \notin c$, where $\tilde{\Lambda}(u)$ is oscillated between two limits (namely $\pm 1/5$) and so $\tilde{\Lambda}(k)$ will also be oscillated (between $4/5$ and $6/5$). Also, it maybe noted that $u \in \ell_\infty$ and $(u_k/k) \in c_0$ while $\tilde{\Lambda}(u)$ and $\Delta(u)$ are oscillated.

Example 3.2.15 Let $\lambda = (1, 2, 4, 6, 9, 12, \cdots)$. That is, for every $k \geq 1$, we have

$$\lambda_k = \begin{cases} (k+1)^2/4\,; & (k \text{ is odd}), \\ (k^2 + 2k)/4\,; & (k \text{ is even}). \end{cases} \qquad u_k = \begin{cases} (k+1)/2\,; & (k \text{ is odd}), \\ (k+2)/2\,; & (k \text{ is even}). \end{cases}$$

Then, for every $n > 1$, we find that

$$\Delta(u_n) = \begin{cases} 1\,; & (n \text{ is even}), \\ 0\,; & (n \text{ is odd}). \end{cases} \qquad \tilde{\Lambda}_n(u) = \begin{cases} (n+1)/(3n)\,; & (n \text{ is even}), \\ n/(3n+3)\,; & (n \text{ is odd}). \end{cases}$$

Hence, we have $u_k/k \to 1/2$, $\tilde{\Lambda}_n(u) \to 1/3$ and so $\tilde{\Lambda}_n(k) \to 2/3$. Also, it must be noted that $u \notin \ell_\infty$, $(u_k/k) \in c$ and $\tilde{\Lambda}(u) \in c$ while $\Delta(u)$ is oscillated.

Example 3.2.16 Let $\lambda = (\lambda_k)$ be given by $\lambda_k = \ln(k+1)$ for all $k \geq 1$. Then, it can easily be seen that

$$\Delta(\lambda_k) = \ln(k+1) - \ln(k) \leq \frac{1}{k} \quad \text{and} \quad v_k = \frac{\ln(k)}{\ln(k+1) - \ln(k)} \geq k\ln(k) \qquad (k \geq 1).$$

Also, for every $n \geq 2$, we find that

$$\sum_{k=2}^{n} \lambda_{k-1} = \sum_{k=2}^{n} \ln(k) \leq (n-1)\ln(n) \leq n\ln(n) \leq v_n$$

and it follows by (3.2.3) that $\tilde{\Lambda}_n(k) \leq 1/\lambda_n$ and so $0 \leq \tilde{\Lambda}_n(k) \leq 1/\lambda_n$ for all $n \geq 2$. But $1/\lambda \in c_0$ and hence $\tilde{\Lambda}(k) \in c_0$ which implies that $\lim_{n\to\infty} \tilde{\Lambda}_n(v) = 1$. Thus $\tilde{\Lambda}(v) \in c$, where $v_k/k \to \infty$ and so $v_k \to \infty$ (as $k \to \infty$).

3.3 Main Results

In this last section, we prove our main results concerning the inclusion relations between the usual difference spaces and the new λ-difference spaces. We essentially characterize the case in which the inclusions $c_0(\Delta) \subset c_0(\Delta^\lambda)$, $c(\Delta) \subset c(\Delta^\lambda)$ and $\ell_\infty(\Delta) \subset \ell_\infty(\Delta^\lambda)$ hold, and then we conclude the necessary and sufficient conditions for their equalities to be satisfied. Also, some important particular cases will be discussed.

At the beginning, we prove the following useful lemma concerning with the matrix transformations on the same sequence space c_0, c or ℓ_∞.

Lemma 3.3.1 *Let $A = [a_{nk}]$ be an infinite matrix such that $\lim_{n\to\infty} a_{nk} = 0$ for every $k \geq 1$. Then, we have the following equivalences:*

(1) $A \in (\ell_\infty, \ell_\infty) \iff A \in (c_0, c_0) \iff \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$.

(2) $A \in (c, c) \iff \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ *and* $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}$ *exists.*

Proof. Since $\lim_{n\to\infty} a_{nk} = 0$ for every $k \geq 1$; part (1) is immediate by using the conditions given in Lemma 1.3.9 and in (3) of Lemma 1.3.11.

Also, part (2) can be obtained from the conditions given in (2) of Lemma 1.3.10 (by noting that conditions (1.3.2) and (1.3.3) are assumed to be satisfied). $\square$

Now, we define a triangle $\hat{A} = [\hat{a}_{nk}]$ as follows:

$$\hat{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & \lambda_1\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) & 0 & 0 & \cdots \\ 0 & \lambda_1\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_3}\right) & \lambda_2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_3}\right) & 0 & \cdots \\ 0 & \lambda_1\left(\frac{1}{\lambda_3} - \frac{1}{\lambda_4}\right) & \lambda_2\left(\frac{1}{\lambda_3} - \frac{1}{\lambda_4}\right) & \lambda_3\left(\frac{1}{\lambda_3} - \frac{1}{\lambda_4}\right) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \qquad (3.3.1)$$

Then, for every $x \in w$, it follows by using the equality given in Lemma 3.2.1, that $\tilde{\Lambda}_n(x) = \hat{A}_n(\Delta(x))$ for all $n \geq 2$ and since $\tilde{\Lambda}_1(x) = x_1 = \Delta(x_1) = \hat{A}_1(\Delta(x))$; we get $\tilde{\Lambda}(x) = \hat{A}(\Delta(x))$ for all $x \in w$. Also, it is clear that $\lim_{n\to\infty} \hat{a}_{n1} = 0$ and we have

$$\lim_{n\to\infty} \hat{a}_{nk} = \lambda_{k-1} \lim_{n\to\infty}\left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right) = 0 \qquad (k \geq 2),$$

$$\sum_{k=1}^{\infty} |\hat{a}_{nk}| = \sum_{k=1}^{\infty} \hat{a}_{nk} = \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)\sum_{k=2}^{n} \lambda_{k-1} \qquad (n \geq 2).$$

Thus, by using (3.2.3), we obtain that

$$\lim_{n\to\infty} \hat{a}_{nk} = 0 \text{ for every } k \geq 1,$$

$$\sum_{k=1}^{\infty} |\hat{a}_{nk}| = \sum_{k=1}^{\infty} \hat{a}_{nk} = \tilde{\Lambda}_n(k) \text{ for every } n \geq 1.$$

Hence, by appling Lemma 3.3.1 to our triangle $\hat{A}$, we find that

$$\hat{A} \in (\ell_\infty, \ell_\infty) \iff \hat{A} \in (c_0, c_0) \iff \sup_n \tilde{\Lambda}_n(k) < \infty \iff \tilde{\Lambda}(k) \in \ell_\infty, \qquad (3.3.2)$$

$$\hat{A} \in (c, c) \iff \lim_{n\to\infty} \tilde{\Lambda}_n(k) \text{ exists } \iff \tilde{\Lambda}(k) \in c. \qquad (3.3.3)$$

Now, by using the usual conventions given in (3.2.1) and Remark 3.2.8, we may begin with the following main result:

Theorem 3.3.2 *We have the following facts:*

(1) *The inclusions* $c_0(\Delta) \subset c_0(\Delta^\lambda)$ *and* $\ell_\infty(\Delta) \subset \ell_\infty(\Delta^\lambda)$ *always hold.*

(2) *The inclusion* $c(\Delta) \subset c(\Delta^\lambda)$ *holds if and only if* $\tilde{\Lambda}\,(t) \in c$ *(or equivalently:* $\tilde{\Lambda}(k) \in c$*), where* $t = (t_k)$ *is either* $u = (u_k)$ *or* $v = (v_k)$*.*

Proof. Let μ be standing for any of the spaces c_0, c or ℓ_∞. Then, we have $\mu(\Delta) \cong \mu$ and so $x \in \mu(\Delta)$ if and only if $\Delta(x) \in \mu$ (by Lemma 2.1.1). Also, the matrices $\tilde{\Lambda}$ and $\hat{A}$ are triangles such that $\tilde{\Lambda}\,(x) = \hat{A}\,(\Delta(x))$ for all $x \in w$. Thus, by using (1) of Lemma 1.3.7, we deduce the following:

$$\begin{aligned} \mu(\Delta) \subset \mu(\Delta^\lambda) &\iff x \in \mu(\Delta^\lambda) \text{ for all } x \in \mu(\Delta) \\ &\iff \tilde{\Lambda}\,(x) \in \mu \text{ for all } x \in \mu(\Delta) \\ &\iff \hat{A}(\Delta(x)) \in \mu \text{ for all } \Delta(x) \in \mu \\ &\iff \hat{A}\,(y) \in \mu \text{ for all } y \in \mu \quad (y = \Delta(x)) \\ &\iff \hat{A} \in (\mu,\, \mu). \end{aligned}$$

Now, to prove (1), we have

$$\ell_\infty(\Delta) \subset \ell_\infty(\Delta^\lambda) \iff \hat{A} \in (\ell_\infty, \ell_\infty) \text{ as well as } c_0(\Delta) \subset c_0(\Delta^\lambda) \iff \hat{A} \in (c_0, c_0)$$

and it follows by (3.3.2) that

$$\ell_\infty(\Delta) \subset \ell_\infty(\Delta^\lambda) \iff c_0(\Delta) \subset c_0(\Delta^\lambda) \iff \tilde{\Lambda}_n(k) \in \ell_\infty\,.$$

But, the condition $\tilde{\Lambda}\,(k) \in \ell_\infty$ is always satisfied by (1) of Lemma 3.2.5. Thus, the inclusions $c_0(\Delta) \subset c_0(\Delta^\lambda)$ and $\ell_\infty(\Delta) \subset \ell_\infty(\Delta^\lambda)$ always hold.

Part (2) can similarly be proved, since $c(\Delta) \subset c(\Delta^\lambda) \iff \hat{A} \in (c, c)$. Thus, it follows from (3.3.3) that $c(\Delta) \subset c(\Delta^\lambda) \iff \hat{A} \in (c,\, c) \iff \tilde{\Lambda}\,(k) \in c$. Finally, by using (4) of Theorem 3.2.7, we deduce the following equivalences:

$$c(\Delta) \subset c(\Delta^\lambda) \iff \tilde{\Lambda}\,(k) \in c \iff \tilde{\Lambda}(t) \in c\,,$$

where t is either u or v, and this completes the proof. □

Corollary 3.3.3 *Let t be either u or v. Then, we have the following facts:*

(1) *If the inclusion $c(\Delta) \subset c(\Delta^{\lambda})$ holds; then the real sequence (t_n/n) is not oscillated (i.e., either $(t_n/n) \in c$ or $t_n/n \to \infty$ as $n \to \infty$) and for every $x \in c(\Delta)$ we have*

$$\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \lim_{n\to\infty} \frac{\Delta(x_n)}{1 + t_n/n} \cdot \tag{3.3.4}$$

(2) *If the real sequence $\Delta(t)$ is not oscillated (i.e., either $\Delta(t) \in c$ or $\Delta(t_n) \to \infty$ as $n \to \infty$); then the inclusion $c(\Delta) \subset c(\Delta^{\lambda})$ holds and for every $x \in c(\Delta)$ we have*

$$\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \lim_{n\to\infty} \frac{\Delta(x_n)}{1 + \Delta(t_n)} \cdot \tag{3.3.5}$$

Proof. By using (2) of Theorem 3.3.2, part (1) is immediate by Corollary 3.2.11, and part (2) follows from Corollary 3.2.13 with help of Corollary 2.1.8 and noting that $x \in c(\Delta) \Longrightarrow \lim_{n\to\infty} \Delta(x_n) = \lim_{n\to\infty} x_n/n$ by (2) of Lemma 1.3.4. □

Further, by combining parts (1) and (2) of Corollary 3.3.3, we deduce the following implications:

$$\Delta(t) \text{ is not oscillated } \Longrightarrow c(\Delta) \subset c(\Delta^{\lambda}) \Longrightarrow (t_k/k) \text{ is not oscillated},$$

$$(t_k/k) \text{ is oscillated } \Longrightarrow c(\Delta) \not\subset c(\Delta^{\lambda}) \Longrightarrow \Delta(t) \text{ is oscillated}$$

but the converse of each implication is not true (see Examples 3.2.14 and 3.2.15 with noting that $c(\Delta) \subset c(\Delta^{\lambda}) \iff \tilde{\Lambda}(t) \in c$).

On other side, we may recall, by Definition 2.2.4, that the regularity of $\tilde{\Lambda}$ over $c(\Delta)$ means that $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \lim_{n\to\infty} \Delta(x_n)$ for all $x \in c(\Delta)$. Thus, the regularity of $\tilde{\Lambda}$ over $c(\Delta)$ implies the inclusion $c(\Delta) \subset c(\Delta^{\lambda})$ (but the converse is not true). Also, it is obvious that $\tilde{\Lambda}$ is regular over $c(\Delta)$ if and only if $\hat{A}$ is regular in the ordinary sense of regularity, where $\hat{A}$ is the triangle defined by (3.3.1). Further, we deduce the following result concerning with the regularity of $\tilde{\Lambda}$ over $c(\Delta)$.

Corollary 3.3.4 *We have the following facts:*

(1) *If the inclusion $c(\Delta) \subset c(\Delta^\lambda)$ holds; then for every $x \in c(\Delta)$ we have*

$$\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \left(\lim_{n\to\infty} \Delta(x_n)\right)\left(\lim_{n\to\infty} \tilde{\Lambda}_n(k)\right)\cdot \tag{3.3.6}$$

(2) *$\tilde{\Lambda}$ is regular over $c(\Delta)$ if and only if $\tilde{\Lambda}(t) \in c_0$ (or equivalently: $\tilde{\Lambda}_n(k) \to 1$ as $n \to \infty$), where t is either u or v.*

Proof. (1) Suppose that $c(\Delta) \subset c(\Delta^\lambda)$. Then $\tilde{\Lambda}(k) \in c$. But $(k) \in c(\Delta)$, where $\Delta(k) = 1$ for all k. Thus, by using (3.3.4) from part (1) of Corollary 3.3.3, it follows that $\lim_{n\to\infty} \tilde{\Lambda}_n(k) = \lim_{n\to\infty} 1/(1 + t_n/n)$ (see also Corollary 3.2.11). Again, let $x \in c(\Delta)$ be given. Then, from (1) of Corollary 3.3.3, we deduce that

$$\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \left(\lim_{n\to\infty} \Delta(x_n)\right)\left(\lim_{n\to\infty} \frac{1}{1 + t_n/n}\right) = \left(\lim_{n\to\infty} \Delta(x_n)\right)\left(\lim_{n\to\infty} \tilde{\Lambda}_n(k)\right)\cdot$$

(2) It is obvious that both of $\tilde{\Lambda}(t) \in c_0$ and the regularity of $\tilde{\Lambda}$ over $c(\Delta)$ imply the inclusion $c(\Delta) \subset c(\Delta^\lambda)$. That is, this inclusion is satisfied in both cases of requirement, and so we can use (3.3.6) from part (1) to deduce that

$$\tilde{\Lambda} \text{ is regular over } c(\Delta) \iff \lim_{n\to\infty} \tilde{\Lambda}_n(k) = 1 \iff \tilde{\Lambda}(t) \in c_0$$

and this stage concludes the proof. □

Corollary 3.3.5 *Let t be either u or v. Then, we have the following facts:*

(1) *If $\tilde{\Lambda}$ is regular over $c(\Delta)$; then $(t_n/n) \in c_0$.*

(2) *If $t \in c_0(\Delta)$; then $\tilde{\Lambda}$ is regular over $c(\Delta)$.*

Proof. (1) If $\tilde{\Lambda}$ is regular over $c(\Delta)$; then $\tilde{\Lambda}(t) \in c_0$ (by (2) of Corollary 3.3.4) which implies that $(t_n/n) \in c_0$ by (2) of Corollary 3.2.12 (see Example 3.2.14 to disprove the converse implication).

(2) If $t \in c_0(\Delta)$; then $\Delta(t) \in c_0$ and so $\tilde{\Lambda}(t) \in c_0$ by using (3.3.5) of Corollary 3.3.3 (or (3) of Corollary 3.2.12). Thus $\tilde{\Lambda}$ is regular over $c(\Delta)$ by (2) of Corollary 3.3.4. □

Corollary 3.3.6 *Let t be either u or v. Then, we have the following facts:*

(1) *If $(t_n/n) \in \ell_\infty$; then the equality $c(\Delta) \cap c_0(\Delta^\lambda) = c_0(\Delta)$ holds.*

(2) *If $t \in \ell_\infty$; then $\ell_\infty \cap c(\Delta^\lambda) \subset c_0(\Delta)$ (in particular, if $t \in \ell_\infty \cap c(\Delta^\lambda)$; then $t \in c_0(\Delta)$).*

(3) *If $t \in \ell_\infty$; then $\ell_\infty \cap c(\Delta^\lambda) = \ell_\infty \cap c_0(\Delta^\lambda) = \ell_\infty \cap c_0(\Delta) = \ell_\infty \cap c(\Delta)$.*

(4) $t \in \ell_\infty \cap c(\Delta^\lambda) \iff t \in \ell_\infty \cap c_0(\Delta^\lambda) \iff t \in \ell_\infty \cap c_0(\Delta) \iff t \in \ell_\infty \cap c(\Delta)$.

Proof. (1) The inclusion $c_0(\Delta) \subset c(\Delta) \cap c_0(\Delta^\lambda)$ is always satisfied (without using the assumption $(t_n/n) \in \ell_\infty$, since $c_0(\Delta) \subset c(\Delta)$ and $c_0(\Delta) \subset c_0(\Delta^\lambda)$). To prove the converse inclusion, let $x \in c(\Delta) \cap c_0(\Delta^\lambda)$ be arbitrary. Then $x \in c(\Delta)$ and $x \in c_0(\Delta^\lambda)$. Thus, from $x \in c_0(\Delta^\lambda)$ we get $(x_n/(n+t_n)) \in c_0$ by (3) of Theorem 3.2.9. Besides,it is obvious that $1/(1+t_n/n) > 0$ for all n and since $(t_n/n) \in \ell_\infty$; we deduce that $(1/(1+t_n/n)) \notin c_0$. Therefore, there exists a positive real number $\alpha > 0$ such that $1/(1+t_n/n) \geq \alpha$ for all n. Hence, we obtain that

$$0 \leq \alpha \left| \frac{x_n}{n} \right| \leq \frac{|x_n|}{n+t_n} \qquad (n \geq 1)$$

and since $(x_n/(n+t_n)) \in c_0$; we get $\lim_{n\to\infty} x_n/n = \lim_{n\to\infty} |x_n/n| = 0$. But $x \in c(\Delta)$ and so $\lim_{n\to\infty} \Delta(x_n) = \lim_{n\to\infty} x_n/n = 0$ (by (2) of Lemma 1.3.4). Thus $x \in c_0(\Delta)$ which implies that $c(\Delta) \cap c_0(\Delta^\lambda) \subset c_0(\Delta)$. Therefore, we deduce that the equality $c(\Delta) \cap c_0(\Delta^\lambda) = c_0(\Delta)$ holds provided that $(t_n/n) \in \ell_\infty$.

(2) Suppose that $t \in \ell_\infty$ which yields $v \in \ell_\infty$ and take any $x \in \ell_\infty \cap c(\Delta^\lambda)$. Then $x \in c_0(\Delta^\lambda)$ (by (1) of Corollary 3.1.8) and so $\tilde{\Lambda}(x) \in c_0$ which implies that

$v\tilde{\Lambda}(x) \in c_0 \subset c_0(\Delta)$ and hence $\Delta(v\tilde{\Lambda}(x)) \in c_0$. Thus, it follows by (3.2.7) that $\Delta(x) - \tilde{\Lambda}(x) \in c_0$ and so $\Delta(x) \in c_0$ (as $\tilde{\Lambda}(x) \in c_0$) which means $x \in c_0(\Delta)$.

(3) Assume that $t \in \ell_\infty$. Then, by using part (2), it follows that $\ell_\infty \cap c(\Delta^\lambda) = \ell_\infty \cap c_0(\Delta)$. Thus, by combining this equality with those equalities given in (2) of Lemma 2.1.4 and (2) of Corollary 3.1.8 we get the required equalities.

Finally, part (4) follows immediately from (3) and this ends the proof. □

Corollary 3.3.7 *Let t be standing for any of the sequences u or v. Then, for every sequence $x = (x_n)$, we have the following facts:*

(1) *If* $\lim_{n\to\infty} x_n/t_n = 0$*; then* $x \in c_0(\Delta^\lambda)$.

(2) *If* $\lim_{n\to\infty} t_n/n = \infty$*; then:* $x \in c_0(\Delta^\lambda) \iff \lim_{n\to\infty} x_n/t_n = 0$.

(3) *If* $\lim_{n\to\infty} t_n/n = \infty$*; then:* $x \in c(\Delta^\lambda) \iff \lim_{n\to\infty} x_n/t_n$ *exists* (*in such case:* $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \lim_{n\to\infty} x_n/t_n$).

Proof. (1) Assume that $\lim_{k\to\infty} x_k/t_k = 0$. Then $x/u \in c_0$ (as $t_k \le u_k$ and so $|x_k|/u_k \le |x_k|/t_k$ for all $k \ge 2$). Thus, we can define a sequence $z \in c_0(\Delta^\lambda)$ by

$$z_n = \frac{x_n}{u_n} - \sum_{k=1}^{n} \frac{x_k}{u_k} \qquad (n \ge 1).$$

To see that, we have $c_0 \subset c_0(\Delta) \subset c_0(\Delta^\lambda)$. Thus, from $x/u \in c_0$ we get $x/u \in c_0(\Delta^\lambda)$ as well as $\left(\sum_{k=1}^{n} x_k/u_k\right) \in c_0(\Delta)$ and so $\left(\sum_{k=1}^{n} x_k/u_k\right) \in c_0(\Delta^\lambda)$. Thus $z \in c_0(\Delta^\lambda)$. On other side, since $x/u \in c_0$; we can use (2.2.9) and (2) of Remark 2.2.9 to define a sequence $y \in c_0(\Delta^\lambda)$ by

$$y_n = \frac{\Delta(\lambda_n \sum_{k=1}^{n} x_k/u_k)}{\Delta(\lambda_n)} \qquad (n \ge 1)$$

which can be written as follows

$$y_n = \sum_{k=1}^{n} \frac{x_k}{u_k} + v_n\left(\frac{x_n}{u_n}\right) \qquad (n \ge 1).$$

But $v_n = u_n - 1$ $(n \geq 1)$ and so we find that $y_n = x_n - z_n$ for all n. That is $y = x - z$ which implies that $x = y + z \in c_0(\Delta^\lambda)$ (as $y, z \in c_0(\Delta^\lambda)$).

(2) Assume that $\lim_{n\to\infty} t_n/n = \infty$ and so $\lim_{n\to\infty} n/t_n = 0$. Then, for every $x \in c_0(\Delta^\lambda)$, we have $\lim_{n\to\infty} x_n/(n + t_n) = 0$ by (3) of Theorem 3.2.9 which implies that $\lim_{n\to\infty}(x_n/t_n)/(1 + n/t_n) = 0$ and hence $\lim_{n\to\infty} x_n/t_n = 0$. Also, the converse implication is immediate by part (1).

(3) Assume that $\lim_{n\to\infty} t_n/n = \infty$. Then, for every $x \in c(\Delta^\lambda)$, we can follow the same technique used in the proof of part (2) to show that $\lim_{n\to\infty} x_n/t_n$ exists. For the converse implication, suppose that $\lim_{n\to\infty} x_n/t_n$ exists, say $L = \lim_{n\to\infty} x_n/t_n$. Then, we have $\lim_{n\to\infty}(x_n - Lt_n)/t_n = 0$ which implies $\lim_{n\to\infty} \tilde{\Lambda}_n(x - Lt) = 0$ by part (1). Thus, we get $\lim_{n\to\infty}(\,\tilde{\Lambda}_n(x) - L\tilde{\Lambda}_n(t)\,) = 0$. But $\lim_{n\to\infty} \tilde{\Lambda}_n(t) = 1$ (as $\tilde{\Lambda}(k) \in c_0$ by (1) of Corollary 3.2.12) and we therefore deduce that $\lim_{n\to\infty}(\,\tilde{\Lambda}_n(x) - L\,) = 0$ and so $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = L = \lim_{n\to\infty} x_n/t_n$ which completes the proof. □

Now, we are going to characterize the cases of identities between the new λ-difference spaces and the usual difference spaces. For this, we need the following:

For any sequence $x = (x_n)$, it follows from (3.2.7) that

$$\Delta(x_n) = \tilde{\Lambda}_n(x) + \Delta(v_n\tilde{\Lambda}_n(x)) = u_n\tilde{\Lambda}_n(x) - v_{n-1}\tilde{\Lambda}_{n-1}(x) \qquad (n \geq 1)$$

which can be written as follows

$$\Delta(x_n) = u_n\tilde{\Lambda}_n(x) - v_{n-1}\tilde{\Lambda}_{n-1}(x) = \hat{B}_n(\,\tilde{\Lambda}(x)\,) \qquad (n \geq 1),$$

where $\hat{B} = [\,\hat{b}_{nk}\,]$ is the triangle defined by

$$\hat{B} = \begin{bmatrix} u_1 & 0 & 0 & 0 & \cdots \\ -v_1 & u_2 & 0 & 0 & \cdots \\ 0 & -v_2 & u_3 & 0 & \cdots \\ 0 & 0 & -v_3 & u_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \qquad (3.3.7)$$

That is $\Delta(x) = \hat{B}(\tilde{\Lambda}(x))$ for all $x \in w$. Then, we prove the following lemma:

Lemma 3.3.8 *If $\hat{B} = [\,\hat{b}_{nk}\,]$ is the triangle given by* (3.3.7) *and $t = (t_n)$ is standing for any of the sequences $u = (u_n)$ or $v = (v_n)$; then we have the following equivalences:*

(1) $\hat{B} \in (\ell_\infty, \ell_\infty) \iff \hat{B} \in (c_0, c_0) \iff t \in \ell_\infty\,.$

(2) $\hat{B} \in (c, c) \iff t \in \ell_\infty \cap c_0(\Delta)\,.$

Proof. Let $\hat{B} = [\,\hat{b}_{nk}\,]$ be the triangle given by (3.3.7). Then, it is obvious that

$$\begin{aligned}
&\lim_{n\to\infty} \hat{b}_{nk} = 0 \ \text{ for every } \ k \geq 1,\\
&\sum_{k=1}^{\infty} |\hat{b}_{nk}| = u_n + v_{n-1} = 1 + v_n + v_{n-1} \qquad (n \geq 1),\\
&\sum_{k=1}^{\infty} \hat{b}_{nk} = u_n - v_{n-1} = 1 + \Delta(v_n) \qquad (n \geq 1)
\end{aligned}$$

and since $\lim_{n\to\infty} \hat{b}_{nk} = 0$ for every $k \geq 1$; we can apply Lemma 3.3.1 to our triangle $\hat{B}$ to obtain the following:

$$\hat{B} \in (\ell_\infty, \ell_\infty) \iff \hat{B} \in (c_0, c_0) \iff v \in \ell_\infty\,,$$

$$\hat{B} \in (c, c) \iff v \in \ell_\infty \cap c(\Delta) \iff v \in \ell_\infty \cap c_0(\Delta)\,,$$

where $\ell_\infty \cap c(\Delta) = \ell_\infty \cap c_0(\Delta)$ by (2) of Lemma 2.1.4. Finally, by using (1) and (4) of Lemma 3.2.6, we conclude the proof. □

Lemma 3.3.9 *Let μ be standing for any of the spaces c_0, c or ℓ_∞. Then, we have the following equivalences:*

(1) *The equality $\mu(\Delta^\lambda) = \mu(\Delta)$ holds $\iff$ the inclusion $\mu(\Delta^\lambda) \subset \mu(\Delta)$ holds.*

(2) *The equality $\mu(\Delta^\lambda) = \mu(\Delta)$ holds $\iff$ $\hat{B} \in (\mu, \mu)$, where $\hat{B}$ is given by* (3.3.7).

Proof. (1) It is clear that $\mu(\Delta^\lambda) = \mu(\Delta) \implies \mu(\Delta^\lambda) \subset \mu(\Delta)$. Also, the converse implication is obvious for $\mu = c_0$ or $\mu = \ell_\infty$, since $\mu(\Delta) \subset \mu(\Delta^\lambda)$ is always satisfied (by (1) of Theorem 3.3.2). So, the implication $c(\Delta^\lambda) \subset c(\Delta) \implies c(\Delta^\lambda) = c(\Delta)$ is left, and to prove it, suppose that $c(\Delta^\lambda) \subset c(\Delta)$ and let's deduce the converse inclusion.

For this, it follows by (2) of Lemma 3.2.5 that $(k+v_k) \in c(\Delta^\lambda)$ and so $(k+v_k) \in c(\Delta)$ (as $c(\Delta^\lambda) \subset c(\Delta)$ by assumption). But $(k) \in c(\Delta)$ and since $(k+v_k) \in c(\Delta)$; we deduce that $(v_k) \in c(\Delta)$, that is $v \in c(\Delta)$ which means that $\Delta(v) \in c$. Thus, by using (2) of Corollary 3.3.3, we find that $c(\Delta) \subset c(\Delta^\lambda)$. Subsequently, we get $c(\Delta^\lambda) = c(\Delta)$.

(2) Let $\hat{B} = [\,\hat{b}_{nk}\,]$ be the triangle given by (3.3.7). Then, we have $\Delta(x) = \hat{B}(\tilde{\Lambda}(x))$ for all $x \in w$. Also, from Theorem 2.2.7 and (2) of Remark 2.2.9, we have $\mu(\Delta^\lambda) \cong \mu$ such that $x \in \mu(\Delta^\lambda) \iff y \in \mu$, where $y = \tilde{\Lambda}(x)$. Therefore, by using part (1) with (1) of Lemma 1.3.7, we deduce the following:

$$\begin{aligned} \mu(\Delta^\lambda) = \mu(\Delta^\lambda) &\iff \mu(\Delta^\lambda) \subset \mu(\Delta) \\ &\iff x \in \mu(\Delta) \ \text{ for all } \ x \in \mu(\Delta^\lambda) \\ &\iff \Delta(x) \in \mu \ \text{ for all } \ \tilde{\Lambda}(x) \in \mu \\ &\iff \hat{B}(\tilde{\Lambda}(x)) \in \mu \ \text{ for all } \ \tilde{\Lambda}(x) \in \mu \\ &\iff \hat{B}(y) \in \mu \ \text{ for all } \ y \in \mu \quad (y = \tilde{\Lambda}(x)) \\ &\iff \hat{B} \in (\mu,\, \mu) \end{aligned}$$

which completes the proof. □

Theorem 3.3.10 *Let $t = (t_n)$ be standing for any of the sequences $u = (u_n)$ or $v = (v_n)$ given by* (3.2.1)*. Then, we have the following facts:*

(1) *The equalities $c_0(\Delta^\lambda) = c_0(\Delta)$ and $\ell_\infty(\Delta^\lambda) = \ell_\infty(\Delta)$ hold if and only if $t \in \ell_\infty$.*

(2) *The equality $c(\Delta^\lambda) = c(\Delta)$ holds if and only if $t \in \ell_\infty \cap c_0(\Delta)$.*

Proof. (1) By using (2) of Lemma 3.3.9, we have

$$c_0(\Delta^\lambda) = c_0(\Delta) \iff \hat{B} \in (c_0, c_0) \quad \text{and} \quad \ell_\infty(\Delta^\lambda) = \ell_\infty(\Delta) \iff \hat{B} \in (\ell_\infty, \ell_\infty)\,,$$

where $\hat{B}$ is the triangle defined by (3.3.7). Thus, from (1) of Lemma 3.3.8, we deduce that $c_0(\Delta^\lambda) = c_0(\Delta) \iff \ell_\infty(\Delta^\lambda) = \ell_\infty(\Delta) \iff t \in \ell_\infty$. In fact, if $t \in \ell_\infty$ and so

$v \in \ell_\infty$; then $x \in c_0(\Delta) \iff x \in c_0(\Delta^\lambda)$ as well as $x \in \ell_\infty(\Delta) \iff x \in \ell_\infty(\Delta^\lambda)$. To see that, we can use the equalities in (3.2.7) which can be written as follows:

$$\Delta(x) - \tilde{\Lambda}(x) = \Delta(v\tilde{\Lambda}(x)) = \tilde{\Lambda}(v\Delta(x)) \qquad (x \in w). \tag{3.3.8}$$

Then, for every $x \in c_0(\Delta)$ and by using (3.3.8) we find that

$$\Delta(x) \in c_0 \Longrightarrow v\Delta(x) \in c_0 \Longrightarrow \tilde{\Lambda}(v\Delta(x)) \in c_0 \Longrightarrow \Delta(x) - \tilde{\Lambda}(x) \in c_0$$

and so $\tilde{\Lambda}(x) \in c_0$ which means that $x \in c_0(\Delta^\lambda)$. Similarly, for any $x \in c_0(\Delta^\lambda)$ we have

$$\tilde{\Lambda}(x) \in c_0 \Longrightarrow v\tilde{\Lambda}(x) \in c_0 \Longrightarrow \Delta(v\tilde{\Lambda}(x)) \in c_0 \Longrightarrow \Delta(x) - \tilde{\Lambda}(x) \in c_0$$

and so $\Delta(x) \in c_0$ which means that $x \in c_0(\Delta)$. That is $x \in c_0(\Delta) \iff x \in c_0(\Delta^\lambda)$ and by using the same technique we can show that $x \in \ell_\infty(\Delta) \iff x \in \ell_\infty(\Delta^\lambda)$.

(2) Similarly, by combining (2) of Lemma 3.3.8 with (2) of Lemma 3.3.9, we get

$$c(\Delta^\lambda) = c(\Delta) \iff \hat{B} \in (c, c) \iff t \in \ell_\infty \cap c_0(\Delta).$$

In fact, if $t \in \ell_\infty \cap c_0(\Delta)$ and so $v \in \ell_\infty \cap c_0(\Delta)$ which implies that $v \in \ell_\infty$ and $\Delta(v) \in c_0$; then it follows by (3.3.8) that $x \in c\left(\Delta^\lambda\right) \iff x \in c\left(\Delta\right)$, where $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \lim_{n\to\infty} \Delta(x_n)$ for every $x \in c(\Delta)$ and for every $x \in c(\Delta^\lambda)$. To see that, we can use (1.1.2) to deduce that

$$\Delta\big(v_k \Delta\left(x_k\right)\big) = v_k\, \Delta\big(\Delta(x_k)\big) + \Delta(x_{k-1})\, \Delta\left(v_k\right) \longrightarrow 0 \;\text{ as }\; k \to \infty \quad (x \in c(\Delta)),$$

$$\Delta\big(v_n\, \tilde{\Lambda}_n(x)\big) = v_n\, \Delta\big(\tilde{\Lambda}_n(x)\big) + \tilde{\Lambda}_{n-1}(x)\, \Delta\left(v_n\right) \longrightarrow 0 \;\text{ as }\; n \to \infty \quad (x \in c(\Delta^\lambda))$$

and we respectively obtain that $\tilde{\Lambda}(v\Delta(x)) \in c_0$ (as $c_0(\Delta) \subset c_0(\Delta^\lambda)$) and $\Delta(v\,\tilde{\Lambda}(x)) \in c_0$. Thus, in both cases we get $\Delta(x) - \tilde{\Lambda}(x) \in c_0$ by using (3.3.8) which implies that $x \in c\left(\Delta^\lambda\right) \iff x \in c\left(\Delta\right)$ such that $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \lim_{n\to\infty} \Delta(x_n)$ for every x in $c(\Delta)$ or in $c(\Delta^\lambda)$. Hence, we get the required equality which ends the proof. □

Now, by using (3.2.1), it can easily be shown that

$$u_{n+1} = \frac{1}{1-(\lambda_n/\lambda_{n+1})} \quad \text{and} \quad \Delta(u_{n+1}) = u_n\, u_{n+1}\, \Delta\Big(\frac{\lambda_n}{\lambda_{n+1}}\Big) \qquad (n \geq 1)$$

and so $u \in \ell_\infty \iff \sup_n(\lambda_n/\lambda_{n+1}) < 1$. Also, if $u \in \ell_\infty$, then: $\Delta(u) \in c_0 \iff \lim_{n\to\infty} \Delta(\lambda_n/\lambda_{n+1}) = 0$. This, with (2) of Theorem 3.3.10, leads us to the following:

The equality $c(\Delta^\lambda) = c(\Delta)$ holds $\iff \sup_n \Big(\frac{\lambda_n}{\lambda_{n+1}}\Big) < 1$ and $\lim_{n\to\infty} \Delta\Big(\frac{\lambda_n}{\lambda_{n+1}}\Big) = 0$.

Remark 3.3.11 From Theorem 3.3.10, we may note the following:

(1) If $t \in c$; then $c(\Delta^\lambda) = c(\Delta)$ (since $c \subset \ell_\infty \cap c_0(\Delta)$ by (3) of Lemma 2.1.4).

(2) By using (2) of above theorem with (4) of Corollary 3.3.6, we deduce that

$$c(\Delta^\lambda) = c(\Delta) \iff t \in \ell_\infty \cap c(\Delta^\lambda) \iff t \in \ell_\infty \cap c_0(\Delta^\lambda) \iff t \in \ell_\infty \cap c(\Delta).$$

(3) For the equalities $\mu(\Delta^\lambda) = \mu(\Delta)$ to be held, the necessary condition $t \in \ell_\infty$ must be satisfied, which is not sufficient for the equality $c(\Delta^\lambda) = c(\Delta)$ (but, if $t \in \ell_\infty$; then $c_0(\Delta) \subset c(\Delta^\lambda) \subset \ell_\infty(\Delta)$).

(4) The equality $c(\Delta^\lambda) = c(\Delta)$ implies the regularity of $\tilde{\Lambda}$ over $c(\Delta)$, but the converse is not true (see Example 3.3.17 for such case when $t \in c_0(\Delta^\lambda) \setminus \ell_\infty$).

Corollary 3.3.12 *Let t be any of the sequences u or v. Then, we have the following:*

(1) *The inclusions $c_0(\Delta) \subset c_0(\Delta^\lambda)$ and $\ell_\infty(\Delta) \subset \ell_\infty(\Delta^\lambda)$ are strict $\iff t \notin \ell_\infty$.*

(2) *The inclusion $c(\Delta) \subset c(\Delta^\lambda)$ strictly holds $\iff t \in c(\Delta^\lambda) \setminus \ell_\infty$.*

Proof. This result is an immediate consequence of Theorems 3.3.2 and 3.3.10. □

In addition, it is well-known that the usual difference spaces $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ are so "large" such that the "most" other sequence spaces are included in these difference spaces. For example, all other sequence spaces defined in Chapter 1 (see Section 1.1.2.4, p. 8) are strictly included in $c_0(\Delta)$ which is the smallest space among the usual difference spaces, but the same is not true for the new λ-difference spaces. To see that, part (1) of Theorem 3.3.2, with help of Lemmas 1.3.1 and 3.1.1, leads us to deduce that the following inclusions are always satisfied:

$$c_0(\Delta) \subset \mu(\Delta^\lambda) \qquad \text{and} \qquad \mu(\Delta) \subset \ell_\infty(\Delta^\lambda)\,, \tag{3.3.9}$$

where μ stands for any of the spaces c_0, c or ℓ_∞. That is $c_0(\Delta)$ is included in all λ-difference spaces and all usual difference spaces are included in $\ell_\infty(\Delta^\lambda)$. Further, although $\ell_\infty \not\subset c_0(\Delta)$, but we have already shown that $\ell_\infty \subset c_0(\Delta^\lambda)$ whenever $\lim_{n\to\infty} t_n = \infty$ (see Theorem 3.1.6), where t is either u or v which are given by (3.2.1). In fact, if $\lim_{n\to\infty} t_n = \infty$; then $c_0(\Delta^\lambda)$ is never included in the largest space of the usual difference spaces, that is $c_0(\Delta^\lambda) \not\subset \ell_\infty(\Delta)$. To see that, let $x = ((-1)^n \sqrt{t_n}\,)$. Then $x \notin \ell_\infty(\Delta)$, but $\lim_{n\to\infty} x_n/t_n = 0$ which implies that $x \in c_0(\Delta^\lambda)$ by (1) of Corollary 3.3.7. Moreover, in the following last result, we show that if $\lim_{n\to\infty} t_n/n = \infty$; then all usual difference spaces are strictly included in the smallest λ-difference spaces, that is $\mu(\Delta) \subset c_0(\Delta^\lambda)$ (the strong case of strict inclusions).

Corollary 3.3.13 *The difference spaces $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ are strictly included in $c_0(\Delta^\lambda)$ if and only if $\tilde{\Lambda}(k) \in c_0$ (or equivalently: $\lim_{n\to\infty} t_n/n = \infty$, where t is any of the sequences u or v).*

Proof. Let μ be any of the spaces c_0, c or ℓ_∞. Then, we have to prove that

$$\mu(\Delta) \subsetneqq c_0(\Delta^\lambda) \iff \tilde{\Lambda}(k) \in c_0 \iff \lim_{n\to\infty} t_n/n = \infty.$$

For this, we have $\tilde{\Lambda}(k) \in c_0 \iff \lim_{n\to\infty} t_n/n = \infty$ by (1) of Corollary 3.2.12. So, it is enough to show that $\mu(\Delta) \subsetneqq c_0(\Delta^\lambda) \iff \tilde{\Lambda}(k) \in c_0$. For, if $\mu(\Delta) \subset c_0(\Delta^\lambda)$ and so $c(\Delta) \subset c_0(\Delta^\lambda)$; then $(k) \in c_0(\Delta^\lambda)$ (as $(k) \in c(\Delta)$) and hence $\tilde{\Lambda}(k) \in c_0$.

Conversely, suppose that $\tilde{\Lambda}(k) \in c_0$ and let $x \in \mu(\Delta)$ be arbitrary. Then $x \in \ell_\infty(\Delta)$ and there must exist a positive real number $M > 0$ such that $|\Delta\,(x_k)| \leq M$ for all k.

Thus, for every $n \geq 2$, it follows, by using (3.2.2) and (3.2.3), that

$$\left|\tilde{\Lambda}_n(x)\right| \leq \frac{1}{\lambda_n v_n} \sum_{k=2}^{n} \lambda_{k-1} \left|\Delta(x_k)\right| \leq \frac{M}{\lambda_n v_n} \sum_{k=2}^{n} \lambda_{k-1} = M\,\tilde{\Lambda}_n(k).$$

Thus, we get $0 \leq |\tilde{\Lambda}_n(x)| \leq M\,\tilde{\Lambda}_n(k)$ for all n, and by passing to the limits when $n \to \infty$ with using the assumption $\tilde{\Lambda}(k) \in c_0$, we get $\tilde{\Lambda}(x) \in c_0$ and so $x \in c_0(\Delta^\lambda)$ which shows that $\mu(\Delta) \subset c_0(\Delta^\lambda)$. Finally, this inclusion must be strict. To see that, let $x = ((-1)^n n)$. Then $x \notin \ell_\infty(\Delta)$ and so $x \notin \mu(\Delta)$, but $x \in c_0(\Delta^\lambda)$ by Corollary 3.3.7 (as $\lim_{n\to\infty} n/t_n = 0$ from the hypothesis and so $\lim_{n\to\infty} x_n/t_n = 0$). □

At the end of this chapter, previous results will be illustrated by various examples concerning with the distinct cases of relation between the spaces $c(\Delta)$ and $c(\Delta^\lambda)$.

Example 3.3.14 The case of strict inclusion $c(\Delta) \subsetneqq c(\Delta^\lambda)$ (and so $\mu(\Delta) \subsetneqq \mu(\Delta^\lambda)$): In this case, we must have $t \notin \ell_\infty$ and $\tilde{\Lambda}(t) \in c$ by (2) of Corollary 3.3.12. Thus (t_n/n) cannot be oscillated while $\Delta(t)$ maybe oscillated. So, we have the following two cases:

I - When $(t_n/n) \in c$ and $t \notin \ell_\infty$: e.g., see Example 3.2.15 for such case, where $u \notin \ell_\infty$, $(u_n/n) \in c$ and $\tilde{\Lambda}(u) \in c$ (but $\Delta(u)$ is oscillated). In particular, if $\Delta(t) \in c$ and $t \notin \ell_\infty$; then $c(\Delta) \subsetneqq c(\Delta^\lambda)$. For example, consider the sequence $\lambda = (\lambda_n)$ defined by $\lambda_n = (n+1)^r$ for $n \geq 1$, where $r > 0$. Then, we have $t_n \to \infty$, $\Delta(t_n) \to 1/r$, $\tilde{\Lambda}_n(t) \to 1/(1+r)$ and $\tilde{\Lambda}_n(k) \to r/(1+r)$ as $n \to \infty$ (also, see Example 3.3.17).

II - When $t_n/n \to \infty$ (and so $t_n \to \infty$) as $n \to \infty$: This is the strong case of strict inclusions in which $\tilde{\Lambda}(k) \in c_0$ (see Corollary 3.3.13). In this case, we have $\mu(\Delta) \subsetneqq c_0(\Delta^\lambda) \subsetneqq c(\Delta^\lambda)$ and so $c(\Delta) \subsetneqq c(\Delta^\lambda)$. e.g., see Example 3.2.16 for such case, where $v_n \to \infty$, $v_n/n \to \infty$ and $\tilde{\Lambda}_n(v) \to 1$ as $n \to \infty$. In particular, if $\Delta(t) \to \infty$; then $c(\Delta) \subsetneqq c(\Delta^\lambda)$. For example, let $\lambda = (\lambda_n)$ be given by $\lambda_n = (2^{2n}-1)/2^{2n-1}$ and so $u_n = (2^{2n}-1)/3$ for all n. Then $u_n \to \infty$, $\Delta(u_n) \to \infty$, $\tilde{\Lambda}_n(u) \to 1$ and $\tilde{\Lambda}_n(k) \to 0$.

Example 3.3.15 The case of identity $c(\Delta^\lambda) = c(\Delta)$ (and so $\mu(\Delta^\lambda) = \mu(\Delta)$): In this case, we must have $t \in \ell_\infty \cap c_0(\Delta)$ by (2) of Theorem 3.3.10. In particular, if $t \in c$; then $c(\Delta^\lambda) = c(\Delta)$. For example, let $\lambda = (\lambda_n)$ be defined by $\lambda_n = (n+1)!$ (or $\lambda_n = a^n$, $a > 1$) for $n \geq 1$. Then, we can show that $t \in \ell_\infty$, $\Delta(t_n) \to 0$, $\tilde{\Lambda}_n(t) \to 0$, $\tilde{\Lambda}_n(k) \to 1$.

Example 3.3.16 The case of non-inclusion $c(\Delta) \not\subset c(\Delta^\lambda)$ (and so $c(\Delta^\lambda) \not\subset c(\Delta)$): In this case, we must have $\tilde{\Lambda}(t) \notin c$ (by (2) of Theorem 3.3.2), i.e. the sequence $\tilde{\Lambda}(t)$ is oscillated and so $\Delta(t)$ must be oscillated while (t_n/n) may not be oscillated (but if (t_n/n) is oscillated; then $c(\Delta) \not\subset c(\Delta^\lambda)$). Thus, with $\tilde{\Lambda}(t) \notin c$, we have two cases:

I - When $t \in \ell_\infty$ and so both of t and $\Delta(t)$ must be oscillated (in this case, we have $c_0(\Delta^\lambda) = c_0(\Delta)$ and $\ell_\infty(\Delta^\lambda) = \ell_\infty(\Delta)$). For example, consider the sequence $\lambda = (a, ab, a^2b, a^2b^2, \cdots)$, where $b > a > 1$. That is, for $k \geq 1$ we have

$$\lambda_k = \begin{cases} a^{(k+1)/2}\, b^{(k-1)/2}\,; & (k \text{ is odd}), \\ a^{k/2}\, b^{k/2}\,; & (k \text{ is even}). \end{cases}$$

Then, for evey $n > 1$, it can easily be shown that

$$u_n = \begin{cases} a/(a-1)\,; & (n \text{ is odd}), \\ b/(b-1)\,; & (n \text{ is even}). \end{cases} \qquad \Delta(u_n) = \begin{cases} (b-a)/[(a-1)(b-1)]\,; & (n \text{ is odd}), \\ -(b-a)/[(a-1)(b-1)]\,; & (n \text{ is even}). \end{cases}$$

$$\tilde{\Lambda}_n(u) = \begin{cases} \dfrac{b-a}{ab-1} + \dfrac{(a-1)(b+1)}{ab-1}\,(ab)^{-(n-1)/2}\,; & (n \text{ is odd}), \\ -\dfrac{b-a}{ab-1} + \dfrac{a(b^2-1)}{ab-1}\,(ab)^{-n/2}\,; & (n \text{ is even}). \end{cases}$$

Thus u, $\Delta(u)$, $\tilde{\Lambda}(u)$ and $\tilde{\Lambda}(k)$ are oscillated while $(u_n/n) \in c_0$ (see Example 3.2.16).

II - When $t \notin \ell_\infty$ and (t_n/n) is oscillated. Then, both of $\Delta(t)$ and $\tilde{\Lambda}(t)$ must be oscillated (in this case, the inclusions $c_0(\Delta) \subset c_0(\Delta^\lambda)$ and $\ell_\infty(\Delta) \subset \ell_\infty(\Delta^\lambda)$ are strict). For example, let $\lambda = (1, 3, 4, 8, 9, \cdots)$. That is, for $k \geq 1$, we have

$$\lambda_k = \begin{cases} (k+1)^2/4\,; & (k \text{ is odd}), \\ (k^2+4k)/4\,; & (k \text{ is even}). \end{cases}$$

Then, for every $n > 1$, it can easily be seen that

$$u_n = \begin{cases} (n+1)^2/4\,; & (n \text{ is odd}), \\ (n+4)/4\,; & (n \text{ is even}). \end{cases} \qquad \Delta(u_n) = \begin{cases} (n^2+n-2)/4\,; & (n \text{ is odd}), \\ (-n^2+n+4)/4\,; & (n \text{ is even}). \end{cases}$$

$$\tilde{\Lambda}_n(u) = \begin{cases} (3n-1)/(3n+3)\,; & (n \text{ is odd}), \\ (-n+4)/(3n)\,; & (n \text{ is even}). \end{cases}$$

Hence $u_n \to \infty$, (u_n/n) is oscillated between $1/4$ and ∞, $\Delta(u)$ is oscillated between $\pm\infty$, $\tilde{\Lambda}(u)$ is oscillated between 1 and $-1/3$, and so is $\tilde{\Lambda}(k)$ (between 0 and $4/3$).

• In Example 3.3.16, where $\tilde{\Lambda}(t) \notin c$ and so $c(\Delta) \not\subset c(\Delta^\lambda)$ as well as $c(\Delta^\lambda) \not\subset c(\Delta)$. We may ask about the intersection $c(\Delta) \cap c(\Delta^\lambda)$, what shall be equal? This question is left as an open problem with noting that $c_0(\Delta) \subset c(\Delta) \cap c(\Delta^\lambda) \subset c(\Delta)$.

Example 3.3.17 Here, we consider the case $t \in c_0(\Delta^\lambda) \setminus \ell_\infty$ which shows that the regularity of $\tilde{\Lambda}$ over $c(\Delta)$ does not imply the equality $c(\Delta^\lambda) = c(\Delta)$ (see (4) of Remark 3.3.11). This example is also a particular case of that given in (I) of Example 3.3.14. In this particular case, we get the same properties of the case $c(\Delta^\lambda) = c(\Delta)$ given in Example 3.3.15 with only one difference, namely $t \notin \ell_\infty$. Thus, we just have the strict inclusion $c(\Delta) \subset c(\Delta^\lambda)$ with $\tilde{\Lambda}$-regularity over $c(\Delta)$. For, let $\lambda = (\lambda_k)$ be defined by

$$\lambda_k = \prod_{j=1}^{k} \left(1 + \sqrt{j} - \sqrt{j-1}\right) \qquad (k \geq 1).$$

Then, it can easily be shown that

$$\Delta(\lambda_1) = 2 \quad \text{and} \quad \Delta(\lambda_k) = \left(\sqrt{k} - \sqrt{k-1}\,\right)\lambda_{k-1} \qquad (k > 1),$$

$$v_1 = 0 \quad \text{and} \quad v_k = \sqrt{k} + \sqrt{k-1} \qquad (k > 1).$$

Thus, we find that $v \notin \ell_\infty$ which implies that $c(\Delta^\lambda) \neq c(\Delta)$. But, it is clear that $\Delta(v) \in c_0$ which implies that $\tilde{\Lambda}(v) \in c_0$ (as $c_0(\Delta) \subset c_0(\Delta^\lambda)$). Therefore, we have the strict inclusion $c(\Delta) \subset c(\Delta^\lambda)$ with $\tilde{\Lambda}$-regularity over $c(\Delta)$.

Chapter 4

KÖTHE-TOEPLITZ DUALITY

4 KÖTHE-TOEPLITZ DUALITY

In the present chapter, we shall conclude the α-, β- and γ-duals for the λ-difference spaces of bounded, convergence and null λ-difference sequences. Also, we study some properties of their duals. This chapter is divided into three sections, the first is devoted to present our used notations and prove some lemmas, the second is to obtain the dual spaces for the usual difference spaces and the last is to deduce the dual spaces for the new λ-difference spaces. The materials of this chapter are part of our research paper* which has been published in the Ijrdo J. Math., on 2022.

4.1 Terminologies

In this first section, we shall display some needed notations and terminologies, and prove some important results about series (bounded, convergent and absolutely convergent series) which will be used in proving the main results in next sections.

For simplicity in notations, here and in the sequel, we assume that $a \in cs$. Then, the series $\sum_{j=1}^{\infty} a_j$ converges and we denote its k-th remainder $\sum_{j=k}^{\infty} a_j$ by $R_k(a)$ or simply R_k for all $k \geq 1$, and so $R = (R_k)$ is the sequence of all those remainders. Besides, the finite sum $\sum_{j=k}^{n} a_j$ will be denoted by $R_k^n(a)$ or simply R_k^n for all $n \geq 1$ and every $k \leq n$. That is, we have

$$R_k := R_k(a) = \sum_{j=k}^{\infty} a_j \quad \text{and} \quad R_k^n := R_k^n(a) = \sum_{j=k}^{n} a_j \qquad (k \geq 1,\, n \geq k). \tag{4.1.1}$$

*A.K. Noman and O.H. Al-Sabri, *Matrix operators on the new spaces of λ-difference sequences*, Ijrdo J. Math., **8**(1) (2022), 1–22.

Thus $R_k = \lim_{n\to\infty} R_k^n$ and $R_k = R_k^n + R_{n+1}$ which implies that

$$\big|\,|R_k^n| - |R_k|\,\big| \le |R_k^n - R_k| = |R_{n+1}| \qquad (1 \le k \le n)$$

and by applying the triangle inequality, we obtain that

$$\left|\sum_{k=1}^{n}|R_k^n| - \sum_{k=1}^{n}|R_k|\right| \le n|R_{n+1}| \qquad (n \ge 1). \tag{4.1.2}$$

Also, for every $n \ge 2$, we have

$$\sum_{k=1}^{n}|a_k| = |a_n| + \sum_{k=1}^{n-1}|R_k^n - R_{k+1}^n| \le \sum_{k=1}^{n}|R_k^n| + \sum_{k=2}^{n}|R_k^n|,$$

that is

$$\sum_{k=1}^{n}|a_k| \le \sum_{k=1}^{n}|R_k^n| + \sum_{k=2}^{n}|R_k^n| \qquad (n \ge 2)$$

and by adding $|R_1^n|$ to both sides, we deduce the following inequality:

$$\sum_{k=1}^{n}|a_k| \;\le\; \sum_{k=1}^{n}|a_k| + \left|\sum_{k=1}^{n}a_k\right| \;\le\; 2\sum_{k=1}^{n}|R_k^n| \qquad (n \ge 1). \tag{4.1.3}$$

Further, we will frequently use the following familiar sum-formula:

$$\sum_{k=r}^{n}s_k\sum_{j=r}^{k}t_j = \sum_{k=r}^{n}t_k\sum_{j=k}^{n}s_j \qquad (1 \le r \le n)$$

which still valid if n is replaced by ∞ provided that series are convergent. Thus, with our assumption $a \in cs$, we obtain that

$$R_{n+1} \;=\; \sum_{m=n+1}^{\infty}\Big(\frac{1}{m} - \frac{1}{m+1}\Big)\sum_{j=n+1}^{m} j a_j \qquad (n \ge 1). \tag{4.1.4}$$

Moreover, for every $x \in w$, we have $x_k = \sum_{j=1}^{k}\Delta(x_j)$ for $k \ge 1$ and it follows that

$$\sum_{k=1}^{n}a_k x_k = \sum_{k=1}^{n}R_k^n\,\Delta(x_k) = \sum_{k=1}^{n}R_k\,\Delta(x_k) - x_n R_{n+1} \qquad (n \ge 1), \tag{4.1.5}$$

$$\sum_{k=1}^{n}k a_k = \sum_{k=1}^{n}R_k^n = \sum_{k=1}^{n}R_k - nR_{n+1} \qquad (n \ge 1), \tag{4.1.6}$$

where (4.1.6) can also be obtained from (4.1.5) by taking $x_k = k$ for all $k \ge 1$.

In addition, let us consider the particular case $a \in \ell_1$ in which $|a| = (|a_k|) \in cs$ and so $a \in cs$. Then, all above relations are valid and we also use (4.1.1) to define $\bar{R}_k^n = R_k^n(|a|)$ for $1 \leq k \leq n$, and $\bar{R} = (\bar{R}_k)$ is the sequence defined by $\bar{R}_k = R_k(|a|)$ for every $k \geq 1$. That is, we have the following additional conventions:

$$\bar{R}_k := R_k(|a|) = \sum_{j=k}^{\infty}|a_j| \quad \text{and} \quad \bar{R}_k^n := R_k^n(|a|) = \sum_{j=k}^{n}|a_j| \qquad (k \geq 1,\, n \geq k). \quad (4.1.7)$$

Thus, from (4.1.6) with $|a|$ instead of a, we find that

$$\sum_{k=1}^{n}|ka_k| = \sum_{k=1}^{n}\bar{R}_k^n = \sum_{k=1}^{n}\bar{R}_k - n\bar{R}_{n+1} \qquad (n \geq 1). \qquad (4.1.8)$$

Now, we may begin with the following basic lemma:

Lemma 4.1.1 *For every $a \in \ell_1$, we have the following facts:*

(1) $\sum_{k=1}^{n}|\bar{R}_k| = \sum_{k=1}^{n}\bar{R}_k$ *and* $\sum_{k=1}^{n}|\bar{R}_k^n| = \sum_{k=1}^{n}\bar{R}_k^n$ *for all n.*

(2) $\bar{R} \in cs \Longleftrightarrow \bar{R} \in \ell_1 \Longleftrightarrow \sum_{k=1}^{\infty}\sum_{j=k}^{\infty}|a_j| < \infty$.

(3) *The sequence $\left(\sum_{k=1}^{n}\bar{R}_k^n\right)_{n=1}^{\infty}$ is increasing of non-negative real numbers.*

(4) *If $\bar{R} \in \ell_1$; then $\left(\sum_{k=1}^{n}\bar{R}_k^n\right)_{n=1}^{\infty} \in \ell_\infty$.*

Proof. (1) It is obvious by (4.1.7) that $\bar{R}_k^n \geq 0$ as well as $\bar{R}_k \geq 0$ for every $k \geq 1$ and all $n \geq k$. Thus $|\bar{R}_k^n| = \bar{R}_k^n$ as well as $|\bar{R}_k| = \bar{R}_k$ for every $k \geq 1$ and all $n \geq k$ which proves (1). Also, part (2) is immediate by (1) with help of (4.1.7).

For (3), it can easily be seen that $\Delta\left(\sum_{k=1}^{n}\bar{R}_k^n\right) = n|a_n| \geq 0$ for all n and so $\left(\sum_{k=1}^{n}\bar{R}_k^n\right)_{n=1}^{\infty}$ is increasing of non-negative real numbers, where $\bar{R}_k^n \geq 0$ for all n.

To prove (4), it is clear by (4.1.7) that $\bar{R}_k^n \leq \bar{R}_k$ for every $k \geq 1$ and all $n \geq k$. Also, if $\bar{R} \in \ell_1$; then we deduce that $\sum_{k=1}^{n}\bar{R}_k^n \leq \sum_{k=1}^{n}\bar{R}_k \leq \sum_{k=1}^{\infty}\bar{R}_k$ for all n (since $\bar{R}_k \geq 0$ for all k). Hence, it follows that $\left(\sum_{k=1}^{n}\bar{R}_k^n\right)_{n=1}^{\infty} \in \ell_\infty$. □

Lemma 4.1.2 *We have the following true implications:*

(1) $(na_n) \in bs \Longrightarrow a \in cs$. *Also* $(\sum_{k=1}^{n} R_k^n) \in \ell_\infty \Longrightarrow a \in cs$.

(2) $(na_n) \in cs \Longrightarrow a \in cs$. *Also* $(\sum_{k=1}^{n} R_k^n) \in c \Longrightarrow a \in cs$.

(3) $(\sum_{k=1}^{n} |R_k^n|) \in \ell_\infty \Longrightarrow a \in \ell_1$. *Also* $(\sum_{k=1}^{n} |R_k^n|) \in c \Longrightarrow a \in \ell_1$.

(4) $(na_n) \in \ell_1 \Longrightarrow a \in \ell_1$. *Also* $(\sum_{k=1}^{n} \bar{R}_k^n) \in \ell_\infty \Longrightarrow a \in \ell_1$.

Proof. For (1), assume $(na_n) \in bs$, and since $(1/n) \in bv_0$; we deduce that $a = (a_n) = (1/n)(na_n) \in cs$ (as $bs^\beta = bv_0$). The second implication is obtained from the first one which help of (4.1.6), where $(\sum_{k=1}^{n} R_k^n) \in \ell_\infty \Longrightarrow (\sum_{k=1}^{n} ka_k) \in \ell_\infty \Longrightarrow (na_n) \in bs$ (in other words: $(\sum_{k=1}^{n} R_k^n) \in \ell_\infty \Longrightarrow (\Delta(\sum_{k=1}^{n} R_k^n)) \in bs \Longrightarrow (na_n) \in bs$, since $\Delta(\sum_{k=1}^{n} R_k^n) = na_n$ for all n). Also, the implications in part (2) are immediate by those given in part (1), as $cs \subset bs$ and $c \subset \ell_\infty$.

For (3), it is obvious by (4.1.3) that

$$\sum_{k=1}^{n} |a_k| \leq 2 \sum_{k=1}^{n} |R_k^n| \leq 2 \sup_n \sum_{k=1}^{n} |R_k^n| \qquad (n \geq 1).$$

Thus, if $(\sum_{k=1}^{n} |R_k^n|) \in \ell_\infty$; then $a \in \ell_1$ which proves the first implication and the second one is immediate by the first.

Finally, the implications in part (4) are immediate by those given in part (1) with $|a|$ instead of a. In other words, we have $|a_k| \leq |ka_k|$ for all k and hence $\sum_{k=1}^{\infty} |a_k| \leq \sum_{k=1}^{\infty} |ka_k|$ which yields the first implication, i.e. $(na_n) \in \ell_1 \Longrightarrow a \in \ell_1$. The last implication can be obtained from (3), where $|R_k^n| \leq \bar{R}_k^n$ $(k \leq n)$ and so $\sum_{k=1}^{n} |R_k^n| \leq \sum_{k=1}^{n} \bar{R}_k^n$ which can also be obtained from the first implication by using (4.1.8), where

$$(\sum_{k=1}^{n} \bar{R}_k^n) \in \ell_\infty \Longrightarrow (\sum_{k=1}^{n} |ka_k|) \in \ell_\infty \Longrightarrow (\sum_{k=1}^{n} |ka_k|) \in c \Longrightarrow (na_n) \in \ell_1$$

(note also that: $\Delta(\sum_{k=1}^{n} \bar{R}_k^n) = |na_n|$ for all n. Thus, we deduce that

$$(\sum_{k=1}^{n} \bar{R}_k^n) \in \ell_\infty \Longrightarrow (\Delta(\sum_{k=1}^{n} \bar{R}_k^n)) \in bs \Longrightarrow (|na_n|) \in bs \Longrightarrow (na_n) \in \ell_1). \qquad \square$$

Lemma 4.1.3 *We have the following true implications:*

(1) $(na_n) \in bs \Longrightarrow (nR_{n+1}) \in \ell_\infty$. *Also* $\left(\sum_{k=1}^n R_k^n\right) \in \ell_\infty \Longrightarrow (nR_{n+1}) \in \ell_\infty$.

(2) $(na_n) \in cs \Longrightarrow (nR_{n+1}) \in c_0$. *Also* $\left(\sum_{k=1}^n R_k^n\right) \in c \Longrightarrow (nR_{n+1}) \in c_0$.

(3) $(na_n) \in \ell_1 \Longrightarrow (n\bar{R}_{n+1}) \in c_0$. *Also* $\left(\sum_{k=1}^n \bar{R}_k^n\right) \in c \Longrightarrow (n\bar{R}_{n+1}) \in c_0$.

Proof. (1) Suppose that $(na_n) \in bs$. Then $a \in cs$ (Lemma 4.1.2) and so the sequence $R = (R_k)$ is well-defined. Also, since $\left(\sum_{j=1}^m ja_j\right) \in \ell_\infty$ (by assumption); there is a real number $M > 0$ such that $|\sum_{j=1}^m ja_j| \leq M$ for every $m \geq 1$. Thus, for all integers $m, n \geq 1$ such that $m > n$, we find that

$$\left|\sum_{j=n+1}^m ja_j\right| = \left|\sum_{j=1}^m ja_j - \sum_{j=1}^n ja_j\right| \leq \left|\sum_{j=1}^m ja_j\right| + \left|\sum_{j=1}^n ja_j\right| \leq 2M$$

and by using (4.1.4) it follows that $(n \geq 1)$

$$|R_{n+1}| \leq \sum_{m=n+1}^\infty \left(\frac{1}{m} - \frac{1}{m+1}\right)\left|\sum_{j=n+1}^m ja_j\right| \leq 2M \sum_{m=n+1}^\infty \left(\frac{1}{m} - \frac{1}{m+1}\right).$$

Thus $|R_{n+1}| \leq 2M/(n+1) < 2M/n$ and hence $n|R_{n+1}| < 2M$ for all $n \geq 1$ which means that $(nR_{n+1}) \in \ell_\infty$. Also, if $\left(\sum_{k=1}^n R_k^n\right) \in \ell_\infty$, then it follows by (4.1.6) that $\left(\sum_{k=1}^n ka_k\right) \in \ell_\infty$ which means that $(na_n) \in bs$ and so $(nR_{n+1}) \in \ell_\infty$ (as we have already shown).

For (2), assume that $(na_n) \in cs$. Then $a \in cs$ and so R exists. Also, since $\left(\sum_{j=1}^m ja_j\right) \in c$ (by hypothesis); for every positive real number $\epsilon > 0$ there is an integer $k_0 > 0$ such that $|\sum_{j=n+1}^m ja_j| < \epsilon$ for all integers m and n satisfying $m > n > k_0$. This, together with (4.1.4) and the same technique used above, leads us to deduce the following for every integer $n > k_0$

$$|R_{n+1}| \leq \sum_{m=n+1}^\infty \left(\frac{1}{m} - \frac{1}{m+1}\right)\left|\sum_{j=n+1}^m ja_j\right| < \epsilon \sum_{m=n+1}^\infty \left(\frac{1}{m} - \frac{1}{m+1}\right).$$

Thus $|R_{n+1}| < \epsilon/(n+1) < \epsilon/n$ and hence $n|R_{n+1}| < \epsilon$ for all $n > k_0$ which means that $(nR_{n+1}) \in c_0$. Also, the other implication is obvious by (4.1.6).

Finally, to prove (3), let $(na_n) \in \ell_1$. Then $a \in \ell_1$ (Lemma 4.1.2) and so $\bar{R} = (\bar{R}_k)$ is well-defined. Therefore, the implications in part (3) are immediate by those of part (2) with $|a|$ instead of a, where $\bar{R}_k^n = R_k^n(|a|)$ for $n \geq k \geq 1$. This ends the proof (note that $(na_n) \in \ell_1 \Longleftrightarrow (|na_n|) \in cs$, and we have $|na_n| = n|a_n|$ for all n). □

Remark 4.1.4 It must be noted that $\bar{R} \in \ell_1$ implies that $(n\bar{R}_{n+1}) \in c_0$ for every $a \in \ell_1$. To see that, we may note that $\bar{R} \in \ell_1 \Longrightarrow (\sum_{k=1}^n \bar{R}_k^n) \in \ell_\infty$ (by (4) of Lemma 4.1.1) $\Longrightarrow (\sum_{k=1}^n \bar{R}_k^n) \in c$ (by (3) of Lemma 4.1.1) $\Longrightarrow (n\bar{R}_{n+1}) \in c_0$ (by (3) of Lemma 4.1.3). In other words, it is clear that if $\bar{R} \in \ell_1$, then $\bar{R}$ is a decreasing sequence of non-negative real numbers such that $\bar{R} \in \ell_1$ and so $(n\bar{R}_{n+1}) \in c_0$ [56].

Lemma 4.1.5 *The following conditions are equivalent to each others:*

(1) $(na_n) \in bs$.

(2) $a \in cs$, $R \in bs$ *and* $(nR_{n+1}) \in \ell_\infty$.

(3) $(\sum_{k=1}^n R_k^n) \in \ell_\infty$.

Proof. We will prove that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1) as follows:

Suppose that (1) is satisfied, that is $(\sum_{k=1}^n ka_k) \in \ell_\infty$. Then $a \in cs$ (by (1) of Lemma 4.1.2) and $(nR_{n+1}) \in \ell_\infty$ (by (1) of Lemma 4.1.3). Thus, it follows by (4.1.6) that $(\sum_{k=1}^n R_k) \in \ell_\infty$ which means that $R \in bs$ and this shows that (1) $\Longrightarrow$ (2).

Also, assume that (2) is satisfied, that is $(nR_{n+1}) \in \ell_\infty$ and $R \in bs$, where $a \in cs$ and so $R = (R_k)$ is well-defined. Then, it follows by (4.1.6) that $(\sum_{k=1}^n R_k^n) \in \ell_\infty$ which is (3).

Lastly, it is clear by (4.1.6) that (3) $\Longrightarrow$ (1), where (3) implies that $(\,\Delta(\sum_{k=1}^{n} R_k^n)\,) \in bs$, but $\Delta(\,\sum_{k=1}^{n} R_k^n\,) = na_n$ for all n which implies (1). This ends the proof (note that: each one of given conditions implies that $a \in cs$). □

Theorem 4.1.6 *The following conditions are equivalent to each others:*

(1) $(\,\sum_{k=1}^{n} |R_k^n|\,) \in \ell_\infty$.

(2) $a \in \ell_1$, $R \in \ell_1$ *and* $(\,\sum_{k=1}^{n} R_k^n\,) \in \ell_\infty$.

(3) $a \in \ell_1$, $R \in \ell_1$ *and* $(nR_{n+1}) \in \ell_\infty$.

(4) $a \in \ell_1$, $R \in \ell_1$ *and* $(na_n) \in bs$.

Proof. It obvious by Lemma 4.1.5 that (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) (since $\ell_1 \subset cs \subset bs$). To see that, it is clear that if $a \in \ell_1$ and $R \in \ell_1$ (and so $a \in cs$ and $R \in bs$); then we find, by Lemma 4.1.5, that

$$(\,\sum_{k=1}^{n} R_k^n\,) \in \ell_\infty \Longleftrightarrow (nR_{n+1}) \in \ell_\infty \Longleftrightarrow (na_n) \in bs$$

which means that (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4). Thus, to prove that given conditions are equivalent, it is remaining to prove that (1) $\Longleftrightarrow$ (2). For this, suppose that (1) is satisfied, that is $(\,\sum_{k=1}^{n} |R_k^n|\,) \in \ell_\infty$. Then $a \in \ell_1$ (by (3) of Lemma 4.1.2) and since

$$\left|\sum_{k=1}^{n} R_k^n\right| \leq \sum_{k=1}^{n} |R_k^n| \leq \sup_n \sum_{k=1}^{n} |R_k^n| \qquad (n \geq 1);$$

we get $(\,\sum_{k=1}^{n} R_k^n\,) \in \ell_\infty$ and so $(nR_{n+1}) \in \ell_\infty$ by Lemma 4.1.5. Therefore, we have $(\,\sum_{k=1}^{n} |R_k^n|\,) \in \ell_\infty$ as well as $(nR_{n+1}) \in \ell_\infty$. Also, by using (4.1.2), we deduce that $(\,\sum_{k=1}^{n} |R_k|\,) \in \ell_\infty$ and so $(\,\sum_{k=1}^{n} |R_k|\,) \in c$ which means that $R \in \ell_1$. Hence, we conclude that (1) $\Longrightarrow$ (2).

Conversely, assume that (2) is satisfied, that is $(\,\sum_{k=1}^{n} R_k^n\,) \in \ell_\infty$ and $R \in \ell_1$, where $a \in \ell_1$ and so $a \in cs$ which means that all terms of R exist. Then, we have $(nR_{n+1}) \in \ell_\infty$

(by Lemma 4.1.5) and $\left(\sum_{k=1}^{n}|R_k|\right) \in \ell_\infty$ (as $R \in \ell_1$). This together with (4.1.2) lead us to conclude that $\left(\sum_{k=1}^{n}|R_k^n|\right) \in \ell_\infty$ which is (1), that is (2) $\Longrightarrow$ (1). Therefore, we deduce that (1) $\Longleftrightarrow$ (2) and this completes the proof, since (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4). □

Lemma 4.1.7 *The following conditions are equivalent to each others:*

(1) $(na_n) \in cs$.

(2) $a \in cs$, $R \in cs$ *and* $(nR_{n+1}) \in c_0$.

(3) $\left(\sum_{k=1}^{n} R_k^n\right) \in c$.

Furthermore, if any one of above conditions is satisfied, then we have

$$\sum_{k=1}^{\infty} k a_k = \lim_{n\to\infty} \sum_{k=1}^{n} R_k^n = \lim_{n\to\infty} \sum_{k=1}^{n} R_k = \sum_{k=1}^{\infty} R_k\,. \tag{4.1.9}$$

Proof. First, it is obvious, by (2) of Lemma 4.1.2, that each one of given conditions implies that $a \in cs$ and so the sequence R is well-defined. Next, to show that these conditions are equivalent, it can easily be proved that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1), but the proof is exactly same as that proof of Lemma 4.1.5 (by using (4.1.6) and (2) of Lemma 4.1.3).

Further, suppose that condition (1), (2) or (3) is satisfied. Then, since these conditions are equivalent; all are satisfied and by going to the limits in all sides of (4.1.6) as $n \to \infty$, we get (4.1.9). This ends the proof. □

Theorem 4.1.8 *The following conditions are equivalent to each others:*

(1) $\left(\sum_{k=1}^{n}|R_k^n|\right) \in \ell_\infty$ *and* $(nR_{n+1}) \in c_0$.

(2) $\left(\sum_{k=1}^{n}|R_k^n|\right) \in c$ *and* $(nR_{n+1}) \in c_0$.

(3) $a \in \ell_1$, $R \in \ell_1$ *and* $(nR_{n+1}) \in c_0$.

(4) $a \in \ell_1$, $R \in \ell_1$ *and* $(na_n) \in cs$.

(5) $a \in \ell_1$, $R \in \ell_1$ *and* $\left(\sum_{k=1}^{n} R_k^n\right) \in c$.

Furthermore, if any one of above conditions is satisfied, then we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} |R_k^n| = \lim_{n\to\infty} \sum_{k=1}^{n} |R_k| = \sum_{k=1}^{\infty} |R_k|. \tag{4.1.10}$$

Proof. It obvious by Lemma 4.1.7 that (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) (as $\ell_1 \subset cs$). Thus, to prove that given conditions are equivalent, it is remaining to prove that (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3). For this, suppose that (1) is satisfied, that is $\left(\sum_{k=1}^{n} |R_k^n|\right) \in \ell_\infty$ and $(nR_{n+1}) \in c_0$, where $a \in \ell_1$ (by (3) of Lemma 4.1.2) and so R is well-defined. Also, since $\left(\sum_{k=1}^{n} |R_k^n|\right) \in \ell_\infty$; we get $R \in \ell_1$ (by Theorem 4.1.6) which means that $\left(\sum_{k=1}^{n} |R_k|\right) \in c$. Therefore, we have $(nR_{n+1}) \in c_0$ as well as $\left(\sum_{k=1}^{n} |R_k|\right) \in c$. This leads us with help of (4.1.2) to deduce that $\left(\sum_{k=1}^{n} |R_k^n|\right) \in c$ which means that (2) is satisfied and hence (1) $\Longrightarrow$ (2). Also, it is trivial that (2) $\Longrightarrow$ (1) and so (1) $\Longleftrightarrow$ (2).

Next, assume that (2) is satisfied, that is $\left(\sum_{k=1}^{n} |R_k^n|\right) \in c$ and $(nR_{n+1}) \in c_0$, where $a \in \ell_1$ (by (3) of Lemma 4.1.2) and so R is well-defined. Then, by using (4.1.2), it can easily be seen that $\left(\sum_{k=1}^{n} |R_k|\right) \in c$ and so $R \in \ell_1$, that is (2) $\Longrightarrow$ (3). Conversely, suppose that (3) holds, that is $a \in \ell_1$, $R \in \ell_1$ and $(nR_{n+1}) \in c_0$. Then, it follows by (4.1.2) that $\left(\sum_{k=1}^{n} |R_k^n|\right) \in c$ which means that (2) holds, that is (3) $\Longrightarrow$ (2) and hence (2) $\Longleftrightarrow$ (3). Consequently, the given conditions are equivalent to each others.

Finally, if any one of these equivalent conditions is satisfied; we have $(nR_{n+1}) \in c_0$, $\left(\sum_{k=1}^{n} |R_k^n|\right) \in c$ and $\left(\sum_{k=1}^{n} |R_k|\right) \in c$. Therefore, from (4.1.2) we get (4.1.10) and this ends the proof. □

Finally, we end this section with the following theorem for which we need to keep in mind those facts mentioned in Lemma 4.1.1 with our notations given by (4.1.7).

Theorem 4.1.9 *The following conditions are equivalent to each others:*

(1) $(\sum_{k=1}^{n} \bar{R}_k^n) \in \ell_\infty$.

(2) $(\sum_{k=1}^{n} \bar{R}_k^n) \in c$.

(3) $(na_n) \in \ell_1$.

(4) $a \in \ell_1$ *and* $\bar{R} \in \ell_1$.

Furthermore, if any one of above conditions is satisfied, then $(n\bar{R}_{n+1}) \in c_0$ *and we have*

$$\sum_{k=1}^{\infty} |ka_k| = \lim_{n\to\infty} \sum_{k=1}^{n} \bar{R}_k^n = \lim_{n\to\infty} \sum_{k=1}^{n} \bar{R}_k = \sum_{k=1}^{\infty} \bar{R}_k. \tag{4.1.11}$$

Proof. First, it is obvious that each of given conditions implies that $a \in \ell_1$ (by (4) of Lemma 4.1.2). Next, to show that these conditions are equivalent, we will prove that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1).

For this, suppose that (1) is satisfied, that is $(\sum_{k=1}^{n} \bar{R}_k^n) \in \ell_\infty$. Then, it follows by (3) of Lemma 4.1.1 that the sequence $(\sum_{k=1}^{n} \bar{R}_k^n)$ is increasing as well as bounded. This implies that $(\sum_{k=1}^{n} \bar{R}_k^n) \in c$ which is (2). Hence (1) $\Longrightarrow$ (2).

Also, let (2) be satisfied, that is $(\sum_{k=1}^{n} \bar{R}_k^n) \in c$. Then, it follows by (4.1.8) that $(\sum_{k=1}^{n} |ka_k|) \in c$ and so $(na_n) \in \ell_1$ which is (3), that is (2) $\Longrightarrow$ (3).

Further, assume that (3) is satisfied, that is $(na_n) \in \ell_1$. Then $a \in \ell_1$ (by (4) of Lemma 4.1.2) and so $\bar{R}$ is well-defined. Further, we have, by assumption, that $(|na_n|) \in cs$ or $(n|a_n|) \in cs$. Thus, it follows, by Lemma 4.1.7 with $|a|$ instead of a, that $R(|a|) \in cs$ and so $\bar{R} \in cs$, where $\bar{R} = R(|a|)$ by (4.1.7). This together with (2) of Lemma 4.1.1 implies that $\bar{R} \in \ell_1$ which means that (4) is satisfied, that is (3) $\Longrightarrow$ (4).

Moreover, it is obvious, by (4) of Lemma 4.1.1, that (4) $\Longrightarrow$ (1). Therefore, all given conditions are equivalent to each others.

Finally, if any of these equivalent conditions is satisfied; then $(\sum_{k=1}^{n}\bar{R}_k^n) \in c$, $(\sum_{k=1}^{n}\bar{R}_k) \in c$ and $(n\bar{R}_{n+1}) \in c_0$ (see (3) of Lemma 4.1.3 and Remark 4.1.4). Hence, by passing to the limits in all sides of (4.1.8) as $n \to \infty$, we get (4.1.11). $\square$

Remark 4.1.10 It must be noted that every condition in Theorems 4.1.6, 4.1.8 and 4.1.9 implies that $a \in \ell_1$. To see that, we have $R \in \ell_1 \Longrightarrow R \in bv_1$ (as $\ell_1 \subset bv_1$) $\Longrightarrow \Delta(R) \in \ell_1 \Longrightarrow a \in \ell_1$, where $|\Delta(R_{k+1})| = |a_k|$ for all k. i.e. $R \in \ell_1 \Longrightarrow a \in \ell_1$ and similarly $\bar{R} \in \ell_1 \Longrightarrow a \in \ell_1$ (see also (3) and (4) of Lemma 4.1.2). Thus, the condition "$a \in \ell_1$" is necessary in Theorem 4.1.9 and must be mentioned. But, in Theorems 4.1.6 and 4.1.8, the condition "$a \in \ell_1$" can be replaced by the weaker condition "$a \in cs$" which is enough to define R by (4.1.1) (and from $R \in \ell_1$ we get $a \in \ell_1$). Similarly, we may note that "$a \in cs$" is a necessary condition in Lemmas 4.1.5 and 4.1.7.

4.2 Duality For $\mu(\Delta)$

In the present section, we apply the results of previous section to obtain some new and known results for the α-, β- and γ-duals of the usual difference spaces.

For simplicity in notations, the usual terminologies of previous chapters will be used throughout. That is, we will use the symbol μ to stand for any one of the spaces c_0, c or ℓ_∞. Thus, by $\mu(\Delta)$ we mean the respective one of the spaces $c_0(\Delta)$, $c(\Delta)$ or $\ell_\infty(\Delta)$, and so the corresponding space of $c_0(\Delta^\lambda)$, $c(\Delta^\lambda)$ or $\ell_\infty(\Delta^\lambda)$ will be denoted by $\mu(\Delta^\lambda)$. Also, by θ, we mean any one of the duality symbols α, β or γ, that is $\theta := \alpha$, β or γ. Thus, the θ-dual of a sequence space X is the α-, β- or γ-dual of X which was defined by (1.1.3) as $X^\theta = \{a \in w : ax \in \langle\theta\rangle \text{ for all } x \in X\}$, where $\langle\alpha\rangle = \ell_1$, $\langle\beta\rangle = cs$ and $\langle\gamma\rangle = bs$. For example, it is known by (3) of Lemma 1.3.5 that $\mu^\theta = \ell_1$, and we are going to find out the θ-duals of the spaces $\mu(\Delta)$ (in this section) and the θ-duals of the spaces $\mu(\Delta^\lambda)$ (in next section). For this, we have $\mu \subset \mu(\Delta)$ and $\mu \subset \mu(\Delta^\lambda)$. Thus, it follows by (2) of Lemma 1.3.5 that $\{\mu(\Delta)\}^\theta \subset \mu^\theta$ and $\{\mu(\Delta^\lambda)\}^\theta \subset \mu^\theta$. That is

$$\{\mu(\Delta)\}^\theta \subset \ell_1 \quad \text{and} \quad \{\mu(\Delta^\lambda)\}^\theta \subset \ell_1.$$

Thus, we assume that $a \in \ell_1$ and we may begin with obtaining θ-duals of the difference spaces $\mu(\Delta)$. So, the notations in (4.1.1) and (4.1.7) will be used. Also, for every $x \in w$, it is known that $x \in \mu(\Delta) \Longleftrightarrow \Delta(x) \in \mu$, and by using (4.1.5) we have

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} R_k^n \, \Delta(x_k) = T_n(\Delta(x)) \qquad (n \geq 1),$$

where $T = [t_{nk}]$ is a triangle defined for all $n, k \geq 1$ by

$$t_{nk} = \begin{cases} R_k^n\,; & (1 \leq k \leq n), \\ 0\,; & (k > n \geq 1). \end{cases}$$

Thus, by using above relation with (1) and (2) of Lemma 1.3.7, we deduce the following:

$$\begin{aligned} a \in \{\mu(\Delta)\}^\gamma &\Longleftrightarrow ax \in bs \text{ for all } x \in \mu(\Delta) \\ &\Longleftrightarrow \Big(\sum\nolimits_{k=1}^{n} a_k x_k\Big) \in \ell_\infty \text{ for all } x \in \mu(\Delta) \\ &\Longleftrightarrow T(\Delta(x)) \in \ell_\infty \text{ for all } \Delta(x) \in \mu \\ &\Longleftrightarrow T(y) \in \ell_\infty \text{ for all } y \in \mu \quad (\, y = \Delta(x)\,) \\ &\Longleftrightarrow T \in (\mu, \ell_\infty) \quad (\text{as } T \text{ is a triangle}). \end{aligned}$$

On other side, Lemma 1.3.9 tells us that $\sup_n \sum_{k=1}^{\infty} |t_{nk}| < \infty$ is the necessary and sufficient condition for $T \in (\mu, \ell_\infty)$. Thus, by above definition of T, it follows that

$$T \in (\mu, \ell_\infty) \Longleftrightarrow \sup_n \sum_{k=1}^{\infty} |t_{nk}| < \infty \Longleftrightarrow \sup_n \sum_{k=1}^{n} |R_k^n| < \infty.$$

Therefore, we deduce that

$$a \in \{\mu(\Delta)\}^\gamma \Longleftrightarrow \Big(\sum\nolimits_{k=1}^{n} |R_k^n|\Big) \in \ell_\infty. \tag{4.2.1}$$

Similarly, we show that $a \in \{\mu(\Delta)\}^{\beta} \Longleftrightarrow T \in (\mu, c)$, and by Lemma 1.3.10 we get

$$a \in \{c_0(\Delta)\}^{\beta} \Longleftrightarrow \left(\sum_{k=1}^{n} |R_k^n|\right) \in \ell_\infty \tag{4.2.2}$$

$$a \in \{\eta(\Delta)\}^{\beta} \Longleftrightarrow \left(\sum_{k=1}^{n} |R_k^n|\right) \in \ell_\infty \text{ and } (nR_{n+1}) \in c_0, \tag{4.2.3}$$

where η stands for any one of the spaces c or ℓ_∞, and so $\eta(\Delta)$ is the respective one of the spaces $c(\Delta)$ or $\ell_\infty(\Delta)$. Similarly, by using Lemma 1.3.8, it can be shown that

$$a \in \{\mu(\Delta)\}^{\alpha} \Longleftrightarrow \left(\sum_{k=1}^{n} \bar{R}_k^n\right) \in \ell_\infty \tag{4.2.4}$$

So, with help of Theorems 4.1.6, 4.1.8 and 4.1.9, we conclude the following result in which we give an equivalent formula for each type of dual spaces of $\mu(\Delta)$.

Theorem 4.2.1 *Let μ be any of the spaces c_0, c or ℓ_∞. Then, we have the following:*

(1) *The α-duals of $\mu(\Delta)$ are given by*

$$\{\mu(\Delta)\}^{\alpha} = \{a \in \ell_1 : \bar{R} \in \ell_1\} = \{a \in \ell_1 : (na_n) \in \ell_1\}.$$

(2) *The β-duals of $\mu(\Delta)$ are given by*

$$\{c_0(\Delta)\}^{\beta} = \{a \in \ell_1 : (nR_{n+1}) \in \ell_\infty \text{ and } R \in \ell_1\}$$
$$= \{a \in \ell_1 : (na_n) \in bs \text{ and } R \in \ell_1\}.$$

$$\{\eta(\Delta)\}^{\beta} = \{a \in \ell_1 : (nR_{n+1}) \in c_0 \text{ and } R \in \ell_1\}$$
$$= \{a \in \ell_1 : (na_n) \in cs \text{ and } R \in \ell_1\},$$

where $\eta(\Delta)$ stands for any of the spaces $c(\Delta)$ or $\ell_\infty(\Delta)$.

(3) *The γ-duals of $\mu(\Delta)$ are given by*

$$\{\mu(\Delta)\}^{\gamma} = \{a \in \ell_1 : (nR_{n+1}) \in \ell_\infty \text{ and } R \in \ell_1\}$$
$$= \{a \in \ell_1 : (na_n) \in bs \text{ and } R \in \ell_1\},$$

where $\bar{R} = (\bar{R}_n)$ and $R = (R_n)$ such that $\bar{R}_n = \sum_{j=n}^{\infty} |a_j|$ and $R_n = \sum_{j=n}^{\infty} a_j$ for all n.

Proof. Part (1) is immediate by combining (4.2.4) with Theorem 4.1.9 (see [32, Theorem 2.1] for the 2^{nd} formula of $\{\mu(\Delta)\}^{\alpha}$).

For part (2), the first two formulae of $\{c_0(\Delta)\}^{\beta}$ are obtained from (4.2.2) with help of Theorem 4.1.6 (see [33, Lemma 3] for the 1^{st} formula of $\{c_0(\Delta)\}^{\beta}$). Also, the second two formulae of $\{\eta(\Delta)\}^{\beta}$ are obtained from (4.2.3) and Theorem 4.1.8 (see [32, Theorem 2.1] for the 2^{nd} formula of $\{\eta(\Delta)\}^{\beta}$).

Lastly, part (3) is immediate by (4.2.1) with Theorem 4.1.6 (see [32, Theorem 2.1] for the 2^{nd} formula). □

Remark 4.2.2 Frrom Theorem 4.2.1, we note that:

(1) $\{c_0(\Delta)\}^{\beta} = \{c_0(\Delta)\}^{\gamma}$, $\{c(\Delta)\}^{\theta} = \{\ell_\infty(\Delta)\}^{\theta}$ for $\theta = \alpha$, β and γ, while $\{c_0(\Delta)\}^{\theta} = \{\eta(\Delta)\}^{\theta}$ for only $\theta = \alpha$ and γ (not β), where $\eta = c$ or ℓ_∞.

(2) The term "$a \in \ell_1$" can equivalently be replaced by "$a \in cs$" in the formulae of β- and γ-duals given in parts (2) and (3) of Theorem 4.2.1, and then it is understood that $a \in \ell_1$ (see Remark 4.1.10). But, in part (1) of Theorem 4.2.1, the term "$a \in \ell_1$" is necessary and must be mentioned in the formulae of α-dual of $\mu(\Delta)$.

Corollary 4.2.3 *We have the following facts:*

(1) *If $a \in \{\mu(\Delta)\}^{\alpha}$; then $(n\bar{R}_{n+1}) \in c_0$, $(\bar{R}_{n+1}\sigma_n(y)) \in c_0$ and $(\bar{R}_{n+1}\sigma_n(|y|)) \in c_0$ for all $y \in \mu$.*

(2) *If $a \in \{\mu(\Delta)\}^{\beta}$; then $(R_{n+1}\sigma_n(y)) \in c_0$ and $(R_{n+1}\sigma_n(|y|)) \in c_0$ for all $y \in \mu$.*

(3) *If $a \in \{\mu(\Delta)\}^{\gamma}$; then $(R_{n+1}\sigma_n(y)) \in \ell_\infty$ and $(R_{n+1}\sigma_n(|y|)) \in \ell_\infty$ for all $y \in \mu$.*

Proof. For (1), let $a \in \{\mu(\Delta)\}^{\alpha}$. Then $a \in \ell_1$ and $\bar{R} \in \ell_1$ (by Theorem 4.2.1) and so $(n\bar{R}_{n+1}) \in c_0$ (by Remark 4.1.4 or Theorem 4.1.9). Thus, for every $y \in \mu \subset \ell_\infty$, we have $|y_n| \leq \|y\|_\infty < \infty$ and so $|\sigma_n(y)| \leq \sigma_n(|y|) \leq n\|y\|_\infty$ for all n. This implies that

$$0 \le \bar{R}_{n+1}|\sigma_n(y)| \le \bar{R}_{n+1}\sigma_n(|y|) \le n\bar{R}_{n+1}\|y\|_\infty \to 0 \text{ as } n \to \infty$$

which proves (1).

For (2), let $a \in \{\mu(\Delta)\}^\beta$ and take any $y \in \mu$ which implies that $(\sigma_n(y)/n) \in \mu$. Then, one of the sequences (nR_{n+1}) or $(\sigma_n(y)/n)$ is bounded and the other tends to zero. Thus $(R_{n+1}\sigma_n(y)) = (nR_{n+1})(\sigma_n(y)/n) \in c_0$, that is $(R_{n+1}\sigma_n(y)) \in c_0$ for all $y \in \mu$ and hence $(R_{n+1}\sigma_n(|y|)) \in c_0$ for all $y \in \mu$ ($y \in \mu \Longrightarrow |y| \in \mu$). Finally, part (3) can be proved same as part (1) with $(nR_{n+1}) \in \ell_\infty$ instead of $(n\bar{R}_{n+1}) \in c_0$. □

Corollary 4.2.4 *We have the following:*

$$\{\mu(\Delta)\}^\beta = \{a \in \ell_1 : R \in \ell_1 \textit{ and } (x_n R_{n+1}) \in c_0 \textit{ for all } x \in \mu(\Delta)\}$$

$$= \{a \in \ell_1 : R \in \ell_1 \textit{ and } (R_{n+1}\sigma_n(|y|)) \in c_0 \textit{ for all } y \in \mu\},$$

$$\{\mu(\Delta)\}^\gamma = \{a \in \ell_1 : R \in \ell_1 \textit{ and } (x_n R_{n+1}) \in \ell_\infty \textit{ for all } x \in \mu(\Delta)\}$$

$$= \{a \in \ell_1 : R \in \ell_1 \textit{ and } (R_{n+1}\sigma_n(|y|)) \in \ell_\infty \textit{ for all } y \in \mu\}.$$

Proof. This result can be proved same as Corollary 4.2.3 (note that: all these formulae are equal to each others when $\mu = c_0$). □

4.3 Duality For $\mu(\Delta^\lambda)$

In the last section, we shall apply the results of previous sections to the new λ-difference spaces $\mu(\Delta^\lambda)$ in order to conclude their α-, β- and γ-duls.

As usual, the notations given by (4.1.1) and (4.1.7) will be used, where $a \in \ell_1$ (as $\{\mu(\Delta^\lambda)\}^\theta \subset \ell_1$). Besides, the sequence $v = (v_k)$ of non-negative real numbers was defined by (3.2.1) as follows:

$$v_k = \frac{\lambda_{k-1}}{\Delta(\lambda_k)} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}} \qquad (k \ge 1). \tag{4.3.1}$$

Further, every sequence $x = (x_k) \in w$ will be connected with another sequence $y = (y_k)$ by the relation $y = \tilde{\Lambda}(x)$, and we then say that y is the sequence connected with x by $y = \tilde{\Lambda}(x)$ which, together with (2.2.7), yields that

$$y_k = \tilde{\Lambda}_k(x) = \Lambda_k(x) - \Lambda_{k-1}(x) \quad \text{and} \quad \Lambda_k(x) = \sigma_k(y) \qquad (k \geq 1).$$

Thus, with help of (2.2.2) and (2.2.9), we have $x_k = \Delta(\lambda_k \sum_{j=1}^{k} y_j)/\Delta(\lambda_k)$ $(k \geq 1)$ which can equivalently be written as in (3.2.6) by using (1.1.2), that is

$$x_k = v_k y_k + \sum_{j=1}^{k} y_j \qquad (k \geq 1). \tag{4.3.2}$$

Now, from Theorem 2.2.7 and (2) of Remark 2.2.9, we deduce the following:

Lemma 4.3.1 *Two sequences x and y are connected by $y = \tilde{\Lambda}(x)$ if and only if* (4.3.2) *is satisfied. In such case: $x \in \mu(\Delta^\lambda)$ if and only if $y \in \mu$. Further, for every $x \in \mu(\Delta^\lambda)$ there exists a unique $y \in \mu$ connected with x by $y = \tilde{\Lambda}(x)$, and conversely for every $y \in \mu$ there exists a unique $x \in \mu(\Delta^\lambda)$ given by* (4.3.2) *and so $y = \tilde{\Lambda}(x)$.*

Here and in what follows, we shall assume that x and y are connected by $y = \tilde{\Lambda}(x)$ which implies the validity of (4.3.2) by which we find that

$$a_k x_k = a_k v_k y_k + a_k \sum_{j=1}^{k} y_j \qquad (k \geq 1)$$

and so we obtain (for every $n \geq 1$) that

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} (a_k v_k + R_k^n) y_k \quad \text{and} \quad \sum_{k=1}^{n} |a_k x_k| \leq \sum_{k=1}^{n} (|a_k v_k| + \bar{R}_k^n) |y_k|$$

which can be used to derive the following relations in which $n \geq 1$:

$$\sum_{k=1}^{n} |a_k x_k| \leq \sum_{k=1}^{n} (|a_k v_k| + \bar{R}_k) |y_k| - \bar{R}_{n+1} \sigma_n(|y|), \tag{4.3.3}$$

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} (a_k v_k + R_k) y_k - R_{n+1} \sigma_n(y), \tag{4.3.4}$$

$$\left| \sum_{k=1}^{n} a_k x_k \right| \leq \sum_{k=1}^{n} |(a_k v_k + R_k) y_k| + |R_{n+1} \sigma_n(y)|. \tag{4.3.5}$$

Furthermore, let $x = (x_k)$ be given by $x_k = k + v_k = \Delta(k\lambda_k)/\Delta(\lambda_k)$ for all k. Then, it follows by (3.2.4) that $y_k = \tilde{\Lambda}_k(x) = 1$ $(k \geq 1)$. Thus, by taking $x_k = k + v_k$ and $y_k = 1$ in (4.3.4) and (4.3.5) for all k, we get the following:

$$\sum_{k=1}^{n}(k + v_k)a_k = \sum_{k=1}^{n}(a_k v_k + R_k^n) = \sum_{k=1}^{n}(a_k v_k + R_k) - nR_{n+1}, \tag{4.3.6}$$

$$\left|\sum_{k=1}^{n}(k + v_k)a_k\right| = \left|\sum_{k=1}^{n}(a_k v_k + R_k^n)\right| \leq \sum\nolimits_{k=1}^{n}|a_k v_k + R_k| + |nR_{n+1}| \tag{4.3.7}$$

and on replacing a by $|a|$ in (4.3.6), we obtain that

$$\sum_{k=1}^{n}(k + v_k)|a_k| = \sum_{k=1}^{n}(|a_k v_k| + \bar{R}_k^n) = \sum_{k=1}^{n}(|a_k v_k| + \bar{R}_k) - n\bar{R}_{n+1}. \tag{4.3.8}$$

On other side, we have $c_0(\Delta^\lambda) \subset \mu(\Delta^\lambda)$ (by Lemma 3.1.1) and from (3.3.9) we have $c_0(\Delta) \subset \mu(\Delta^\lambda)$. Therefore, we deduce the following satisfied inclusions:

$$\{\mu(\Delta^\lambda)\}^\theta \subset \{c_0(\Delta)\}^\theta \subset \{c_0(\Delta)\}^\gamma \quad \text{and} \quad \{\mu(\Delta^\lambda)\}^\theta \subset \{c_0(\Delta^\lambda)\}^\theta \subset \{c_0(\Delta^\lambda)\}^\gamma. \tag{4.3.9}$$

Now, we prove the following result which shows that $a \in \{\mu(\Delta)\}^\theta$ and $av \in \ell_1$ are necessary conditions in order that $a \in \{\mu(\Delta^\lambda)\}^\theta$, where $\theta = \alpha$, β or γ, and $\mu = c_0$, c or ℓ_∞ (and it will be shown latter that these conditions are also sufficient).

Lemma 4.3.2 *We have the following facts:*

(1) *If $a \in \{\mu(\Delta^\lambda)\}^\theta$; then $av = (a_k v_k) \in \ell_1$.*

(2) *The inclusion $\{\mu(\Delta^\lambda)\}^\theta \subset \{\mu(\Delta)\}^\theta$ always holds.*

Proof. For (1), let $a \in \{\mu(\Delta^\lambda)\}^\theta$ be arbitrary. Then, it follows by (4.3.9) that $a \in \{c_0(\Delta)\}^\gamma$ as well as $a \in \{c_0(\Delta^\lambda)\}^\gamma$. Thus $a \in \ell_1$ and $R \in \ell_1$ (as $a \in \{c_0(\Delta)\}^\gamma$ by Theorem 4.2.1). Also, for any $y \in c_0$, let $x = (x_k)$ be given by (4.3.2). Then $x \in c_0(\Delta^\lambda)$ and since $a \in \{c_0(\Delta^\lambda)\}^\gamma$; we get $ax \in bs$ and so $(\sum_{k=1}^{n} a_k x_k) \in \ell_\infty$. Further, it follows by (3) of Corollary 4.2.3 that $(R_{n+1}\sigma_n(y)) \in \ell_\infty$ (since $y \in c_0$ and $a \in \{c_0(\Delta)\}^\gamma$). Therefore, we have shown that $(\sum_{k=1}^n a_k x_k) \in \ell_\infty$ as well as $(R_{n+1}\sigma_n(y)) \in \ell_\infty$ which together with (4.3.4) imply that $(\sum_{k=1}^n (a_k v_k + R_k) y_k) \in \ell_\infty$ and this means that $(av + R)y \in bs$ for all $y \in c_0$ (as $y \in c_0$ was arbitrary). Hence, we deduce that $av + R \in c_0^\gamma = \ell_1$ and so $av \in \ell_1$ (as $R \in \ell_1$).

Next, to prove (2), it is obvious that given inclusion is trivially satisfied when $\theta = \alpha$ or γ. To see that, we have $\{c_0(\Delta)\}^\theta = \{\mu(\Delta)\}^\theta$ when $\theta = \alpha$ or γ (by Theorem 4.2.1), but $\{\mu(\Delta^\lambda)\}^\theta \subset \{c_0(\Delta)\}^\theta$ by (4.3.9) and so $\{\mu(\Delta^\lambda)\}^\theta \subset \{\mu(\Delta)\}^\theta$ for $\theta = \alpha$ or γ. On other side, consider the case $\theta = \beta$. Then, it is clear by (4.3.9) that $\{c_0(\Delta^\lambda)\}^\beta \subset \{c_0(\Delta)\}^\beta$. Thus, it is remaining to show that $\{\eta(\Delta^\lambda)\}^\beta \subset \{\eta(\Delta)\}^\beta$, where $\eta = c$ or ℓ_∞. For this, take any $a \in \{\eta(\Delta^\lambda)\}^\beta$. Then $av \in \ell_1$ (by part (1)) and from (4.3.9) we find that $a \in \{c_0(\Delta)\}^\beta$ and so $a \in \ell_1$ as well as $R \in \ell_1$ by Theorem 4.2.1. Besides, we have $\tilde{\Lambda}(k + v_k) = e \in \eta$ (Lemma 3.2.5) which means that $(k + v_k) \in \eta(\Delta^\lambda)$ and since $a \in \{\eta(\Delta^\lambda)\}^\beta$; we obtain that $(k + v_k)a \in cs$, that is $(ka_k + a_k v_k) \in cs$ and hence $(ka_k) \in cs$ (as $av \in \ell_1 \subset cs$). Therefore, we have already shown that $a \in \ell_1$, $(ka_k) \in cs$ and $R \in \ell_1$ which leads us with Theorem 4.2.1 to deduce that $a \in \{\eta(\Delta)\}^\beta$. This means that $\{\eta(\Delta^\lambda)\}^\beta \subset \{\eta(\Delta)\}^\beta$ which completes the proof. □

Theorem 4.3.3 *For $\theta = \alpha$, β or γ, the θ-duals of the spaces $\mu(\Delta^\lambda)$ are given by*

$$\{\mu(\Delta^\lambda)\}^\theta = \{\mu(\Delta)\}^\theta \cap \{a \in w : av \in \ell_1\},$$

where $v = (\lambda_{k-1}/\Delta(\lambda_k))$ and μ stands for any one of the spaces c_0, c or ℓ_∞.

Proof. We will prove that $\{\mu(\Delta^\lambda)\}^\theta = D^\theta$, where $D^\theta = \{\mu(\Delta)\}^\theta \cap \{a \in w : av \in \ell_1\}$. For this, it is obvious by Lemma 4.3.2 that $\{\mu(\Delta^\lambda)\}^\theta \subset D^\theta$. Thus, we have to prove the converse inclusion. So, let $a \in D^\theta$ be arbitrary and let's show that $a \in \{\mu(\Delta^\lambda)\}^\theta$.

For, take any $x \in \mu(\Delta^\lambda)$ and let $y = (y_k)$ be the sequence connected by $y = \tilde{\Lambda}(x)$. Then $y \in \mu$ and since $a \in D^\theta$; we have $av \in \ell_1$ and $a \in \{\mu(\Delta)\}^\theta$. Therefore, we have three distinct cases which are $\theta = \alpha$, $\theta = \beta$ and $\theta = \gamma$.

In the first case ($\theta = \alpha$), we have $a \in \{\mu(\Delta)\}^\alpha$ and so $\bar{R} \in \ell_1$ (by Theorem 4.2.1), where $a \in \ell_1$. In such case, we have $y \in \mu$, $av \in \ell_1$ and $\bar{R} \in \ell_1$. Thus, we deduce that $\left(\sum_{k=1}^{n}(|a_k v_k| + \bar{R}_k)|y_k|\right) \in c$ and $(\bar{R}_{n+1}\sigma_n(|y|)) \in c_0$ by (1) of Corollary 4.2.3. Therefore, it follows by (4.3.3) that $\left(\sum_{k=1}^{n}|a_k x_k|\right) \in \ell_\infty$ and so $\left(\sum_{k=1}^{n}|a_k x_k|\right) \in c$ which means that $ax \in \ell_1$ for all $x \in \mu(\Delta^\lambda)$ and hence $a \in \{\mu(\Delta^\lambda)\}^\alpha$ which implies that $D^\alpha \subset \{\mu(\Delta^\lambda)\}^\alpha$.

Similarly, in the second case ($\theta = \beta$), we have $a \in \{\mu(\Delta)\}^\beta$ and so $R \in \ell_1$ (by Theorem 4.2.1) as well as $(R_{n+1}\sigma_n(y)) \in c_0$ by (2) of Corollary 4.2.3, where $a \in \ell_1$. Also, since $y \in \mu \subset \ell_\infty$, $av \in \ell_1$ and $R \in \ell_1$, we deduce that $\left(\sum_{k=1}^{n}(a_k v_k + R_k)y_k\right) \in c$ and it follows by (4.3.4) that $\left(\sum_{k=1}^{n}a_k x_k\right) \in c$ which means that $ax \in cs$ for all $x \in \mu(\Delta^\lambda)$ and so $a \in \{\mu(\Delta^\lambda)\}^\beta$ which implies that $D^\beta \subset \{\mu(\Delta^\lambda)\}^\beta$.

Finally, in the third case ($\theta = \gamma$), we have $y \in \mu \subset \ell_\infty$, $av \in \ell_1$ and $R \in \ell_1$ ($a \in \ell_1$). Thus, we deduce that $\left(\sum_{k=1}^{n}|(a_k v_k + R_k)y_k|\right) \in \ell_\infty$ and $(R_{n+1}\sigma_n(y)) \in \ell_\infty$. Hence, it follows by (4.3.5) that $\left(\sum_{k=1}^{n}a_k x_k\right) \in \ell_\infty$ which means that $ax \in bs$ for all $x \in \mu(\Delta^\lambda)$ and so $a \in \{\mu(\Delta^\lambda)\}^\gamma$ which implies that $D^\gamma \subset \{\mu(\Delta^\lambda)\}^\gamma$. Consequently $D^\theta \subset \{\mu(\Delta^\lambda)\}^\theta$ which yields the equality $\{\mu(\Delta^\lambda)\}^\theta = D^\theta$ and we have done. □

Furthermore, let $\rho = \beta$ or γ. Then, it follows by Theorem 4.3.3 that

$$a \in \{\mu(\Delta^\lambda)\}^\rho \Longleftrightarrow a \in \{\mu(\Delta)\}^\rho \text{ and } av \in \ell_1$$

which can equivalently be written as follows:

$$a \in \{\mu(\Delta^\lambda)\}^\rho \Longleftrightarrow a \in \{\mu(\Delta)\}^\rho \text{ and } av + R \in \ell_1 .$$

To see that, it is obvious that if $R \in \ell_1$; then: $av \in \ell_1 \Longleftrightarrow av + R \in \ell_1$. That is $av \in \ell_1 \Longleftrightarrow av + R \in \ell_1$ (provided that $R \in \ell_1$). Besides, we have $R \in \ell_1$ in both sides of above equivalence, since each of $a \in \{\mu(\Delta)\}^\rho$ or $a \in \{\mu(\Delta^\lambda)\}^\rho$ implies $R \in \ell_1$ by Theorem 4.2.1 and Lemma 4.3.2. This leads us to conclude the following:

Corollary 4.3.4 *For $\rho = \beta$ or γ, the ρ-duals of the spaces $\mu(\Delta^\lambda)$ are given by*

$$\left\{\mu(\Delta^\lambda)\right\}^\rho = \{\mu(\Delta)\}^\rho \cap \{a \in w : \, av + R(a) \in \ell_1\},$$

where $R(a)$ and v are respectively given by (4.1.1) *and* (4.3.1).

Theorem 4.3.5 *Let μ be any of the spaces c_0, c or ℓ_∞. Then, we have the following:*

(1) *The α-duals of $\mu(\Delta^\lambda)$ are given by*

$$\begin{aligned}\{\mu(\Delta^\lambda)\}^\alpha &= \{a \in \ell_1 : \, \bar{R} \in \ell_1 \text{ and } av \in \ell_1\} \\ &= \{a \in \ell_1 : (ka_k) \in \ell_1 \text{ and } av \in \ell_1\}.\end{aligned}$$

(2) *The β-duals of $\mu(\Delta^\lambda)$ are given by*

$$\begin{aligned}\{c_0(\Delta^\lambda)\}^\beta &= \{a \in \ell_1 : \, (kR_{k+1}) \in \ell_\infty, \, R \in \ell_1 \text{ and } av \in \ell_1\} \\ &= \{a \in \ell_1 : \, (ka_k) \in bs, \, R \in \ell_1 \text{ and } av \in \ell_1\}.\end{aligned}$$

$$\begin{aligned}\{\eta(\Delta^\lambda)\}^\beta &= \{a \in \ell_1 : \, (kR_{k+1}) \in c_0, \, R \in \ell_1 \text{ and } av \in \ell_1\} \\ &= \{a \in \ell_1 : \, (ka_k) \in cs, \, R \in \ell_1 \text{ and } av \in \ell_1\},\end{aligned}$$

where $\eta(\Delta^\lambda)$ stands for any of the spaces $c(\Delta^\lambda)$ or $\ell_\infty(\Delta^\lambda)$.

(3) *The γ-duals of $\mu(\Delta^\lambda)$ are given by*

$$\begin{aligned}\{\mu(\Delta^\lambda)\}^\gamma &= \{a \in \ell_1 : \, (kR_{k+1}) \in \ell_\infty, \, R \in \ell_1 \text{ and } av \in \ell_1\} \\ &= \{a \in \ell_1 : \, (ka_k) \in bs, \, R \in \ell_1 \text{ and } av \in \ell_1\},\end{aligned}$$

where $v = (\lambda_{k-1}/\Delta(\lambda_k))$, $\bar{R} = (\bar{R}_k)$ and $R = (R_k)$ such that $\bar{R}_k = \sum_{j=k}^{\infty} |a_j|$ and $R_k = \sum_{j=k}^{\infty} a_j$ for all $n \geq 1$.

Proof. It is immediate by combining Theorems 4.2.1 and 4.3.3 . □

Remark 4.3.6 From Theorem 4.3.5, we note the following:

(1) $\{c_0(\Delta^\lambda)\}^\beta = \{c_0(\Delta^\lambda)\}^\gamma$, $\{c(\Delta^\lambda)\}^\theta = \{\ell_\infty(\Delta^\lambda)\}^\theta$ for $\theta = \alpha$, β and γ, while $\{c_0(\Delta^\lambda)\}^\theta = \{\eta(\Delta^\lambda)\}^\theta$ for only $\theta = \alpha$ and γ, where $\eta = c$ or ℓ_∞.

(2) By Remarks 4.1.10 and 4.2.2, the term "$a \in \ell_1$" is necessary and must be mentioned in the formulae of the α-duals of $\mu(\Delta^\lambda)$ given in part (1) of Theorem 4.3.5, and so it is not redundant or superfluous. But in parts (2) and (3) of Theorem 4.3.5, the term "$a \in \ell_1$" can be replaced by "$a \in cs$" in the formulae of the β- and γ-duals of $\mu(\Delta^\lambda)$.

Moreover, by using Theorem 3.2.9 and Corollary 4.2.4, we deduce the following:

Corollary 4.3.7 *We have the following:*

$$\{\mu(\Delta^\lambda)\}^\alpha = \{a \in \ell_1 : (\,(k+v_k)a_k\,) \in \ell_1\} = \{(\,a_k/(k+v_k)\,) : a = (a_k) \in \ell_1\},$$

$$\{\mu(\Delta^\lambda)\}^\beta = \{a \in \ell_1 : R \in \ell_1,\ av \in \ell_1 \text{ and } (x_n R_{n+1}) \in c_0 \text{ for all } x \in \mu(\Delta)\}$$

$$= \{a \in \ell_1 : R \in \ell_1,\ av \in \ell_1 \text{ and } (R_{n+1}\sigma_n(|y|)) \in c_0 \text{ for all } y \in \mu\},$$

$$\{\mu(\Delta^\lambda)\}^\gamma = \{a \in \ell_1 : R \in \ell_1,\ av \in \ell_1 \text{ and } (x_n R_{n+1}) \in \ell_\infty \text{ for all } x \in \mu(\Delta)\}$$

$$= \{a \in \ell_1 : R \in \ell_1,\ av \in \ell_1 \text{ and } (R_{n+1}\sigma_n(|y|)) \in \ell_\infty \text{ for all } y \in \mu\}.$$

Corollary 4.3.8 *We have the following facts:*

(1) *If $a \in \{\mu(\Delta^\lambda)\}^\alpha$; then $(n\bar{R}_{n+1}) \in c_0$ and we have the following:*

$$\sum_{k=1}^{\infty}(k+v_k)|a_k| = \lim_{n\to\infty}\sum_{k=1}^{n}(|a_k v_k| + \bar{R}_k^n) = \sum_{k=1}^{\infty}(|a_k v_k| + \bar{R}_k),$$

$$\sum_{k=1}^{\infty}|a_k x_k| \le \lim_{n\to\infty}\sum_{k=1}^{n}(|a_k v_k| + \bar{R}_k^n)|y_k| = \sum_{k=1}^{\infty}(|a_k v_k| + \bar{R}_k)|y_k|,$$

where $x \in \mu(\Delta^\lambda)$ and $y = \tilde{\Lambda}(x)$.

(2) *If $a \in \{\mu(\Delta^\lambda)\}^\gamma$; then we have the following:*

$$\sup_n \sum_{k=1}^{n} |a_k v_k + R_k^n| \leq \sum_{k=1}^{\infty} |a_k v_k + R_k| + \sup_n |nR_{n+1}| < \infty,$$

$$\sup_n \left| \sum_{k=1}^{n} (k + v_k) a_k \right| \leq \sum_{k=1}^{\infty} |a_k v_k + R_k| + \sup_n |nR_{n+1}| < \infty,$$

$$\sup_n \left| \sum_{k=1}^{n} a_k x_k \right| \leq \sum_{k=1}^{\infty} |(a_k v_k + R_k) y_k| + \sup_n |R_{n+1} \sigma_n(y)| < \infty,$$

where $x \in \mu(\Delta^\lambda)$ and $y = \tilde{\Lambda}(x)$.

Proof. We have $\{\mu(\Delta^\lambda)\}^\theta \subset \{\mu(\Delta)\}^\theta$ by Lemma 4.3.2. Thus, by using Corollaries 4.2.3 and 4.3.7, we deduce this result, where part (1) is immediate by (4.3.3) and (4.3.8), and part (2) is obtained from (4.3.5) and (4.3.7). □

Corollary 4.3.9 *We have the following facts:*

(1) *If $a \in \{\mu(\Delta^\lambda)\}^\beta$; then for every $x \in \mu(\Delta^\lambda)$ with $y = \tilde{\Lambda}(x)$, we have*

$$\sum_{k=1}^{\infty} a_k x_k = \lim_{n \to \infty} \sum_{k=1}^{n} (a_k v_k + R_k^n) y_k = \sum_{k=1}^{\infty} (a_k v_k + R_k) y_k.$$

(2) *In particular, if $a \in \{\eta(\Delta^\lambda)\}^\beta$ ($\eta = c$ or ℓ_∞); then we have the additional equalities:*

$$\sum_{k=1}^{\infty} (k + v_k) a_k = \lim_{n \to \infty} \sum_{k=1}^{n} (a_k v_k + R_k^n) = \sum_{k=1}^{\infty} (a_k v_k + R_k),$$

$$\lim_{n \to \infty} \sum_{k=1}^{n} |a_k v_k + R_k^n| = \lim_{n \to \infty} \sum_{k=1}^{n} |a_k v_k + R_k| = \sum_{k=1}^{\infty} |a_k v_k + R_k|.$$

Proof. It is same as the proof of Corollary 4.3.8, part (1) is obtained from (4.3.4), and part (2) is immediate by (4.3.6) and noting that

$$\big| |a_k v_k + R_k^n| - |a_k v_k + R_k| \big| \leq |R_k^n - R_k| = |R_{n+1}| \quad (1 \leq k \leq n).$$ □

Remark 4.3.10 It must be noted that Theorem 4.3.5 is reduced to Theorem 4.2.1 with $v = 0$. That is, the θ-duals of $\mu(\Delta)$, as in Theorem 4.2.1, can be obtained from

θ-duals of $\mu(\Delta^\lambda)$, as in Theorem 4.3.5, with assuming that $v_k = 0$ for all k (Corollary 4.3.7 is also reduced to Corollary 4.2.4). So, similar results of those in Corollaries 4.3.8 and 4.3.9 can be obtained for $\mu(\Delta)$ instead of $\mu(\Delta^\lambda)$ by taking $v_k = 0$ with $y_k = \Delta(x_k)$ instead of $y_k = \tilde{\Lambda}_k(x)$ for all k, where $x \in \mu(\Delta)$ in place of $x \in \mu(\Delta^\lambda)$. For instance, if $a \in \{\mu(\Delta)\}^\beta$; then for every $x \in \mu(\Delta)$, the relations given in Corollary 4.3.9 are satisfied with $v_k = 0$ and $y_k = \Delta(x_k)$ for all k, and the same for $a \in \{\eta(\Delta)\}^\beta$.

Corollary 4.3.11 *If $v \in \ell_\infty$; then $\{\mu(\Delta^\lambda)\}^\theta = \{\mu(\Delta)\}^\theta$ for $\theta = \alpha$, β and γ.*

Proof. It is enough to show that if $v \in \ell_\infty$; then $\{\mu(\Delta)\}^\theta \subset \{a \in w : av \in \ell_1\}$ and so $\{\mu(\Delta^\lambda)\}^\theta = \{\mu(\Delta)\}^\theta$ by Theorem 4.3.3 (in such case, the condition $av \in \ell_1$ is redundant and so $\{\mu(\Delta^\lambda)\}^\theta$ is reduced to $\{\mu(\Delta)\}^\theta$). For this, suppose that $v \in \ell_\infty$. Then, for every $a \in \{\mu(\Delta)\}^\theta$, we have $a \in \ell_1$ (by Theorem 4.2.1) which implies $av \in \ell_1$ (as $v \in \ell_\infty$). Thus, we deduce that $\{\mu(\Delta)\}^\theta \subset \{a \in w : av \in \ell_1\}$ when $v \in \ell_\infty$. □

Finally, at the end of this chapter, we may observe the following:

(1) Obviously, the inclusions $c_0(\Delta) \subset c_0(\Delta^\lambda)$ and $\ell_\infty(\Delta) \subset \ell_\infty(\Delta^\lambda)$ imply both of inclusions $\{c_0(\Delta^\lambda)\}^\theta \subset \{c_0(\Delta)\}^\theta$ and $\{\ell_\infty(\Delta^\lambda)\}^\theta \subset \{\ell_\infty(\Delta)\}^\theta$. This is compatible with Lemma 4.3.2, but Lemma 4.3.2 tells us also that $\{c(\Delta^\lambda)\}^\theta \subset \{c(\Delta)\}^\theta$ while the inclusion $c(\Delta) \subset c(\Delta^\lambda)$ need not be held. The justification can be understood in light of the equalities $\{c(\Delta)\}^\theta = \{\ell_\infty(\Delta)\}^\theta$ and $\{c(\Delta^\lambda)\}^\theta = \{\ell_\infty(\Delta^\lambda)\}^\theta$. To see that, we have $\{c(\Delta^\lambda)\}^\theta = \{\ell_\infty(\Delta^\lambda)\}^\theta \subset \{\ell_\infty(\Delta)\}^\theta = \{c(\Delta)\}^\theta$.

(2) If $v \in \ell_\infty$; then $c_0(\Delta^\lambda) = c_0(\Delta)$ and $\ell_\infty(\Delta^\lambda) = \ell_\infty(\Delta)$ (Theorem 3.3.10). This implies that $\{c_0(\Delta^\lambda)\}^\theta = \{c_0(\Delta)\}^\theta$ and $\{\ell_\infty(\Delta^\lambda)\}^\theta = \{\ell_\infty(\Delta)\}^\theta$ which is compatible with Corollary 4.3.11, but Corollary 4.3.11 tells us also that $\{c(\Delta^\lambda)\}^\theta = \{c(\Delta)\}^\theta$ while the equality $c(\Delta^\lambda) = c(\Delta)$ need not be satisfied. Again, this can similarly be justified.

Chapter 5

CERTAIN MATRIX OPERATORS

5 CERTAIN MATRIX OPERATORS

The last chapter is devoted to characterize some new matrix classes and matrix operators related to our λ-difference spaces $\mu(\Delta^\lambda)$ of bounded, convergent and null difference sequences, where μ stands for any of the spaces c_0, c or ℓ_∞. That is, the necessary and sufficient conditions for an infinite matrix A to act on, into and between our spaces will be deduced. This chapter is divided into three sections, the first is to characterize matrix operators on our spaces, the second is for matrix operators into these spaces and the last is for matrix operators between them. The materials of this chapter are part of our research paper* which has been published in the Ijrdo J. Math. 2022.

5.1 Matrix Operators On $\mu(\Delta^\lambda)$

In this section, we obtain the necessary and sufficient conditions for an infinite matrix A to act on the λ-difference spaces $\mu(\Delta^\lambda)$.

Every infinite matrix $A = [a_{nk}]$ will be associated with another infinite matrix $\tilde{A} = [\tilde{a}_{nk}]$, called as the *matrix associated with* A, which can be defined in terms of A as follows:

$$\tilde{a}_{nk} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}} a_{nk} + \sum_{j=k}^{\infty} a_{nj} \qquad (n, k \geq 1), \tag{5.1.1}$$

where $A_n \in \ell_1$ for all $n \geq 1$. That is, the associated matrix $\tilde{A} = [\tilde{a}_{nk}]$ can be defined with help of our conventions given by (4.1.1) and (4.3.1) as follows:

$$\tilde{a}_{nk} = v_k a_{nk} + R_k(A_n) = v_k a_{nk} + R_{nk} \qquad (n, k \geq 1),$$

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where $v = (v_k) = (\lambda_{k-1}/\Delta(\lambda_k))$, $A_n = (a_{nk})_{k=1}^{\infty}$ is the n-th row sequence in A $(n \geq 1)$ and $[R_{nk}]$ is an infinite matrix defined via A by $R_{nk} = R_k(A_n)$ for all $n, k \geq 1$, that is

$$v_k = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}} \quad \text{and} \quad R_{nk} = \sum_{j=k}^{\infty} a_{nj} \qquad (n, k \geq 1).$$

Further, we assume the sequences $x, y \in w$ are connected by the relation $y = \tilde{\Lambda}(x)$. Thus $x \in \mu(\Delta^{\lambda}) \iff y \in \mu$ (see Lemma 4.3.1). So, by using (4.3.2) with the same technique by which the relation (4.3.4) has been derived, we obtain that

$$\sum_{k=1}^{m} a_{nk} x_k = \sum_{k=1}^{m} \tilde{a}_{nk} y_k - R_{n,\, m+1}\, \sigma_m(y) \qquad (n, m \geq 1).$$

Moreover, if $A_n \in \{\mu(\Delta^{\lambda})\}^{\beta}$ for every $n \geq 1$; then it follows, by (2) of Theorem 4.3.5, that $A_n \in \ell_1$, $vA_n \in \ell_1$ and $R(A_n) = (R_{nk})_{k=1}^{\infty} \in \ell_1$ for all n. Also, we must have $\tilde{A}_n \in \ell_1$ and $\lim_{m\to\infty} R_{n,\, m+1} \sigma_m(y) = 0$ for all n and every $y \in \mu$ (see Corollaries 4.2.3 and 4.3.4), where $\tilde{A}_n = (\tilde{a}_{nk})_{k=1}^{\infty}$ is the n-th row sequence in the associated matrix $\tilde{A}$ for each $n \geq 1$, that is

$$\tilde{A}_n = vA_n + R(A_n) = \big(a_{nk} v_k + R_{nk}\big)_{k=1}^{\infty} \qquad (n \geq 1).$$

Thus, by going to the limits in both sides of above equality as $m \to \infty$, we get the following (see (1) of Corollary 4.3.9):

$$\sum_{k=1}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} \tilde{a}_{nk} y_k \qquad (n \geq 1) \tag{5.1.2}$$

which means that $A_n(x) = \tilde{A}_n(y)$ for all n, and so $A(x) = \tilde{A}(y)$ for all $x \in \mu(\Delta^{\lambda})$ and $y \in \mu$ which are connected by $y = \tilde{\Lambda}(x)$. This also means that $A(x) \in X$ for every $x \in \mu(\Delta^{\lambda})$ if and only if $\tilde{A}(y) \in X$ for every $y \in \mu$, where X is any sequence space. Thus, we immediately deduce the following useful results which will be used to characterize matrix operators on the λ-difference spaces.

Lemma 5.1.1 *For any infinite matrix A, let $\tilde{A}$ be its associated matrix defined by (5.1.1). Then, for each $n \geq 1$, we have $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ if and only if $A_n \in \{\mu(\Delta)\}^\beta$ and $\tilde{A}_n \in \mu^\beta$, where $\mu^\beta = \ell_1$ ($\mu = c_0$, c or ℓ_∞). Further, if $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$; then $A(x) = \tilde{A}(y)$ for all $x \in \mu(\Delta^\lambda)$ and $y \in \mu$ which are connected by $y = \tilde{\Lambda}(x)$.*

Proof. Let $n \geq 1$. Then, by using Theorem 4.3.5 and Corollary 4.3.4 with A_n instead of a, we deduce the following:

$$A_n \in \{\mu(\Delta^\lambda)\}^\beta \Longleftrightarrow A_n \in \{\mu(\Delta)\}^\beta \text{ and } vA_n + R(A_n) \in \ell_1$$

$$\Longleftrightarrow A_n \in \{\mu(\Delta)\}^\beta \text{ and } \tilde{A}_n \in \mu^\beta,$$

where $\mu^\beta = \ell_1$ (as $\mu = c_0$, c or ℓ_∞) and $\tilde{A}_n = vA_n + R(A_n)$ for all n. Further, if $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$; then it follows by (5.1.2) that $A(x) = \tilde{A}(y)$ for all $x \in \mu(\Delta^\lambda)$ and $y \in \mu$ which are connected by $y = \tilde{\Lambda}(x)$. This ends the proof. □

Theorem 5.1.2 *For any sequence space X and every infinite matrix A, the following statements are equivalent to each others:*

(1) $A \in (\mu(\Delta^\lambda), X)$.

(2) $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ *for every* $n \geq 1$ *and* $\tilde{A}(y) \in X$ *for all* $y \in \mu$.

(3) $A_n \in \{\mu(\Delta)\}^\beta$ *for every* $n \geq 1$ *and* $\tilde{A} \in (\mu, X)$.

Proof. Suppose that (1) is satisfied, that is $A \in (\mu(\Delta^\lambda), X)$. Then $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$ and $A(x) \in X$ for all $x \in \mu(\Delta^\lambda)$ (Lemma 1.3.6). Thus, for every $y \in \mu$, let $x = (x_k)$ be given by (4.3.2). Then $x \in \mu(\Delta^\lambda)$ such that $y = \tilde{\Lambda}(x)$ and so $\tilde{A}(y) \in X$ (as $A(x) = \tilde{A}(y)$ by Lemma 5.1.1) and since $y \in \mu$ was arbitrary; we find that $\tilde{A}(y) \in X$ for all $y \in \mu$. Hence, we have $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$ and $\tilde{A}(y) \in X$ for all $y \in \mu$ which is (2), that is (1) $\Longrightarrow$ (2).

Further, assume that (2) is satisfied, that is $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$ and $\tilde{A}(y) \in X$ for all $y \in \mu$. This, together with Lemma 5.1.1, implies that $A_n \in \{\mu(\Delta)\}^\beta$ and $\tilde{A}_n \in \mu^\beta$ for every $n \geq 1$ as well as $\tilde{A}(y) \in X$ for all $y \in \mu$. Hence, we deduce that $A_n \in \{\mu(\Delta)\}^\beta$ for every $n \geq 1$ and $\tilde{A} \in (\mu, X)$ which is (3), that is (2) $\Longrightarrow$ (3).

Finally, suppose that (3) is satisfied, that is $A_n \in \{\mu(\Delta)\}^\beta$ for every $n \geq 1$ and $\tilde{A} \in (\mu, X)$. This means that $A_n \in \{\mu(\Delta)\}^\beta$ and $\tilde{A}_n \in \mu^\beta$ for every $n \geq 1$ as well as $\tilde{A}(y) \in X$ for all $y \in \mu$. Hence, it follows by Lemma 5.1.1 that $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$. Besides, for every $x \in \mu(\Delta^\lambda)$, let $y = \tilde{\Lambda}(x)$. Then $y \in \mu$ and $A(x) = \tilde{A}(y)$ by (5.1.2) which implies that $A(x) \in X$ for all $x \in \mu(\Delta^\lambda)$. Therefore, we have $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$ and $A(x) \in X$ for all $x \in \mu(\Delta^\lambda)$. This means that $A \in (\mu(\Delta^\lambda), X)$ which is (1), that is (3) $\Longrightarrow$ (1). This completes the proof. □

Now, by using (5.1.1), let's consider the following conditions (note that: the first five conditions are obtained from Lemma 5.1.1 while the last eight conditions are obtained from Lemmas 1.3.8, 1.3.9, 1.3.10 and 1.3.11):

$$\sum_{k=1}^{\infty} |a_{nk}| \text{ converges for every } n \geq 1 \tag{5.1.3}$$

$$\left(k \sum_{j=k+1}^{\infty} a_{nj} \right)_{k=1}^{\infty} \in \ell_\infty \text{ for every } n \geq 1 \tag{5.1.4}$$

$$\left(k \sum_{j=k+1}^{\infty} a_{nj} \right)_{k=1}^{\infty} \in c_0 \text{ for every } n \geq 1 \tag{5.1.5}$$

$$\sum_{k=1}^{\infty} \left| \sum_{j=k}^{\infty} a_{nj} \right| \text{ converges for every } n \geq 1 \tag{5.1.6}$$

$$\sum_{k=1}^{\infty} |\tilde{a}_{nk}| \text{ converges for every } n \geq 1 \tag{5.1.7}$$

$$\sup_n \sum_{k=1}^{\infty} |\tilde{a}_{nk}| < \infty \tag{5.1.8}$$

$$\lim_{n\to\infty} \tilde{a}_{nk} = \tilde{a}_k \text{ exists for every } k \geq 1 \tag{5.1.9}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \tilde{a}_{nk} = \tilde{a} \text{ exists} \tag{5.1.10}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |\tilde{a}_{nk} - \tilde{a}_k| = 0 \tag{5.1.11}$$

$$\lim_{n\to\infty} \tilde{a}_{nk} = 0 \text{ for every } k \geq 1 \tag{5.1.12}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \tilde{a}_{nk} = 0 \tag{5.1.13}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |\tilde{a}_{nk}| = 0 \tag{5.1.14}$$

$$\sup_{K\in\mathcal{K}} \sum_{n=1}^{\infty} \left| \sum_{k\in K} \tilde{a}_{nk} \right|^p < \infty \text{ for } p \geq 1, \tag{5.1.15}$$

where $\mathcal{K}$ stands for the collection of all non-empty finite subsets of positive integers. Then, by using Theorems 4.2.1, 4.3.5 and Lemma 5.1.1, we find that

$$A_n \in \{c_0(\Delta)\}^\beta \text{ for every } n \geq 1 \Longleftrightarrow (5.1.3), (5.1.4) \text{ and } (5.1.6) \text{ are satisfied,}$$

$$A_n \in \{\eta(\Delta)\}^\beta \text{ for every } n \geq 1 \Longleftrightarrow (5.1.3), (5.1.5) \text{ and } (5.1.6) \text{ are satisfied,}$$

$$A_n \in \{c_0(\Delta^\lambda)\}^\beta \text{ for every } n \geq 1 \Longleftrightarrow (5.1.3), (5.1.4), (5.1.6) \text{ and } (5.1.7) \text{ are satisfied,}$$

$$A_n \in \{\eta(\Delta^\lambda)\}^\beta \text{ for every } n \geq 1 \Longleftrightarrow (5.1.3), (5.1.5), (5.1.6) \text{ and } (5.1.7) \text{ are satisfied,}$$

where η stands for any of the spaces c or ℓ_∞ (also, it is obvious by Corollary 4.2.3 that conditions (5.1.4) and (5.1.5), in above equivalences, can be replaced by only one condition, viz: $\lim_{k\to\infty} \sigma_k(y) \sum_{j=k+1}^{\infty} a_{nj} = 0$ for all $y \in \mu$ and every $n \geq 1$). Therefore, by using Theorem 5.1.2 with help of Lemmas 1.3.8, 1.3.9, 1.3.10 and 1.3.11 characterizing matrix operators between the classical sequence spaces, we can immediately deduce the following new consequences characterizing matrix operators on the spaces $\mu(\Delta^\lambda)$ of λ-difference sequences.

Corollary 5.1.3 *For an infinite matrix A, we have the following:*

(1) $A \in (c_0(\Delta^\lambda), \ell_\infty)$ *if and only if* (5.1.3), (5.1.4), (5.1.6) *and* (5.1.8) *are satisfied.*

(2) $A \in (\eta(\Delta^\lambda), \ell_\infty)$ *if and only if* (5.1.3), (5.1.5), (5.1.6) *and* (5.1.8) *are satisfied.*

Proof. This result follows from Theorem 5.1.2 and Lemma 1.3.9, since $\tilde{A} \in (\mu, \ell_\infty) \Longleftrightarrow$ (5.1.8) holds (note that: $(c(\Delta^\lambda), \ell_\infty) = (\ell_\infty(\Delta^\lambda), \ell_\infty)$ by part (2), as $\eta = c$ or ℓ_∞). □

Corollary 5.1.4 *For an infinite matrix A, we have the following:*

(1) $A \in (c_0(\Delta^\lambda), c)$ *if and only if* (5.1.3), (5.1.4), (5.1.6), (5.1.8) *and* (5.1.9) *are satisfied. Further, if* $A \in (c_0(\Delta^\lambda), c)$*; then* $\lim_{n\to\infty} A_n(x) = \sum_{k=1}^{\infty} \tilde{a}_k y_k$ *for every* $x \in \mu(\Delta^\lambda)$*, where* $y_k = \tilde{\Lambda}_k(x)$ *and* $\tilde{a}_k = \lim_{n\to\infty} \tilde{a}_{nk}$ *for all* k*.*

(2) $A \in (c(\Delta^\lambda), c)$ *if and only if* (5.1.3), (5.1.5), (5.1.6), (5.1.8), (5.1.9) *and* (5.1.10) *are satisfied. Further, if* $A \in (c(\Delta^\lambda), c)$*; then* $\lim_{n\to\infty} A_n(x) = L(\tilde{a} - \sum_{k=1}^{\infty} \tilde{a}_k) + \sum_{k=1}^{\infty} \tilde{a}_k y_k$ *for every* $x \in \mu(\Delta^\lambda)$*, where* $y = \tilde{\Lambda}(x)$*,* $L = \lim_{k\to\infty} \tilde{\Lambda}_k(x)$ *and* $\tilde{a} = \lim_{n\to\infty} \sum_{k=1}^{\infty} \tilde{a}_{nk}$*.*

(3) $A \in (\ell_\infty(\Delta^\lambda), c)$ *if and only if* (5.1.3), (5.1.5), (5.1.6), (5.1.8), (5.1.9) *and* (5.1.11) *are satisfied. Further, if* $A \in (\ell_\infty(\Delta^\lambda), c)$*; then* $\lim_{n\to\infty} A_n(x) = \sum_{k=1}^{\infty} \tilde{a}_k y_k$ *for every* $x \in \mu(\Delta^\lambda)$*, where* $y = \tilde{\Lambda}(x)$*.*

Proof. It is immediate by Theorem 5.1.2 and Lemma 1.3.10 with noting that: (1) $\tilde{A} \in (c_0, c) \Longleftrightarrow$ (5.1.8) and (5.1.9) are satisfied. (2) $\tilde{A} \in (c, c) \Longleftrightarrow$ (5.1.8), (5.1.9) and (5.1.10) hold. (3) $\tilde{A} \in (\ell_\infty, c) \Longleftrightarrow$ (5.1.8), (5.1.9) and (5.1.11) are satisfied. □

Corollary 5.1.5 *For an infinite matrix A, we have the following:*

(1) $A \in (c_0(\Delta^\lambda), c_0)$ *if and only if* (5.1.3), (5.1.4), (5.1.6), (5.1.8) *and* (5.1.12) *are satisfied.*

(2) $A \in (c(\Delta^\lambda), c_0)$ *if and only if* (5.1.3), (5.1.5), (5.1.6), (5.1.8), (5.1.12) *and* (5.1.13) *are satisfied.*

(3) $A \in (\ell_\infty(\Delta^\lambda), c_0)$ *if and only if* (5.1.3), (5.1.5), (5.1.6) *and* (5.1.14) *are satisfied.*

Proof. This follows by Theorem 5.1.2 and Lemma 1.3.11 with observing that: (1) $\tilde{A} \in (c_0, c_0) \iff$ (5.1.8) and (5.1.12) satisfied. (2) $\tilde{A} \in (c, c_0) \iff$ (5.1.8), (5.1.12) and (5.1.13) hold. (3) $\tilde{A} \in (\ell_\infty, c_0) \iff$ (5.1.14) holds. □

Corollary 5.1.6 *Let A be an infinite matrix. Then, for every $p \geq 1$, we have:*

(1) $A \in (c_0(\Delta^\lambda), \ell_p)$ *if and only if* (5.1.3), (5.1.4), (5.1.6), (5.1.7) *and* (5.1.15) *are satisfied.*

(2) $A \in (\eta(\Delta^\lambda), \ell_p)$ *if and only if* (5.1.3), (5.1.5), (5.1.6), (5.1.7) *and* (5.1.15) *are satisfied.*

Proof. This follows from Theorem 5.1.2 and Lemma 1.3.8 with noting that: $\tilde{A} \in (\mu, \ell_p) \iff$ (5.1.15) holds (note that: $(c(\Delta^\lambda), \ell_p) = (\ell_\infty(\Delta^\lambda), \ell_p)$ for $p \geq 1$ by part (2), as $\eta = c$ or ℓ_∞). □

Further, in the light of Remark 4.3.10, it must be noted that Corollaries 5.1.3, 5.1.4, 5.1.5 and 5.1.6 can be reduced, with assumption $v = 0$, to characterize matrix operators on the usual difference spaces $\mu(\Delta)$ as follows:

Corollary 5.1.7 *The necessary and sufficient conditions for an infinite matrix A in order to belong to any of the classes $(\mu(\Delta), \ell_\infty)$, $(\mu(\Delta), c)$, $(\mu(\Delta), c_0)$ or $(\mu(\Delta), \ell_p)$ are those conditions given respectively in Corollary 5.1.3, 5.1.4, 5.1.5 or 5.1.6 by removing condition (5.1.6) and taking $\tilde{a}_{nk} = R_{nk} = \sum_{j=k}^{\infty} a_{nj}$ for all $n, k \geq 1$, where $p \geq 1$.*

Finally, it is obvious that Corollary 5.1.7 has various consequences concerning with the particular cases of the space μ, where μ is any of the spaces c_0, c or ℓ_∞ (also, in Corollary 5.1.7, the condition (5.1.6) is redundant, as it is reduced to (5.1.7) and implied by (5.1.8)).

5.2 Matrix Operators Into $\mu(\Delta^{\lambda})$

In this section, we conclude the necessary and sufficient conditions for an infinite matrix A to act from any sequence space into the λ-difference spaces $\mu(\Delta^{\lambda})$.

For this, we will apply the useful result in part (3) of Lemma 1.3.7 to the new spaces $\mu(\Delta^{\lambda})$. This leads us to the following theorem:

Theorem 5.2.1 *Let X be a sequence space and for any infinite matrix $A = [a_{nk}]$ define the matrix $B = [b_{nk}]$ by*

$$b_{nk} = \frac{\Delta(\lambda_n)}{\lambda_n}\, a_{nk} + \Delta\Big(\frac{1}{\lambda_n}\Big)\sum_{j=0}^{n-1}\Delta(\lambda_j)\, a_{jk} \qquad (n, k \geq 1).$$

Then $A \in (X, \mu(\Delta^{\lambda}))$ if and only if $B \in (X, \mu)$, where μ stands for any of the spaces c_0, c or ℓ_∞, and $\mu(\Delta^{\lambda})$ is the respective one of the spaces $c_0(\Delta^{\lambda})$, $c(\Delta^{\lambda})$ or $\ell_\infty(\Delta^{\lambda})$.

Proof. This result is immediate by (3) of Lemma 1.3.7 with using (2.2.6), where $B = \tilde{\Lambda}\, A$. □

In the particular case of Theorem 5.2.1, if X is any of the classical sequence spaces; then we obtain the following corollary:

Corollary 5.2.2 *Let A be an infinite matrix and define the matrix $B = [b_{nk}]$ by*

$$b_{nk} = \frac{\Delta(\lambda_n)}{\lambda_n}\, a_{nk} + \Delta\Big(\frac{1}{\lambda_n}\Big)\sum_{j=0}^{n-1}\Delta(\lambda_j)\, a_{jk} \qquad (n, k \geq 1).$$

Then A belongs to any one of the classes $(c_0, \mu(\Delta^{\lambda}))$, $(c, \mu(\Delta^{\lambda}))$, $(\ell_\infty, \mu(\Delta^{\lambda}))$ or $(\ell_p, \mu(\Delta^{\lambda}))$ if and only if B belongs to the respective one of the classes (c_0, μ), (c, μ), (ℓ_∞, μ) or (ℓ_p, μ), where $p \geq 1$ and μ stands for any of the spaces c_0, c or ℓ_∞.

More precisely, by using the conditions given in Lemmas 1.3.9, 1.3.10, 1.3.11 and 1.3.12 which characterize the matrix classes (c_0, μ), (c, μ), (ℓ_∞, μ) and (ℓ_p, μ), where $1 \leq p < \infty$, we obtain the conditions:

$$\sup_n \sum_{k=1}^{\infty} |b_{nk}| < \infty \tag{5.2.1}$$

$$\lim_{n\to\infty} b_{nk} = b_k \text{ exists for every } k \geq 1 \tag{5.2.2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |b_{nk} - b_k| = 0 \tag{5.2.3}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{nk} = b \text{ exists} \tag{5.2.4}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |b_{nk}| = 0 \tag{5.2.5}$$

$$\lim_{n\to\infty} b_{nk} = 0 \text{ for every } k \geq 1 \tag{5.2.6}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{nk} = 0 \tag{5.2.7}$$

$$\sup_{n,k} |b_{nk}| < \infty \tag{5.2.8}$$

$$\sup_n \sum_{k=1}^{\infty} |b_{nk}|^q < \infty \qquad (\, q = p/(p-1)\,). \tag{5.2.9}$$

Now, with help of Lemmas 1.3.9, 1.3.10, 1.3.11 and 1.3.12, we immediately deduce the following corollaries in which $B = [b_{nk}]$ is the triangle defined in Corollary 5.2.2.

Corollary 5.2.3 *We have* $(c_0, \ell_\infty(\Delta^\lambda)) = (c, \ell_\infty(\Delta^\lambda)) = (\ell_\infty, \ell_\infty(\Delta^\lambda))$, *and* $A \in (\mu, \ell_\infty(\Delta^\lambda))$ *if and only if* (5.2.1) *holds.*

Proof. This result is immediate by Lemma 1.3.9 and Corollary 5.2.2. □

Corollary 5.2.4 *We have the following:*

(1) $A \in (c_0, c(\Delta^\lambda))$ *if and only if* (5.2.1) *and* (5.2.2) *hold. Further, if* $A \in (c_0, c(\Delta^\lambda))$*; then* $\lim_{n\to\infty} \tilde{\Lambda}_n(A(x)) = \sum_{k=1}^{\infty} b_k x_k$ *for all* $x \in c_0$.

(2) $A \in (c, c(\Delta^\lambda))$ *if and only if* (5.2.1), (5.2.2) *and* (5.2.4). *Furthermore, if* $A \in (c, c(\Delta^\lambda))$*; then* $\lim_{n\to\infty} \tilde{\Lambda}_n(A(x)) = L\big(b - \sum_{k=1}^{\infty} b_k\big) + \sum_{k=1}^{\infty} b_k x_k$ *for all* $x \in c$, *where* $L = \lim_{k\to\infty} x_k$.

(3) $A \in (\ell_\infty, c(\Delta^\lambda))$ *if and only if* (5.2.1), (5.2.2) *and* (5.2.3) *hold. Furthermore, if* $A \in (\ell_\infty, c(\Delta^\lambda))$*; then* $\lim_{n\to\infty} \tilde{\Lambda}_n(A(x)) = \sum_{k=1}^{\infty} b_k x_k$ *for all* $x \in \ell_\infty$.

Proof. It follows from Lemma 1.3.10 and Corollary 5.2.2. □

Corollary 5.2.5 *We have the following:*

(1) $A \in (\ell_\infty, c_0(\Delta^\lambda))$ *if and only if* (5.2.5) *holds.*

(2) $A \in (c, c_0(\Delta^\lambda))$ *if and only if* (5.2.1), (5.2.6) *and* (5.2.7) *hold.*

(3) $A \in (c_0, c_0(\Delta^\lambda))$ *if and only if* (5.2.1) *and* (5.2.6) *hold.*

Proof. This result is obtained by Lemma 1.3.11 and Corollary 5.2.2. □

Corollary 5.2.6 *We have the following:*

(1) $A \in (\ell_1, \ell_\infty(\Delta^\lambda))$ *if and only if* (5.2.8) *holds.*

(2) $A \in (\ell_1, c(\Delta^\lambda))$ *if and only if* (5.2.2) *and* (5.2.8) *hold.*

(3) $A \in (\ell_1, c_0(\Delta^\lambda))$ *if and only if* (5.2.6) *and* (5.2.8) *hold.*

Proof. This result is immediate by Lemma 1.3.12 and Corollary 5.2.2. □

Corollary 5.2.7 *Let* $1 < p < \infty$ *and* $q = p/(p-1)$*. Then, we have the following:*

(1) $A \in (\ell_p, \ell_\infty(\Delta^\lambda))$ *if and only if* (5.2.9) *holds.*

(2) $A \in (\ell_p, c(\Delta^\lambda))$ *if and only if* (5.2.2) *and* (5.2.9) *hold.*

(3) $A \in (\ell_p, c_0(\Delta^\lambda))$ *if and only if* (5.2.6) *and* (5.2.9) *hold.*

Proof. It is immediate by Lemma 1.3.13 and Corollary 5.2.2. □

5.3 Matrix Operators Between $\mu(\Delta^\lambda)$

In this final section, we apply our main results to some particular cases. Also, we conclude the necessary and sufficient conditions for an infinite matrix A to act between the new λ-difference spaces.

First, we will apply (3) of Lemma 1.3.7 to the main results in previous section in order to characterize the matrix operators acting from $\mu(\Delta^\lambda)$ into the matrix domains of triangles, and then we will obtain the characterizations of matrix operators between two λ-difference spaces as particular cases.

At the beginning, by using (3) of Lemma 1.3.7, we conclude the following theorem:

Theorem 5.3.1 *Let X be a sequence space, $T = [t_{nk}]$ a triangle and for any infinite matrix $A = [a_{nk}]$ define the matrix $B = [b_{nk}]$ by*

$$b_{nk} = \sum\nolimits_{j=1}^{n} t_{nj}\, a_{jk} \qquad (n, k \geq 1).$$

Then $A \in (\mu(\Delta^\lambda), X_T)$ if and only if $B \in (\mu(\Delta^\lambda), X)$, where μ stands for any of the spaces c_0, c or ℓ_∞, and $\mu(\Delta^\lambda)$ is $c_0(\Delta^\lambda)$, $c(\Delta^\lambda)$ or $\ell_\infty(\Delta^\lambda)$, respectively.

Proof. This result is immediate by (3) of Lemma 1.3.7, where $B = TA$. □

Now, by combining Theorems 5.1.2 and 5.3.1, we can obtain various consequences concerning with the particular cases of the space X and the triangle T. For instance, we have $cs_0 = (c_0)_\sigma$, $cs = (c)_\sigma$, $bs = (\ell_\infty)_\sigma$, $c_0(\Delta) = (c_0)_\Delta$, $c(\Delta) = (c)_\Delta$, $\ell_\infty(\Delta) = (\ell_\infty)_\Delta$ and $bv_p = (\ell_p)_\Delta$ for $p \geq 1$. Therefore, we deduce the following particular cases:

Corollary 5.3.2 *Let A be an infinite matrix and define the matrices $[b_{nk}]$ and $[\tilde{b}_{nk}]$ by*

$$b_{nk} = a_{nk} - a_{n-1,k} \quad \textit{and} \quad \tilde{b}_{nk} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\, b_{nk} + \sum\nolimits_{j=k}^{\infty} b_{nj} \qquad (n, k \geq 1),$$

where the series $\sum_{k=1}^{\infty} b_{nk}$ *converge for all* n*. Then, the necessary and sufficient conditions in order that* A *belongs to any one of the classes* $(\mu(\Delta^{\lambda}), \ell_{\infty}(\Delta))$, $(\mu(\Delta^{\lambda}), c(\Delta))$, $(\mu(\Delta^{\lambda}), c_0(\Delta))$ *or* $(\mu(\Delta^{\lambda}), bv_p)$ *are those conditions given respectively in Corollary* 5.1.3, 5.1.4, 5.1.5 *or* 5.1.6 *provided that the entries* a_{nk} *and* $\tilde{a}_{nk}$ *are respectively replaced by* b_{nk} *and* $\tilde{b}_{nk}$ *for all* $n, k \geq 1$, *where* $p \geq 1$.

Corollary 5.3.3 *Let* A *be an infinite matrix and define the matrices* $[b_{nk}]$ *and* $[\tilde{b}_{nk}]$ *by*

$$b_{nk} = \sum_{j=1}^{n} a_{jk} \quad \text{and} \quad \tilde{b}_{nk} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\, b_{nk} + \sum_{j=k}^{\infty} b_{nj} \qquad (n, k \geq 1),$$

where the series $\sum_{k=1}^{\infty} b_{nk}$ *converge for all* n*. Then, the necessary and sufficient conditions in order that* A *belongs to any one of the classes* $(\mu(\Delta^{\lambda}), bs)$, $(\mu(\Delta^{\lambda}), cs)$ *or* $(\mu(\Delta^{\lambda}), cs_0)$ *are those conditions given respectively in Corollary* 5.1.3, 5.1.4 *or* 5.1.5 *provided that the entries* a_{nk} *and* $\tilde{a}_{nk}$ *are respectively replaced by* b_{nk} *and* $\tilde{b}_{nk}$ *for all* $n, k \geq 1$.

Finally, we conclude our work with the following corollaries characterizing matrix operators between two sequence spaces of λ-type. For this, let $\lambda' = (\lambda'_k)$ be a strictly increasing sequence of positive real numbers (λ and λ' need not be equal). Then, we define $\mu(\Delta^{\lambda'}) = (\mu)_{\Lambda'}$, where Λ' is the triangle defined by (2.2.4) with λ' instead of λ. So, we deduce the following consequences:

Corollary 5.3.4 *Let* A *be an infinite matrix and define the matrices* $[b_{nk}]$ *and* $[\tilde{b}_{nk}]$ *by*

$$b_{nk} = \frac{\Delta(\lambda'_n)}{\lambda'_n}\, a_{nk} + \Delta\Big(\frac{1}{\lambda'_n}\Big)\sum_{j=0}^{n-1} \Delta(\lambda'_j)\, a_{jk} \qquad (n, k \geq 1),$$

$$\tilde{b}_{nk} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\, b_{nk} + \sum_{j=k}^{\infty} b_{nj} \qquad (n, k \geq 1),$$

where the series $\sum_{k=1}^{\infty} b_{nk}$ *converge for all* $n \geq 1$*. Then, the necessary and sufficient conditions in order that* A *belongs to any one of the classes* $(\mu(\Delta^{\lambda}), \ell_{\infty}(\Delta^{\lambda'}))$,

$(\mu(\Delta^\lambda), c(\Delta^{\lambda'}))$ *or* $(\mu(\Delta^\lambda), c_0(\Delta^{\lambda'}))$ *are those conditions given respectively in Corollary* 5.1.3, 5.1.4 or 5.1.5 *provided that* a_{nk} *and* $\tilde{a}_{nk}$ *are respectively replaced by* b_{nk} *and* $\tilde{b}_{nk}$ *for all* $n, k \geq 1$.

Corollary 5.3.5 *Let* A *be an infinite matrix and define the matrices* $[b_{nk}]$ *and* $[\hat{b}_{nk}]$ *by*

$$b_{nk} = \frac{1}{\lambda'_n}\sum\nolimits_{j=1}^{n}\Delta(\lambda'_j)\, a_{jk} \quad \text{and} \quad \tilde{b}_{nk} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\, b_{nk} + \sum\nolimits_{j=k}^{\infty} b_{nj} \qquad (n, k \geq 1),$$

where the series $\sum_{k=1}^{\infty} b_{nk}$ *converge for all* n. *Then, the necessary and sufficient conditions in order that* A *belongs to any one of the classes* $(\mu(\Delta^\lambda), \ell_\infty^{\lambda'})$, $(\mu(\Delta^\lambda), c^{\lambda'})$, $(\mu(\Delta^\lambda), c_0^{\lambda'})$ *or* $(\mu(\Delta^\lambda), \ell_p^{\lambda'})$ *are those conditions given respectively in Corollary* 5.1.5, 5.1.6, 5.1.7 or 5.1.8 *provided that* a_{nk} *and* $\tilde{a}_{nk}$ *are respectively replaced by* b_{nk} *and* $\tilde{b}_{nk}$ *for all* $n, k \geq 1$, *where* $1 \leq p < \infty$.

CONCLUSION

CONCLUSION

The λ-sequence spaces have proved their useful in some subjects of analysis with various applications in operator theory and measure of non-compactness [10, 49], in the spectral theory [71, 72], in summability theory [14, 62] and in the theory of double sequences [1, 20, 42, 57, 73] and many researchers and authors, around the world, are working in these interesting spaces of λ-type (e.g., see [9, 15, 21, 23, 29, 75]).

Now, by adding our contribution to the literature, the new λ-difference spaces of bounded, convergent and null difference sequences have been introduced, their topological, algebraic and isomorphic properties have been studied, their inclusion relations and Schauder bases have been established, their Köthe-Toeplitz dual spaces have been constructed and their matrix operators characterized. This gives an open scope and a new area for additional future research studies. For instance, the study of compact operators and some fixed point theorems and spectrum theorems on our new spaces with some applications in differential equations and numerical analysis (see [10, 39, 50, 58, 59, 61, 71, 72] for such studies).

So, at the end of this thesis, I suggest the researchers to continue in study of our new sequence spaces and their matrix transformations to solve many open problems still left in the literature of the theory of λ-sequence spaces.

LIST OF SYMBOLS

$\mathbb{K}$	the scalar field $\mathbb{R}$ or $\mathbb{C}$
n, k	positive integers
x, y	sequences
x_k	k-term of x
$\Delta(x)$	difference sequence of x
$\sigma(x)$	sum sequence of x
w	the space of all sequences
X, Y	sequence spaces
$\|\cdot\|$	norm
X^θ	Köthe-Toeplitz duals of X
X^α	α-dual of X
X^β	β-dual of X
X^γ	γ-dual of X
A, B	matrices
a_{nk}	entries of A
$A(x)$	A-transform of x
Δ	band matrix of difference
σ	sum matrix
(X, Y)	matrix class
X_A	matrix domain of A in X

λ	λ-sequence
Λ	λ-matrix
$\tilde{\Lambda}$	$\tilde{\lambda}$-matrix
X^λ	λ-sequence space
ℓ_∞	space of bounded sequences
c	space of convergent sequences
c_0	space of null sequences
ℓ_p	space of sequences associated with p-absolutely convergent series
bv_p	space of sequences with p-bounded variation
bs	space of sequences associated with bounded series
cs	space of sequences associated with convergent series
cs_0	space of sequences associated with null series
$\ell_\infty(\Delta)$	space of bounded difference sequences
$c(\Delta)$	space of convergent difference sequences
$c_0(\Delta)$	space of null difference sequences
$\ell_\infty(\Delta^\lambda)$	λ-difference space of bounded difference sequences
$c(\Delta^\lambda)$	λ-difference space of convergent difference sequences
$c_0(\Delta^\lambda)$	λ-difference space of null difference sequences
μ	the space c_0, c or ℓ_∞
$\mu(\Delta)$	the space $\ell_\infty(\Delta)$, $c(\Delta)$ or $c_0(\Delta)$
$\mu(\Delta^\lambda)$	the space $\ell_\infty(\Delta^\lambda)$, $c(\Delta^\lambda)$ or $c_0(\Delta^\lambda)$
η	the space c or ℓ_∞
$\eta(\Delta)$	the space $c(\Delta)$ or $\ell_\infty(\Delta)$
$\eta(\Delta^\lambda)$	the space $c(\Delta^\lambda)$ or $\ell_\infty(\Delta^\lambda)$

LIST OF PUBLICATIONS

[1] A.K. Noman and O.H. Al-Sabri, *On the new λ-difference spaces of convergent and bounded sequences*, Albaydha Univ. J., **3**(2) (2021), 18–30.

[2] A.K. Noman and O.H. Al-Sabri, *Matrix operators on the new spaces of λ-difference sequences*, Ijrdo J. Math., **8**(1) (2022), 1–22.

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ملخص الرسالة

في هذه الرسالة، قدمنا تعريفاً لفضاءات الفرق الجديدة للمتتاليات ذات النمط λ وذلك بواسطة الفضاءات الكلاسيكية لمتتاليات الفرق المحدودة والمتقاربة. أيضاً، قمنا بدراسة الخواص الجبرية والتوبولوجية لتلك الفضاءات الجديدة مع علاقاتها الآيزومورفية وقواعد شاودر لها وعلاقات الاحتواء المتعلقة بها. بالإضافة إلى ذلك، أوجدنا الفضاءات الثانوية لتلك الفضاءات واستنتجنا عدد من النتائج الجديدة لتوصيف المؤثرات المصفوفية التي تؤثر على هذه الفضاءات وبينها. علاوةً على ذلك، قمنا بمناقشة العديد من الحالات الخاصة لبعض النتائج الأساسية وحصلنا على عدد من الاستنتاجات الهامة.

بسم الله الرحمن الرحيم

﴿وَقُلِ اعْمَلُوا فَسَيَرَى اللَّهُ عَمَلَكُمْ وَرَسُولُهُ وَالْمُؤْمِنُونَ وَسَتُرَدُّونَ

إِلَى عَالِمِ الْغَيْبِ وَالشَّهَادَةِ فَيُنَبِّئُكُم بِمَا كُنتُمْ تَعْمَلُونَ﴾

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المؤثرات المصفوفية على بعض المجالات المصفوفية الجديدة في فضاءات الفرق للمتتاليات

رســالة ماجســتير

مقدمة كجزء من متطلبات الحصول على درجة الماجستير في الرياضيات

تقـدم بها الطـالب

عمـر حسـين سـالم الصبري

تحت إشـراف الدكتـور

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