SOME CONTRIBUTIONS TO HUB PARAMETERS OF GRAPHS

A THESIS

SUBMITTED AS PARTIAL FULFILMENT OF THE REQUIREMENTS FOR

THE AWARD OF THE MASTER DEGREE IN

MATHEMATICS

AL-BAYDHA University

(Department of Mathematics)

BY

Abdu-Alkafi Saead Qaid Sanad

UNDER THE GUIDANCE OF

Dr. Sultan Senan Mahde

Faculty of Education and Science - Rada'a

Al-Baydha University

Al-Baydha, Yemen

March, 2022

DEDICATED

This Thesis is lovingly dedicated to the dearest people of my heart...

To the first teacher in life, source of all that is important to me, for Muhamad is messenger of god.

To my first love, my mother.

for years of unwavering support and guidance my beloved father.

To the pure soul of my brother, the martyr muhamad.

To my beloved, sisters and brothers.

To my love and happiness, To my wife.

To every piece of my soul, my children.

To the best teacher in my life, source of my strengths,

To the secret of my strength in life, all my friends.

To the secret of my success all, my students.

To all of them

I say,

I love you so much.

CERTIFICATE

This is to certify that the thesis "SOME CONTRIBUTIONS TO HUB PARAMETERS OF GRAPHS" is the result of research work done by Mr Abdu-Alkafi Saead Qaid Sanad under my supervision.

The thesis has been the outcome of the original investigation in his area of research for a period for a year. The thesis as a whole or any part of it has not been submitted in part or in full to any other institution for the award of any other degree/diploma/associate ship/fellowship.

Signature of the Research Supervisor and Date

(Dr. Sultan Senan Mahde)

ACKNOWLEDGMENTS

Praise is to Allah, Lord of the Heavens, the Earth, who gave me the potential and ability to complete this thesis.

First of all, I would like to express my sincere gratitude to my Guide Dr. Sultan Senan Mahde, Assistant Professor, Department of Mathematics, Radaa - Al-Baydha University, for all the assistance he provided me, and for the guidance, and continuous encouragement greatly in reaching the end of this thesis. I have expended energy in shaping my research competence, politically reviewing my work and making valuable suggestions. And all the basic information from beginning to end is all thanks, appreciation and respect.

Secondly, I would like to thank every doctor who taught me or contributed to my teaching from university doctors in the introductory stage, especially Prof. Dr. Abdullah Al-Sherai. I also extend my full appreciation and thanks to the doctors and their support and encouragement to finish this work, and also thanks to the University Administration represented by the President of the University, Prof. Ahmed Al-Arami, for his effective contribution to the success and approval of the program.

I also thank my colleagues at work for providing all the support and continuous encouragement and solving all the difficulties I face.

My special thanks and appreciation to my beloved wife, Mrs. A. Al-

Hanomi, for her constant care and support. She has always made me feel that "I could do anything that I want to achieve", and my lovely daughter malak and my son omar, whose natural smile and compassion always extended to me the real relief.

Finally, my ineffable gratitude goes to my whole family, my parents, wife, children, brothers and sisters for all their love, support and patience. Without their support and encouragement, I cannot concentrate on my work and overcome the difficulties.

Abdu-Alkafi Saead Qaid Sanad

Table of Contents

List of Figures

LIST OF SYMBOLS

$V(G)$	vertex-set of a graph G
$E(G)$	edge-set of a graph G
$\|V(G)\|$	order of a graph G
$\|E(G)\|$	size of a graph G
uv	edge joining vertices u and v
$deg_G(u)$	degree of a vertex u in a graph G
$\triangle(G)$	maximum degree of a graph G
$\delta(G)$	minimum degree of a graph G
$\alpha(G)$	vertex covering number of a graph G
$\beta(G)$	maximum independent set of vertices of a graph G
$\kappa(G)$	connectivity of a graph G
$\chi(G)$	chromatic number of a graph G
$\overline{G}$	complement of a graph G
$\langle C\rangle$	subgraph induced by set C
K_n	complete a graph of order n
P_n	path of order n
C_n	cycle of order n

$d(u, v)$	distance between u and v in a graph G
$K_{m,n}$	complete bipartite a graph of order $m + n$
$K_{1,n}$	star graph of order $n + 1$
$S_{n,m}$	double star
$L(G)$	line graph of a graph G
$S'(G)$	splitting graph of a graph G
$G + H$	join of graphs G and H
$G \circ H$	corona of graphs G and H
$G \cup H$	union of graphs G and H
$G \times H$	Cartesian product of graphs G and H
W_n	wheel of order $n + 1$
F_n	fan graph of order $n + 1$
$h(G)$	hub number of a graph G
$h_h(G)$	hop hub number of graph G
$I(G)$	integrity of a graph G
$HI(G)$	hub-integrity of a graph G
$H_hI(G)$	hop hub-integrity of a graph G
$\gamma(G)$	domination number of a graph G
$\gamma_t(G)$	total domination number of a graph G
$\gamma_h(G)$	hop domination number of a graph G

CHAPTER 1

INTRODUCTION

1.1 Introduction

The field of graph theory is one of the ever growing branch of mathematics. Graph theory, in its essence can be described as the study of relations on finite sets, which are visualized with vertices and edges in a two dimensional plane. Graph theory is intimately related to many branches of mathematics, including group theory, probability, numerical analysis, matrix theory, topology, operational research, combinatorics and many more.

The origin of graph theory can be traced back to Eulers [28] work on the Konigsberg bridges problem (1735), which subsequently led to the concept of an Eulerian graph. Euler studied the problem of Konigsberg bridge and constructed a structure to solve the problem called Eulerian graph [12].

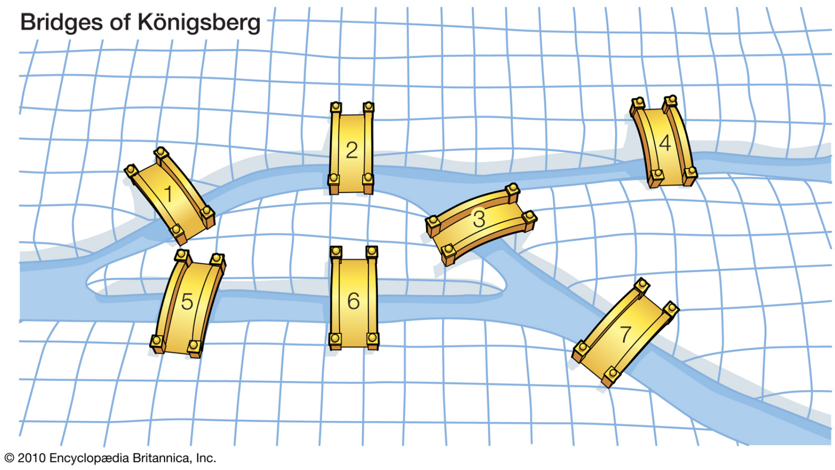

Figure 1.1: The bridge of Konigsberg

In the recent years, great attention have been paid to the modeling and analysis of the spread of belief or influence in complex networks. Various problems in social and virtual networks such as world wide web or models of distributed computing can be formalized in terms of the spread of influence, for example:

- Elections in societies.

Figure 1.2: Elections in societies

- Spread of disease among people.

Figure 1.3: Spread of disease

- Spread of virus in world wide web or any web of computers.

Figure 1.4: Virus in web

- Spread of opinions across social networking sites like Twitter, Facebook and etc.

Figure 1.5: Social media

A network in all of these examples which is simply consisted of a set of elements (e.g. agents in social networks or computing units in distributed computing systems), and some relationships or interactions between these elements can be conveniently modeled by a graph $G(V, E)$, whose vertex set $V(G)$ represent the elements and edges $E(G)$ represent the links of the network, for more detail we refer to [28].

1.2 Basic definitions and terminologies of a graph

In this section, some basic definitions are collected from various books including the books by Harary [28], Charttrand and Lesniak [16], and Bondy and Murthy [14]. Additional definitions will be given as they are needed.

Definition 1.2.1. *[28] A graph G consists of a finite nonempty set $V = V(G)$ of n vertices together with a prescribed set E of m unordered pairs of distinct vertices of V. Each pair $e = \{u, v\}$ of vertices in E is an edge of G, u and v joined by e. We write $e = (u, v)$ or $e = uv$ and say that u and v are* ***adjacent*** *vertices, u and e are* ***incident*** *with each other, as are v and e. If two edges e_1 and e_2 of G are distinct and incident with a common vertex, then they are* ***adjacent*** *edges. The* ***order*** *of G, denoted by $|V(G)| = n$, is the number of vertices in G. The* ***size*** *of G, denoted by $|E(G)| = m$, is the number of edges in G.*

Definition 1.2.2. *[14] If vertex set and edge set of G are finite, then G is* ***finite****. A finite graph G having no loops or multiple edges is called a* ***simple graph****.*

Example 1.2.3. *In Figure 1.6, a graph G is an example of a simple graph with vertex* *set* $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ *and edge-set* $E(G) = \{e_1, e_2, e_3, e_4, e_5\}$*. The order of* G *is* $|V(G)| = 5$ *and the size of* G *is* $|E(G)| = 5$.

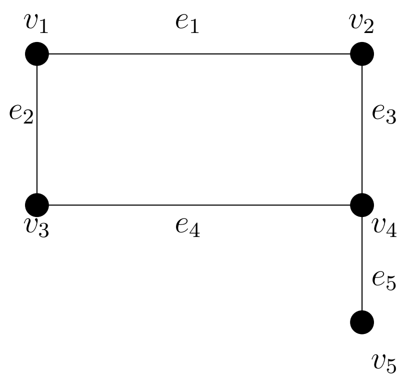

Figure 1.6: Simple graph

Definition 1.2.4. *[33] A* ***null graph*** *or a* ***totally disconnected graph*** *is a graph with nonempty vertex set and no edges.*

Example 1.2.5. *In Figure 1.7, a graph* G *is an example of a null graph.*

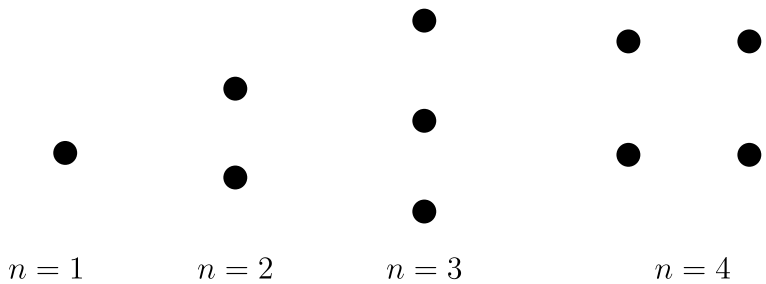

Figure 1.7: Null graph

Definition 1.2.6. *[16] The **degree** of a vertex v is defined to be the number of edges incident with v in a graph G, and is denoted by $d_G(v)$ or $deg(v)$. The **minimum** degree of a graph G is denoted by $\delta(G)$ and $\Delta(G)$ denotes the **maximum** degree of G. A vertex of degree zero is called an **isolated** vertex or an isolate.*

Example 1.2.7. *In Figure 1.8, a graph G is a simple graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $deg(v_1) = 1$, $deg(v_3) = deg(v_4) = 2$, $deg(v_2) = 4$, $deg(v_5) = 3$, then $\delta(G) = 1$ and $\Delta(G) = 4$.*

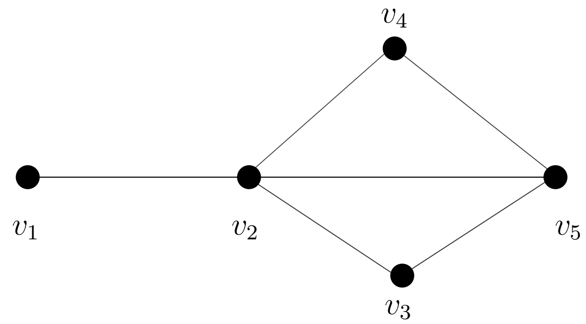

Figure 1.8: G

Definition 1.2.8. *[33] A graph $H = (V(H), E(H))$ is a **subgraph** of a graph $G = (V(G), E(G))$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(G) \neq V(H)$ or $E(G) \neq E(H)$, then we say that H is a **proper** subgraph of G or G is a **super** graph of H. A subgraph H of G is called a **spanning** subgraph of G if $V(G) = V(H)$. If $C \subseteq V(G)$, then the induced subgraph $[C]$ of G is the graph with vertex set C and such that $uv \in E(C)$ wheneveru, $v \in C$ and $uv \in E(G)$.*

Example 1.2.9. *In Figure 1.9, H_1 and H_2 are subgraphs of G . Moreover, H_1 is a spanning subgraph of G and H_2 is an induced subgraph of G.*

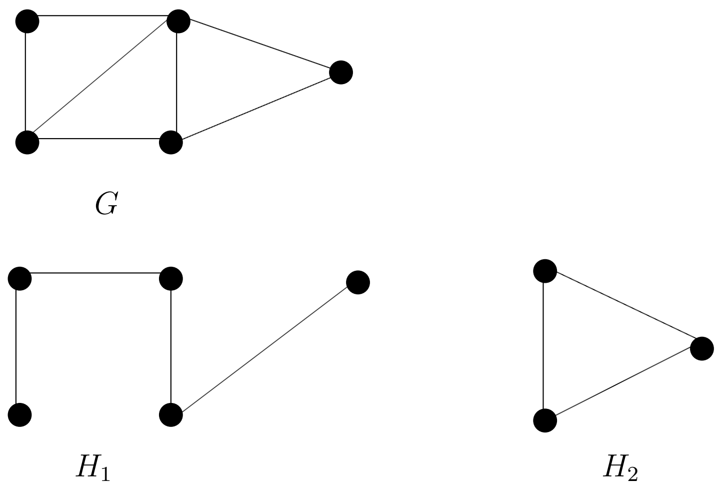

Figure 1.9: A graph G with its subgraphs H_1 and H_2

Definition 1.2.10. *A path is a graph with vertex-set $\{v_1, v_2, ..., v_n\}$ and edge set $\{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$, where the v_i, $1 \leq i \leq n$ is are all distinct, and is denoted by $P_n = \{v_1, v_2, ..., v_n\}$. If G is a graph and u and v are vertices of G, then a path from vertex u to vertex v is oftentimes called a $u-v$ path. The cycle $C_n = \{c_1, c_2, ..., c_n\}$ is the graph of order $n \geq 3$ with vertices $c_1, c_2, ..., c_n$ and edges $c_1c_2, c_2c_3, ..., c_{n-1}c_n, c_nc_1$. The distance $d(u, v)$ in G of two vertices u, v is the length of a shortest $u-v$ path in G.*

Example 1.2.11. *Figure 1.10 shows the path $P_5 = \{v_1, v_2, v_3, v_4, v_5\}$ of order 5 and the cycle $C_5 = \{u_1, u_2, u_3, u_4, u_5\}$ of order 5.*

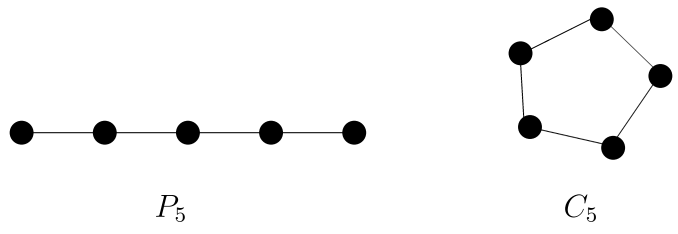

Figure 1.10: P_5 and C_5

Definition 1.2.12. *[16, 28] The* ***distance*** *between two vertices u and v in a graph G is the number of edges in the shortest $u - v$ path and is denoted by $d(u, v)$. The* ***eccentricity*** *$e(v)$ of a vertex $v \in V$ in a connected graph G is defined as $e(v) = max\{d(v, u) : u \in V(G)\}$. The* ***radius*** *of G is the minimum eccentricity of the vertices, and is denoted by $r(G)$. The* ***diameter*** *$diam(G)$ of a connected graph G is defined as $diam(G) = max\{e(v) : v \in V(G)\}$.*

Example 1.2.13. *In Figure 1.8, a graph G is a simple graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $d(v_1, v_4) = 2$, $e(v_1) = 2$, $r(G) = 1$, $diam(G) = 2$.*

Definition 1.2.14. *[28] A graph G is said to be* ***connected*** *if there exists a $u - v$ path for every two vertices $u, v \in V(G)$. A graph G which is not connected is said to be* ***disconnected****. A* ***component*** *of a graph G is a subgraph which is maximal with respect to the property of being connected. A* ***cut vertex*** *of a graph G is a vertex whose deletion increases the number of components, and a* ***bridge*** *is such an edge.*

Definition 1.2.15. *[28] Two graphs G and H are* ***isomorphic****, denoted by $G = H$ or $G \cong H$, if there exists a one-to-one correspondence between vertex sets of G and H which preserves adjacency.*

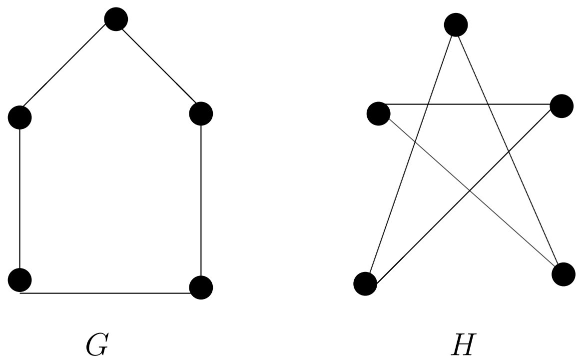

Figure 1.11: $G \cong H$

Definition 1.2.16. *[28] The* ***complement*** *$\overline{G}$ of a graph G has $V(G)$ as its vertex set, two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A graph G is called* ***self-complementary*** *if G and $\overline{G}$ are isomorphic. If every pair of vertices of G are adjacent, then G is called a* ***complete graph****, and it is denoted by K_n with n vertices.*

Example 1.2.17. *In Figure 1.12, K_2, K_3 and K_4 are a complete graph .*

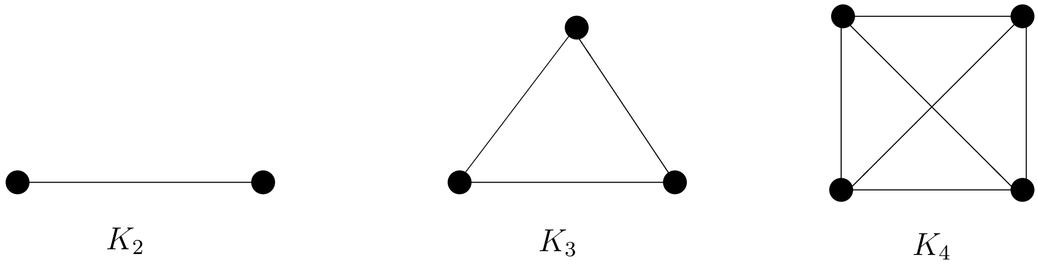

Figure 1.12: Complete graph G

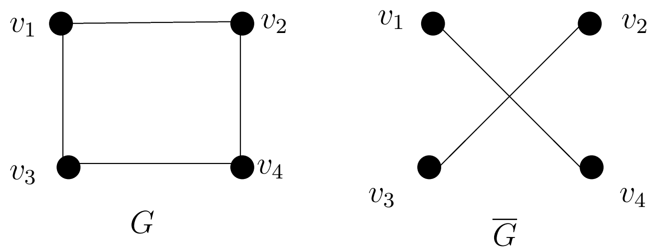

Figure 1.13: Complement $\overline{G}$ of a graph G

Definition 1.2.18. *[28] A graph G is called a* ***bipartite graph*** *if the vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. Furthermore, if every vertex of V_1 is joined to every vertex of V_2, then G is a* ***complete bipartite graph****. The complete bipartite graph with two partite sets V_1 and V_2 of vertices such that $|V_1| = p$, and $|V_2| = q$ is denoted by $K_{p,q}$. The graph $K_{1,n-1}$ is a star.*

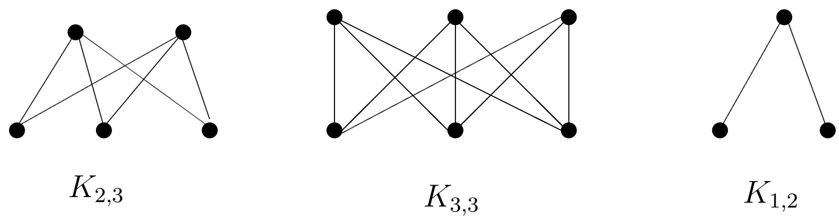

Figure 1.14: Complete bipartite graph $K_{1,2}$, $K_{2,3}$, $K_{3,3}$

Definition 1.2.19. *[28] A graph G is called an* ***acyclic*** *graph or* ***forest*** *if it has no cycles. A connected acyclic graph is called a* ***tree****.*

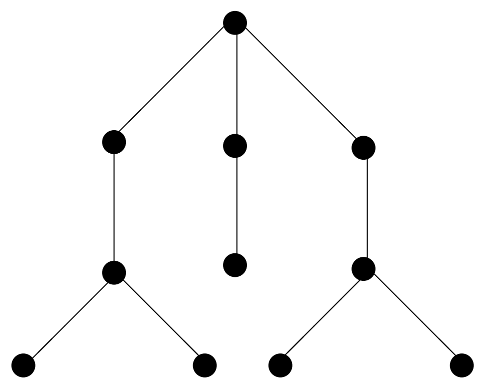

Figure 1.15: Tree

Definition 1.2.20. *[27] A* ***double star*** *$S_{p,q}$ is a tree with exactly two vertices that are not pendant vertices, with one adjacent to p pendant vertices and the other to q pendant vertices.*

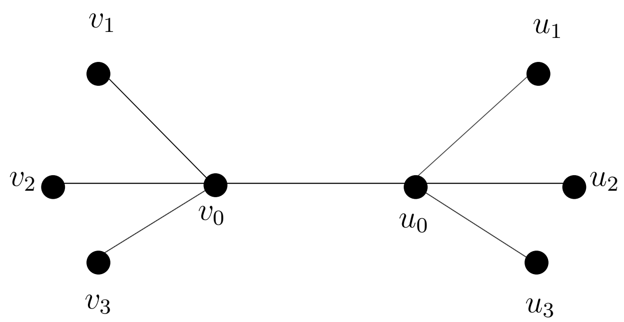

Figure 1.16: Double star $S_{3,3}$

Definition 1.2.21. *[28] Let G_1 and G_2 be two graphs with disjoint vertex sets V_1 and V_2, and edge sets E_1 and E_2, respectively. Then*

1. *their* ***union*** *$G_1 \cup G_2$ is the graph having vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$.*

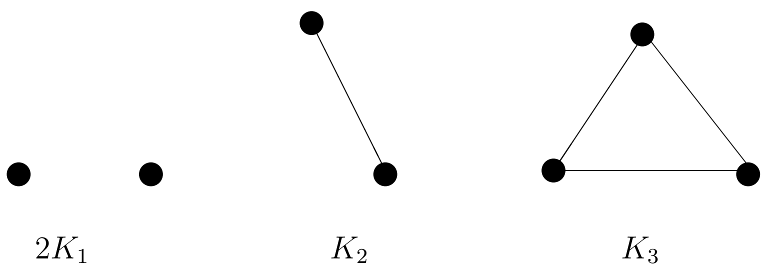

Figure 1.17: Union $2K_1 \cup K_2 \cup K_3$

2. *their* ***join*** *$G_1 + G_2$ is the graph consisting of $G_1 \cup G_2$ with all edges joining V_1 with V_2.*

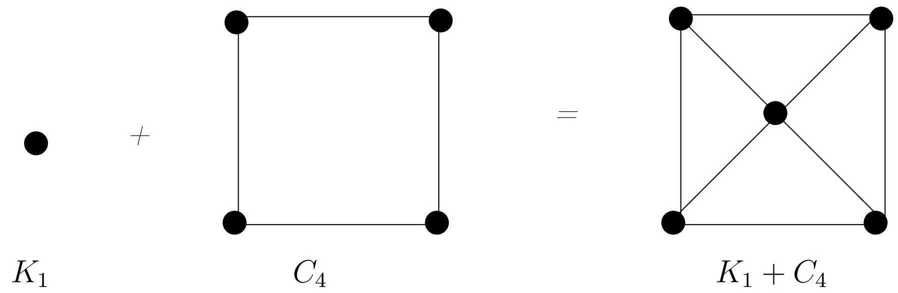

Figure 1.18: Join $K_1 + C_4 = W_{1,4}$

3. *The* ***(Cartesian)product*** $G_1 \times G_2$ *of graphs* G_1 *and* G_2 *has* $V(G_1) \times V(G_2)$ *as its vertex set and* (u_1, u_2) *is adjacent to* (v_1, v_2) *if either* $u_1 = v_1$ *and* u_2 *is adjacent to* v_2 *or* $u_2 = v_2$ *and* u_1 *is adjacent to* v_1*.*

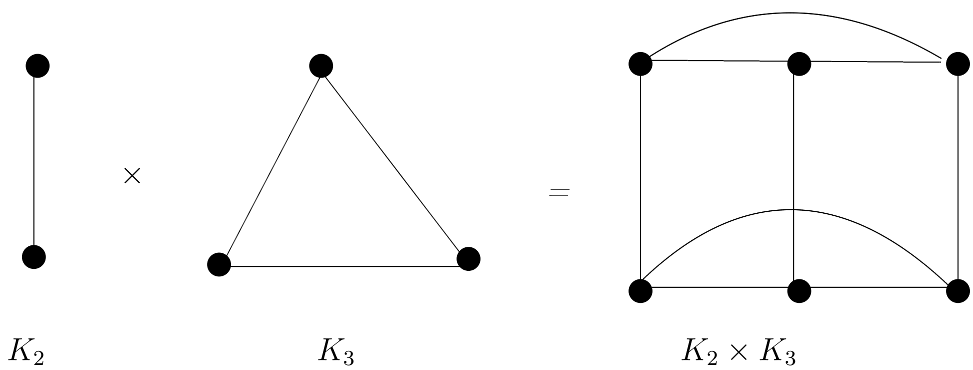

Figure 1.19: Cartesian product $K_2 \times K_3$

Definition 1.2.22. *[22] The* ***Corona*** $G_1 \circ G_2$ *of two graphs* G_1 *and* G_2 *is the graph* G *obtained by taking one copy of* G_1 *(which has* v_1 *vertices) and* v_1 *copies of* G_2*, and* *then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.*

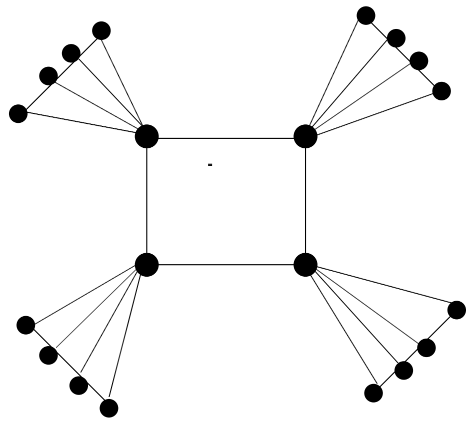

Figure 1.20: Corona $C_4 \circ P_4$

Definition 1.2.23. *[28] The join $K_1 + C_{n-1}$ of vertex disjoint graphs K_1 and C_{n-1} is said to be a **wheel**, and denoted by $W_{1,n-1}$.*

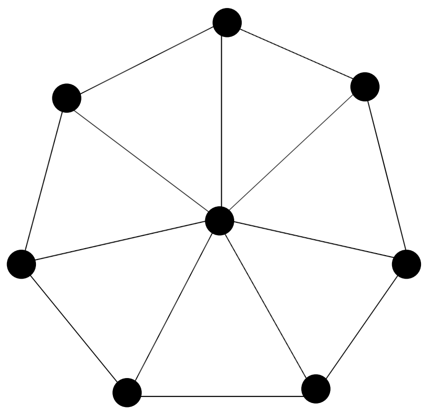

Figure 1.21: Wheel $W_{1,7}$

Definition 1.2.24. *[28] A vertex and an edge are said to* ***cover*** *each other if they are incident. A set of vertices which covers all the edges of a graph G is called a* ***vertex cover*** *for G. The* ***vertex covering number*** *$\alpha(G)$ of G is the minimum number of vertices in a vertex cover.*

Example 1.2.25. *Let G be a graph is shown in Figure 1.22, such that $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. The sets $S_1 = \{v_2, v_5\}$, and $S_2 = \{v_1, v_3, v_4, v_6\}$, are vertex cover sets but the set $S_1 = \{v_2, v_5\}$ is vertex covering number and $\alpha(G) = 2$.*

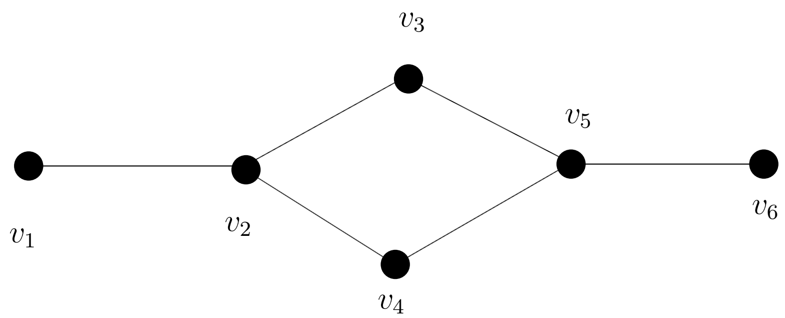

Figure 1.22: Simple graph of G

Definition 1.2.26. *[28] A set S of vertices (or edges) in G is* ***independent*** *if no two vertices (or edges) in S are adjacent. The* ***vertex independence number*** *of a graph G, denoted as $\beta(G)$ is the maximum cardinality of an independent set of vertices.*

Example 1.2.27. *The sets $S_1 = \{v_3, v_4\}$, $S_1 = \{v_2, v_5\}$ and $S_3 = \{v_1, v_3, v_4, v_6\}$ of a graph G in Figure 1.22 are independent set of G. The set $S_3 = \{v_1, v_3, v_4, v_6\}$ is* *vertex independence number of a graph G such that $\beta(G) = 4$.*

Definition 1.2.28. *[28] The **connectivity** $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph.*

Example 1.2.29. *The sets $S_1 = \{v_2\}$, $S_2 = \{v_3, v_4\}$, $S_3 = \{v_2, v_5\}$ and $S_4 = \{v_3, v_4, v_6\}$ of a graph G in Figure 1.22 are connectivity set of G. The set $S_3 = \{v_2\}$ is the minimum number of a graph G such that $\kappa(G) = 1$.*

Definition 1.2.30. *[28] An assignment of colors to the vertices of a graph G so that no two adjacent vertices have the same color is called a **proper coloring** or **coloring** of a graph G. For each color in a coloring of a graph G, the set of all vertices which receive that color is independent and is called a color class. A **proper coloring** of G that has a minimum number of color classes is called a $\chi(G) -$ coloring and the number of color classes in such a coloring is $\chi(G)$ the **chromatic number** of G.*

Example 1.2.31. $\chi(G) = 2$ *of a graph G as shown in Figure 1.23.*

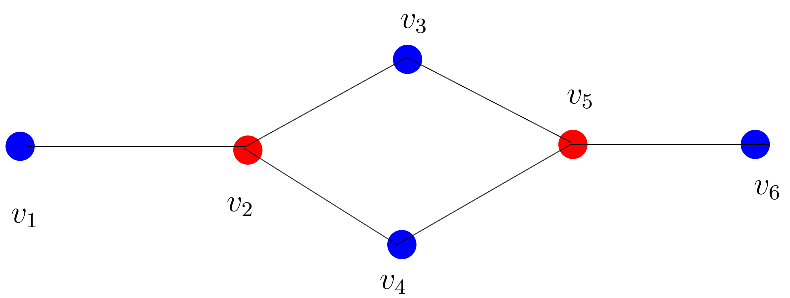

Figure 1.23: G

Definition 1.2.32. *[28] The* ***line graph*** *$L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G.*

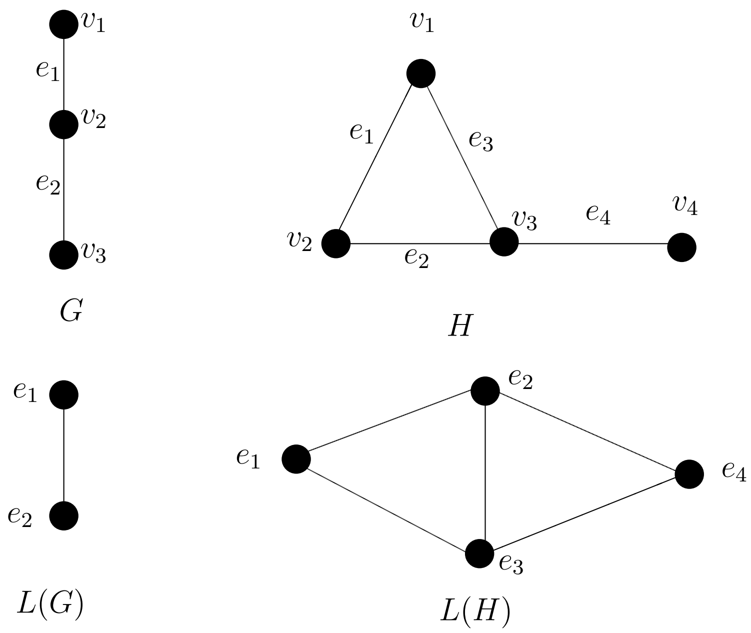

Figure 1.24: G, $L(G)$, H and $L(H)$

Definition 1.2.33. *[33] A tree is called a* ***binary tree*** *if it has one vertex of degree 2 and each of the remaining vertices of degree 1 or 3.*

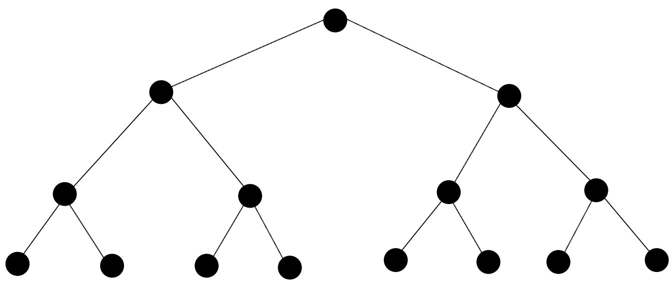

Figure 1.25: Binary tree

Definition 1.2.34. *[29] A set $S \subseteq V(G)$ is called a **dominating set** of G if each vertex of $V - S$ is adjacent to at least one vertex of S. The **domination number** of G denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G.*

Example 1.2.35. *The sets $S_1 = \{v_7, v_6\}$ and $S_2 = \{v_7, v_6, v_5\}$ of a graph G in Figure 1.26 are dominating sets of G. The set $S_1 = \{v_7, v_6\}$ is a domination number of G, such that $\gamma(G) = 2$.*

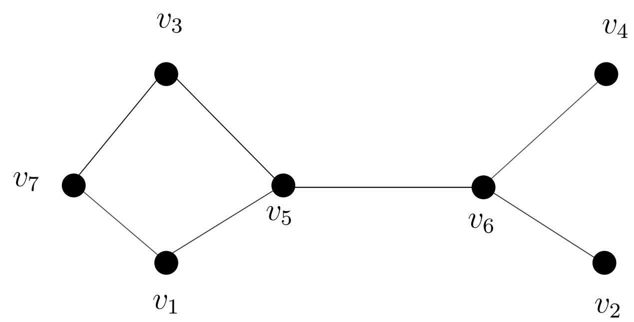

Figure 1.26: G

Definition 1.2.36. *[48] For a graph G the **splitting graph** $S'(G)$ of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$, where $N(v)$ and $N(v')$ are the neighborhood sets of v and v', respectively.*

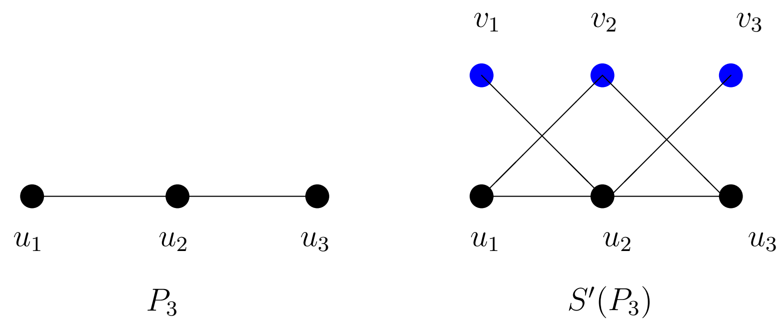

Figure 1.27: Splitting graph of P_3

1.3 Background of the study

In this section, we display a historical background for the theory of hub parameters. A hub set in a graph theory was introduced by Walsh [56] in 2006, and he started by this imagination: we have a graph G which represents the buildings in a large industrial complex, with an edge between two buildings if it is an easy walk from one to the other. The corporation is considering to implement a rapid-transit system, and wants to place its stations in buildings (which will then be used only for this purpose) so that to travel between two non-adjacent buildings (which are not stations), one need only walk to an adjacent station, take the RTS, and walk to the desired building. The corporation would like to implement this plan as cheaply as possible, which involves converting as few buildings as possible into transit stations. After this imagination he introduced the concept of hub number.

In 2011, Peter Johnson, Peter Slater and M. Walsh, have studied the connected hub number and the connected domination number [30], they characterize the graphs G for which $\gamma_c(G) = h_c(G) + 1$. Also they developed and proved many results.

In 2014, Veena Mathad, Ali Mohammed Sahal and S. Kiran, introduced the concept of the total hub number $h_t(G)$ of graphs [44], and they computed the total hub number $h_t(G)$ of several classes of graphs. Also they determined bounds in terms of other graph parameters.

In 2015, E. C. Cuaresma. and R. N. Paluga, have studied the hub number of some graphs [19], they give the results for the hub numbers of the join and corona of two connected graphs, cartesian product of two complete graphs, also cartesian product of a non-complete connected graphs and a complete graphs and the cartesian product of two paths P_n and P_k for $n \geq 4$ and $k = 2, 3$.

A set $S \subseteq V(G)$ is called a dominating set of G if each vertex of $V - S$ is adjacent to at least one vertex of S. The domination number of a graph G denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G [29]. A dominating set D is a connected dominating set of G if the subgraph $< D >$ induced by D, is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G which we denote by $\gamma_c(G)$.

In 2015, Natarajan and S. K. Ayyaswamy [5] introduced a new distance related domination parameter called the hop domination number of a graph. As defined in [5], a subset S of $V(G)$ is a hop dominating set of G if for every $v \in V(G) - S$, there exists $u \in S$ such that $d(u, v) = 2$. In [6], bounds on the hop domination number of a tree were investigated.

In 2018, Shadi and Veena [31, 32] introduced the restrained hub number in graphs. There are several measures of the reliability of a communication network. An elegant and simple one is called the integrity of the network.

The concept of integrity was introduced by Barefoot, Entringer and Swart in 1987 [9]. The motivation is as follows. Model the network as a graph. To disrupt the network a terrorist attempts to remove a small set of vertices or (edges) such that the remaining connected components are small. Formally, the integrity of a graph G with vertex set V is defined as $I(G) = \min_{S \subset V}\{|S| + m(G - S)\}$, where $m(G - S)$ denotes the order of the largest component of $G - S$.

In 2015, Sultan et al. [34] have introduced the concept of hub-integrity of a graph G as a new measure of vulnerability which is defined as follows. The hub-integrity of a graph G denoted by $HI(G)$ is defined by, $HI(G) = \{min|S| + m(G - S)\}$; S is a hub set of G, where $m(G-S)$ is the order of a maximum component of $G-S$. For more details on the hub-integrity see [35, 36, 37].

In [23], Goddard added many results and developed some generalizations. Many results on the integrity of specific families of graphs, bounds for the integrity, maximal and minimal graphs of given integrity, relationships between integrity and other parameters, and computational complexity are studied in [25] . They discuss variations and generalizations of the concept.

We have integrated the concept of integrity and the concept of a hub set of a graph G, to get a new concept. Motivated by this, we introduce **hop hub-integrity of a graph** $H_hI(G)$ as a new measure of the stability of a graph G in **Chapter 3**. We calculate the hop hub-integrity of some standard graphs, also we study some properties of $H_hI(G)$. Furthermore, the relation between $H_hI(G)$ and some parameters is determined, the characterization for H_hI of a tree is obtained. In addition, bounds on $H_hI(G)$ are established. Also, H_hI of line graph is presented. Finally, hop Hub-integrity polynomial of graphs is discussed.

1.4 Research Objectives

In this study, aim is to add the following contributions :

- Introducing the concept of hop hub number in a graph.
- Establishing bounds or exact values of the hop hub numbers for some standard graphs.
- defining and study some new hub parameters of graphs. For instance, we introduce the concept of hop hub integrity of graphs.
- Introducing hop hub polynomial of a connected graph G.
- Introducing the concept of hop hubtic number in a graph and determining bounds or exact values of the hop hubtic numbers for some standard graphs.

1.5 Methodology

We collect the existing research materials related to hub number of a graph. We study the hub number of a graph. We explore a new concept related to hub number of graphs namely hop hub number in graphs. We present the methods are planned to use in our research, for proving the results or to explore new ideas or concepts in our research area. To compute hop hub number for several classes of graphs conditional probabilistic method is used. Also, to determine bounds and relations of hop hub number and other graph parameters we employ the standard methods of proofs namely direct method and contradiction method. To explore these new concepts of graphs we are employing the following methods : Example or Counterexample method, Comparison method.

1.6 Short outline of the thesis

In this thesis, our main objective is to investigate the measures of vulnerability of a graph, that is to know how many vertices can still communicate, after removal of some vertices or edges of a graph, this means, one can determine the extent to which the graph retains certain properties after the removal of a number of vertices or edges.

In Chapter 1, we introduce the basic definitions of a graph and short survey for some concepts used in this thesis.

In Chapter 2, we introduce a new parameter of a hub theory in graph G, namely, hop hub of a graph, we determine the hop hub number of some standard graphs. Also upper and lower bounds for $h_h(G)$ are obtained. we discuss some of its properties hop hub number of line graph.

In Chapter 3, we introduce a new measure of the stability of a graph G namely, hop hub-integrity. The hop hub-integrity of some graphs is obtained. The relations between hop hub-integrity and other parameters are determined and some properties of hop hub-integrity of line graphs.

In Chapter 4, some of properties of hop hub number of splitting graph are obtained and some properties of hop hub-integrity of splitting graph are obtained.

In Chapter 5, we determine the hop hubtic number of some standard graphs. Also we obtain bounds for $h_\xi(G)$. In this chapter we introduce hop hub polynomial of a connected graph G. The hop hub polynomial of a connected graph G of order n is the polynomial $H_h(G,x) = \sum_{i=h_h(G)}^{|V(G)|} h_h(G,i)x^i$, where $h_h(G,i)$ denotes the number of hop hub sets of G of cardinality i and $h_h(G)$ is the hop hub number of G. We obtain hop hub polynomial of some special classes of graphs.

CHAPTER 2

Hop hub number of graphs

Reference [49] is based on this chapter.

2.1 Introduction

A hub set in a graph theory was introduced by Walsh [56] in 2006, and he started by this imagination: we have a graph G which represents the buildings in a large industrial complex, with an edge between two buildings if it is an easy walk from one to the other. The corporation is considering to implement a rapid-transit system, and wants to place its stations in buildings (which will then be used only for this purpose) so that to travel between two non-adjacent buildings (which are not stations), one need only walk to an adjacent station, take the RTS, and walk to the desired building. The corporation would like to implement this plan as cheaply as possible, which involves converting as few buildings as possible into transit stations. After this imagination he introduced the concept of hub number.

Definition 2.1.1. *[56] Suppose that $H \subseteq V(G)$ and take v, u be any two vertices. The H-path between v and u is a path where all intermediate vertices are from H. (This includes the degenerate cases where the path consists of the single edge vu or a single vertex v if $v = u$, call such an $H-$path trivial.)*

Definition 2.1.2. *[56] A hub set in a graph G is a set H of vertices in G such that any two vertices outside H are connected by a path whose all internal vertices lie in H. The hub number of G, denoted $h(G)$, is the minimum size of a hub set in G.*

Theorem 2.1.3. *[56]*

(a) *For any complete graph* K_n, $h(K_n) = 0$.

(b) *For any path* P_n *with* $n \geq 3$, $h(P_n) = n - 2$.

(c) *For any cycle* C_n, $h(C_n) = n - 3$.

(d) *For the star* $K_{1,n-1}$, $h(K_{1,n-1}) = 1$.

(e) *For the double star* $S_{p,q}$, $h(S_{p,q}) = 2$.

In 2011, Peter Johnson, Peter Slater and M. Walsh, have studied the connected hub number and the connected domination number [30], they characterize the graphs G for which $\gamma_c(G) = h_c(G) + 1$. Also they developed and proved many results.

Definition 2.1.4. *[30] A connected set in* G *is a vertex set* F *such that the subgraph of* G *induced by* F, $G[F]$ *is connected. The connected hub number of* G*, denoted* $h_c(G)$*, is the minimum size of a connected hub set in* G.

In 2014, Veena Mathad, Ali Mohammed Sahal and S. Kiran, introduced the concept of the total hub number of graphs $h_t(G)$ [44], and they computed the total hub number $h_t(G)$ of several classes of graphs. Also they determined bounds in terms of other graph parameters.

Definition 2.1.5. *[44] Let* G *be a graph. A total hub set* S *of* G *is a subset of* $V(G)$ *such that every pair of vertices (whether adjacent or nonadjacent) of* $V - S$ *are connected by a path, whose all intermediate vertices are in* S*. The total hub number* $h_t(G)$ *is then defined to be the minimum cardinality of a total hub set of* G.

Theorem 2.1.6. *[44]*

(a) *For any complete graph K_n, $n \geq 2$ $h_t(K_n) = 1$.*

(b) *For any path P_n with $n \geq 3$, $h_t(P_n) = n - 2$.*

(c) *For any cycle C_n, $n \geq 6$ $h_t(C_n) = n - 3$.*

(d) *For the wheel $W_{1,n-1}$, $h_t(W_{1,n-1}) = 1$.*

(e) *For the double star $S_{p,q}$, $p, q \geq 1$ $h_t(S_{p,q}) = 2$.*

In 2015, Edilberto Cuaresma and Rolando Paluga, have studied the hub number of some graphs [19], they give the results for the hub numbers of the join and corona of two connected graphs, cartesian product of two complete graphs, also cartesian product of a non-complete connected graphs and a complete graphs and the cartesian product of two paths P_n and P_k for $n \geq 4$ and $k = 2, 3$.

Theorem 2.1.7. *[19] For any connected graphs G and H,*

$$h(G+H) = \begin{cases} 0, & \textit{if } G \textit{ and } H \textit{ are complete,} \\ 1, & \textit{if } G \textit{ complete and } H \textit{ non - complete,} \\ min\{h(G), h(H), 2\}, & \textit{if } G \textit{ and } H \textit{ are both non - complete.} \end{cases}$$

In 2015, Natarajan and S. K. Ayyaswamy [5] introduced a new distance related domination parameter called the hop domination number of a graph, and they computed hop domination number of a graph $\gamma_h(G)$ of several classes of graphs. In [6] also they determined bounds on the hop domination number of a tree were investigated.

Definition 2.1.8. *[5] A set $S \in V$ of a graph G is a hop dominating set(hd-set, in short) of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hd-set of G is called the hop domination number and is denoted by $\gamma_h(G)$.*

Theorem 2.1.9. *[5]*

(a) *For any complete graph K_n, $\gamma_h(K_n) = n$.*

(b) *For any path P_n with $n = 6r$, $\gamma_h(P_n) = 2r$.*

(c) *For any cycle C_n, $n = 6r$, $\gamma_h(C_n) = 2r$.*

(d) *For the wheel $W_{1,n-1}$, $\gamma_h(W_{1,n-1}) = 3$.*

(e) *For a complete bipartite graph $K_{p,q}$, $\gamma_h(K_{p,q}) = 2$.*

In 2021, Ali Mohammed Sahal, introduced the concept of the doubly connected hub number of graphs $h_{cc}(G)$ [**?**]. And he computed the doubly connected hub number for several classes of graphs, bounds in terms of other graph parameters are also determined.

Definition 2.1.10. *[**?**] Let G be a connected graph. A doubly connected hub set S of G is a subset of $V(G)$ such that any pair of vertices of $V - S$ are connected by a path, whose all intermediate vertices are in S and both $\langle S \rangle$ and $\langle V(G) - S \rangle$ are connected. The cardinality of the minimum doubly connected hub set in G is the doubly connected hub number and is denoted by $h_{cc}(G)$.*

Remark 2.1.1. [?] Notice that

(1) For any complete graph K_n, $h_{cc}(K_n) = h_c(K_n) = 0$;

(2) For any double star $S_{p,q}$, $h_{cc}(S_{p,q}) = p + q + 1$;

(3) For any cycle C_n, $h_{cc}(C_n) = h_c(C_n) = n - 3$;

(4) For any complete bipartite graph $K_{p,q}$,

$$h_{cc}(K_{p,q}) = \begin{cases} 0, & \text{if } p = q = 1 \ ; \\ 1, & \text{if } q = 1 \text{ and } p = 2 \text{ or } q = 2 \text{ and } p \geq 2 \ ; \\ 2, & \text{if } p, q \geq 3. \end{cases}$$

(5) For the wheel $W_{1,n-1}$, $n \geq 4, h_{cc}(W_{1,n-1}) = 1$.

Using the concept of hop domination number $\gamma_h(G)$ of a graph G and the definition of the hub number $h(G)$ of a graph G, motivated by this, we introduce the concept of hop hub number of a graph G as a new parameter of a graph.

The following results will be useful in the proof of our results.

Theorem 2.1.11. *[56] $h(C_n) = n - 3$.*

Theorem 2.1.12. *[11] For any complete graph G, $\chi(G) = n$.*

Theorem 2.1.13. *[56] Let T be a tree with n vertices and l levels. Then $h(T) = n - l$.*

Theorem 2.1.14. *[56] Let S be a subset of $V(G)$. Then G/S is complete if and only if S is a hub set of G.*

Theorem 2.1.15. *[56] For any graph G, $h(G) + 1 \geq \gamma(G)$.*

2.2 The hop hub number of graphs

Definition 2.2.1. *A hub set S is a hop hub set of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop hub set of G is called the hop hub number and is denoted by $h_h(G)$.*

By the definition of hop hub number we obtain the obvious bound $h_h(G) \geq h(G)$.

Proposition 2.2.2. *The hop hub numbers of some specific classes of graphs are as below:*

1. *For any path P_n,*

$$h_h(P_n) = \begin{cases} 2, & \text{if } n = 2, \\ 2, & \text{if } n = 3, \\ n - 2, & \text{if } n \geq 4. \end{cases}$$

2. *For any complete graph K_n, $h_h(K_n) = n$.*

3. *For the wheel graph $W_{1,n-1}$,*

$$h_h(W_{1,n-1}) = \begin{cases} 4, & \text{if } n = 4, \\ 3, & \text{if } n \geq 5, \end{cases}$$

4. *For the complete bipartite graph $K_{p,q}$, $h_h(K_{p,q}) = 2$.*

5. *For the double star* $S_{p,q}$*,* $h_h(S_{p,q}) = 2$*.*

6. *For any cycle* C_n*,*

$$h_h(C_n) = \begin{cases} 2, & \text{if } n = 4, \\ 3, & \text{if } n = 3, \\ n-3, & \text{if } n \geq 5. \end{cases}$$

Proof. 1. We have the following cases:

Case 1: When n = 2, suppose that $\{v_1, v_2\}$ be the vertices of path P_2 see Figure 2.1, then $S = \{v_1\}$ is not hop hub set because $d(v_1, v_2) = 1$ and $S = \{v_2\}$ is not hop hub set because $d(v_1, v_2) = 1$, then $S = \{v_1, v_2\}$ is hop hub set, so $h_h(P_2) = 2$.

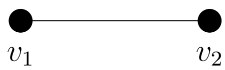

Figure 2.1: Path P_2

Case 2: When n = 3, consider $\{v_1, v_2, v_3\}$ be the vertices of path P_3 see Figure 2.2, suppose that $S = \{v_i\}$, $1 \leq i \leq 3$ is a hop hub set of P_3, because there exist v_j, $1 \leq j \leq 3$, $i \neq j$ such that $d(v_i, v_j) \neq 2$, so $S = \{v_i\}$ is not hop hub set. Consider $S = \{v_1, v_2\}$ is hop hub set of P_3, since $v_3 \in V(P_3) - S$, there exists $v_1 \in S$ such that $d(v_1, v_3) = 2$ or $S = \{v_2, v_3\}$ is hop hub set, since

$v_1 \in V(P_3) - S$, there exists $v_3 \in S$ such that $d(v_1, v_3) = 2$. Then S is a hop hub set of P_3 and it is clear that S is a minimum hop hub set, therefore $h_h(P_3) = 2$.

Figure 2.2: Path P_3

Case 3: When $n \geq 4$, suppose that $\{v_1, v_2, \cdots, v_n\}$, a vertex set of P_n, and $S = \{v_2, v_3, \cdots, v_{n-1}\}$, a hop hub set of P_n such that $|S| = n - 2$.

To show that S is a minimum hop hub set of P_n, if v_i, $2 \leq i \leq n - 1$ is removed form set S, then there does not exist S-path between v_1 and v_n. Thus S is minimum hop hub set. Also since $h_h(P_n) \geq h(P_n)$ and $h(P_n) = n - 2$ then $h_h(P_n) = n - 2$, this complete the proof.

2. Since all vertices in K_n are adjacent and the distance between them equal one, so we must choose all vertices as hop hub set of K_n, and hence $h_h(K_n) = n$.

3. The following cases are considered:

   Case 1: When $n = 4$, then $W_{1,3} \cong K_4$, since $h_h(K_4) = 4$ we get the resuit.

   Case 2: When $n \geq 5$. Let $V(W_{1,n-1}) = \{v, v_1, v_2, \cdots, v_{n-1}\}$ and consider $S = \{v, v_1, v_2\}$ a S-set of $W_{1,n-1}$. We claim that S is a minimum hop hub set of $W_{1,n-1}$, for any $x, y \in V - S$ there exists S-path between them. Now if v is removed form S, there is no S-path between any two vertices outside the set S. Thus the set S is not hub set, and removal of v_1 or v_2 form S, leads to existence some vertex $v_i \in V - S$ such that $d(v_1, v_i) = 1$ or $d(v_2, v_i) = 1$. Thus S is a minimum hop hub set. Hence, $h_h(W_{1,n-1}) = 3$.

4. Let $V(K_{p,q}) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_q\}$. Consider $S = \{v_1, u_1\}$ is a hop hub set of $K_{p,q}$ such that $|S| = 2$. To show that S is a minimum hop hub set of $K_{p,q}$, if we remove v_1 of S, the set $S = \{u_1\}$ is not hop hub set because there exist $v_i, 2 \leq i \leq p$ such that $d(v_i, u) \neq 2$ and this does not achieve the definition of hop hub set. If we remove u_1 of S, the set $S = \{v_1\}$ is not hop hub set because there exist $u_i, 2 \leq i \leq q$ such that $d(u_i, v) \neq 2$ and this does not achieve the definition of hop hub set. Therefore, $h_h(K_{p,q}) = 2$.

5. Let $V(S_{p,q}) = \{v, v_1, v_2, \ldots, v_p, u, u_1, u_2, \ldots, u_q\}$. Consider $S = \{v, u\}$ is hub set of $S_{p,q}$ by Theorem 2.1.3, $h(S_{p,q}) = 2$, we prove S-set is hop hub set of $S_{p,q}$, any $v_i \in V(S_{p,q}) - S$ there exist $u \in S$ such that $d(v_i, u) = 2$ and any $u_i \in V(S_{p,q}) - S$ there exists $v \in S$ such that $d(u_i, v) = 2$, then S is a hop hub set of $S_{p,q}$, and $h_h(G) \geq h(G)$, then S-set is minimum hop hub set of $S_{p,q}$, and $h_h(S_{p,q}) = 2$.

6. We have the following cases:

   Case 1: For $n = 3$, since $C_3 \cong K_3$, we get the result by Proposition 2.2.2 part

2, $h_h(C_3) = 3$.

Case 2: When $n = 4$, consider $S = \{v_1, v_2\}$ is a hop hub set of C_4 as shown in Figure 2.3. It is clear that S is a minimum hop hub set, so $h_h(C_4) = 2$.

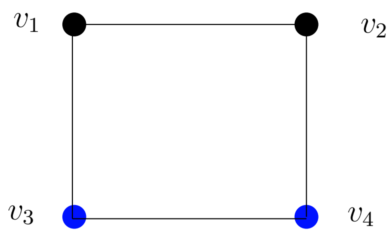

Figure 2.3: Cycle C_4

Case 3: When $n \geq 5$. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Consider $S = \{v_4, v_5, \ldots, v_n\}$ is a hop hub set of C_n, such that $|S| = n-3$. To show that S is a minimum hop hub set of C_n. By Theorem 2.1.11, $h(C_n) = n - 3$ and since $h_h(C_n) \geq h(C_n)$, also any vertex $v \in V(C_n) - S$ there exist v_i, $4 \leq i \leq n$ such that $d(v_i, v) = 2$, it follows that S is a minimum hop hub set of C_n, and hence $h_h(C_n) = n - 3$.

□

Theorem 2.2.3. *For any connected graph G, $2 \leq h_h(G) \leq n$.*

Proof. Suppose that $h_h(G) < 2$, then $h_h(G) = 1$ and $S = \{v\}$, by the definition of hop hub set there exists $u \in V(G) - S$, such that $d(u, v) = 2$, also there exists u_1

adjacent of both u and v, but $S = \{v\}$ only, then $u_1 \in S$, a contraction the definition of hop hub set and upper bound is achieved of $G = K_n$. $\square$

Theorem 2.2.4. *Let G be a disconnected graph having $M_1, M_2, ..., M_l$ components. Then $h_h(G) = \min_{1\leq t\leq l}\{X_t\}$, where $X_t = h_h(M_t) + \sum_{i=1,i\neq t}^{l} |V(M_i)|$.*

Proof. Any hop hub set S of a graph G, must contains all the vertices of $t-1$ components, and the vertices of hop hub set of the remaining component. To show that S is minimum. The union of all components except one and taking the hop hub set of the remaining component, we can compute all hop hub sets of G, and more detailed $S = \bigcup_{i=1,i\neq j}^{t} M_i \cup H_h^t$, where H_h^t is a hop hub set of M_t.

Let $X_t = h_h(M_t) + \sum_{i=1,i\neq t}^{l} |V(M_i)|$, then $\min_{1\leq t\leq l}\{X_t\} = h_h(G)$. $\square$

Theorem 2.2.5. *For any graph G, $\gamma(G) \leq h_h(G) + 1$.*

Proof. By Theorem 2.1.15 and since $h_h(G) \geq h(G)$, we get the result $h_h(G) + 1 \geq \gamma(G)$. $\square$

Remark 2.2.1. In general, the inequality $h_h(G') \leq h_h(G)$ is not true for a subgraph G' of G, for the graph G and a subgraph G' shown in Figure 2.4, we have $h_h(G) = 2$, while $h_h(G') = 4$.

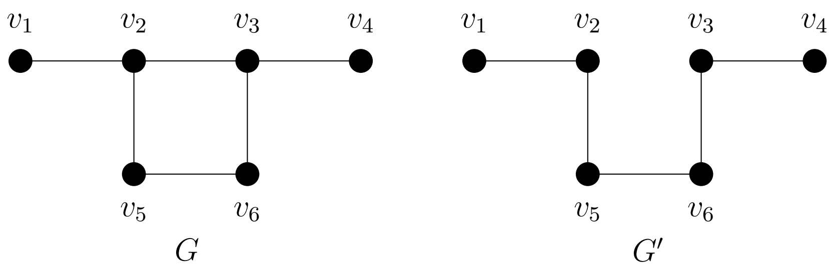

Figure 2.4: G and G'

Theorem 2.2.6. *Let G be a connected graph of order n, $h_h(G) = n$ if and only if $G \cong K_n$.*

Proof. Suppose that $h_h(G) = n$, this means that all vertices of at graph G are adjacent and hence $G \cong K_n$.

Conversely, if $G \cong K_n$, the proof follow form Proposition 2.2.2. □

Lemma 2.2.7. *Let S be a subset of $V(G)$. Then G/S is complete if and only if S is a hop hub set of G.*

Proof. By definition of hop hub set and since any hop hub set is hub set of G. By Theorem 2.1.14 it follows that G/S is complete graph. □

Theorem 2.2.8. *If G is complete graph, then $\chi(G) = h_h(G)$.*

Proof. Assume that G is complete graph and by Proposition 2.2.2, and Theorem 2.1.12, we get the result. $\square$

Remark 2.2.2. The converse of Theorem 2.2.8 is not true, for example $G = K_{1,3}$ as shown in Figure 2.5.

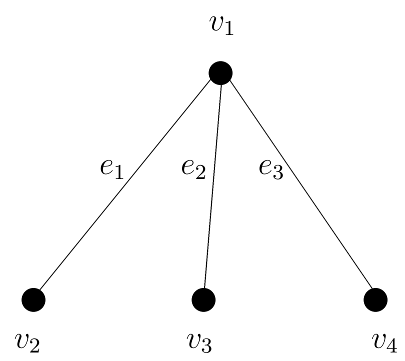

Figure 2.5: $K_{1,3}$

Note that $\chi(G) = 2$ and $h_h(G) = 2$, but G is not complete.

Theorem 2.2.9. *For any connected graph G, if $h_h(G) = 2$ then $diam(G) \leq 3$.*

Proof. Let $h_h(G) = 2$, we prove $diam(G) \leq 3$, suppose $diam(G) > 3$. Then by definition of $diam(G)$, there exists a path between five vertices at least and $h_h(P_5) = 3$, but this contradiction that $h_h(G) = 2$, then $diam(G) \leq 3$. $\square$

Remark 2.2.3. The converse of Theorem 2.2.9 is not true. For example K_3 such that $diam(K_3) = 1$ and $h_h(K_3) = 3$.

Theorem 2.2.10. *Let T be a tree, $h_h(T) = 2$ if and only if $diam(T) \leq 3$.*

Proof. Suppose that $h_h(T) = 2$, then $diam(T) \leq 3$ by Theorem 2.2.9.

Converse, suppose that $diam(T) \leq 3$ and we prove $h_h(T) = 2$. Since $diam(T) \leq 3$ and tree has not closed path, then the largest distance in T contains four vertices. Let T be P_4 since $h_h(P_4) = 2$ by Proposition 2.2.2, and without loss of generally, then $h_h(T) = 2$. □

Theorem 2.2.11. *For any connected graph G, $h_h(G) \geq h(G)$ and the inequality is sharp if $G \cong T$, and $h(G) \geq 3$.*

Proof. Form definition of hop hub set of G, $h_h(G) \geq h(G)$ and if $h(G) \geq 3$ and $G \cong T$ then $diam(G) \geq 3$ and for any vertex $u \in V - S$ there exists vertex $v \in S$ such that $d(u, v) = 2$, then $h_h(G) = h(G)$. □

Lemma 2.2.12. *Let T be a tree with n vertices and l leaves and i internal vertices, then $h_h(T) = h(T) = n - l$ such that $i \geq 3$.*

Proof. Since $i \geq 3$, then $h(T) \geq 3$. By using Theorem 2.2.11 and by Theorem 2.1.13, we get $h_h(T) = n - l$. □

2.3 Hop hub number of line graphs

Definition 2.3.1. *[28] The line graph $L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G.*

Proposition 2.3.2. *The hop hub number of some specific classes of graphs are as below*

1. *For any path P_n, $h_h(L(P_n)) = n - 3$.*

2. *For any cycle C_n,*

$$h_h(L(C_n)) = \begin{cases} 2, & \textit{if } n = 4, 5, \\ 3, & \textit{if } n = 3, \\ n-3, & \textit{if } n \geq 6. \end{cases}$$

3. *For any double star $S_{p,q}$, $h_h(L(S_{p,q})) = 3$.*

Proof. 1. Since $L(P_n) \cong P_{n-1}$, and by Proposition 2.2.2, then $h_h(L(P_n)) = n - 3$.

2. Since $L(C_n) \cong C_n$, and by Proposition 2.2.2, then $h_h(L(C_n)) = h_h(C_n)$.

3. The graph $L(S_{p,q})$ consists of two complete graphs of orders p, q respectively, and the vertex e that is adjacent to all vertices in $L(S_{p,q})$. The number of vertices of $L(S_{p,q})$ is $p+q = n$. The graphs $S_{p,q}$ and $L(S_{p,q})$ are shown in Figure 2.6. Consider $S = \{e, e_p, e'_q\}$ a hop hub set of $L(S_{p,q})$. Since e is adjacent to all vertices in $L(S_{p,q})$, for any vertex $e_i \in L(S_{p,q}), 1 \leq i \leq p-1$, there exist $e'_q \in S$ such that $d(e_i, e'_q) = 2$ and also for any vertex $e_j \in L(S_{p,q}), 1 \leq j \leq q-1$, there exists $e_p \in S$ such that $d(e'_j, e_p) = 2$. If we remove it from the graph $L(S_{p,q})$, there is no S-path between the vertices that are not adjacent. So S is a minimum hop hub set. Therefore $h_h(L(S_{p,q})) = 3$.

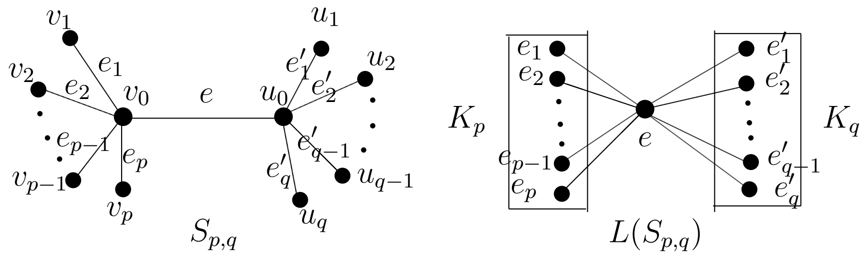

Figure 2.6: $S_{p,q}$ and $L(S_{p,q})$

$\square$

CHAPTER 3

Hop hub-integrity of graphs

Reference [42] is based on this chapter.

3.1 Introduction

In communication networks, we require greater degrees of stability or less vulnerability. The vulnerability measures resistance of the network to the disruption in operation after the failure of certain stations or communication links.

The stability of a communication network is of prime importance to network designers. As the network starts losing links or nodes, ultimately there is a loss in its efficiency. Thus, communication networks must be assembled to be as stable as possible, not only with respect to the initial interruption, but also with respect to the possible reconstruction of the network. The communication network can be represented as an undirected graph. Tree, hypercube and star graph are popular communication networks. If we model a network through graph, then there are many graph theoretical parameters to describe the stability of communication networks. Most notably, the vertex-connectivity and edge-connectivity have been frequently used. The best known measure of reliability of a graph is its vertex-connectivity $\kappa(G)$ defined to be the minimum number of vertices whose removal results in a disconnected or trivial graph. The difficulty with these parameters is that they do not consider what remains after the graph is disconnected. Consequently, a number of other parameters have recently been introduced in order to attempt to survive with this difficulty. The connectivity of the two different graphs may be same, but the orders of theirs largest components need not be same. Then they may differ in respect to stability. Now, how can we measure this property? The idea behind the answer is the concept of integrity, which is different from connectivity. The concept of integrity was introduced as a measure of graph vulnerability.

In [9], Barefoot, Entringer, and Swart introduced the integrity as a useful measure of the "vulnerability" of a graph G, and it is defined as follows.

Definition 3.1.1. *[9] $I(G) = \min\{|S| + m(G - S) : S \subseteq V(G)\}$, where $m(G - S)$ denotes the order of the largest component of $G - S$.*

Theorem 3.1.2. *[7] The integrity of some specific classes of graphs are as below:*

(a) *For any complete graph K_n, $I(K_n) = n$.*

(b) *For any path P_n, $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$.*

(c) *For any cycle C_n, $I(C_n) = \lceil 2\sqrt{n} \rceil - 1$.*

(d) *For the star $K_{1,n-1}$, $I(K_{1,n-1}) = 2$.*

(e) *For a complete bipartite graph $K_{p,q}$, $I(K_{p,q}) = 1 + min\{p, q\}$.*

(F) *For the null graph, $I(\overline{K_n}) = 1$.*

The integrity can be described as a measurement of connectivity of a graph. $|S|$ measures the volume of action needed to deteriorate or break a graph, and $m(G-S)$ is a measure of how much of the graph is yet intact. If we think of the graph as modeling a network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. The authors in [9] compared integrity, connectivity, toughness and binding number for several classes of graphs. They concluded that integrity is the most suitable measure of vulnerability because it is most capable to distinguish between graphs that intuitively should have various measures of vulnerability.

In 1987, Barefoot et al. [10] investigated the integrity of trees and powers of cycles.

Chartrand et al. [15] introduced the mean integrity of a graph, denoted $j(G)$, and is defined as $j(G) = min_{S \subseteq V(G)}\{|S| + \overline{m}(G - S)\}$, where, $\overline{m}(G - S) = \frac{\sum_{i=1}^{k} p_i^2}{\sum_{i=1}^{k} p_i}$ and $p_1, p_2, ..., p_k$ are the orders of components of $G - S$.

In 1988, Goddard and Swart [23] have introduced the concept of the integrity of the join, union, product and composition of two graphs and the integrity of a graph and its complement.

The integrity of a small class of regular graphs is studied by Atici and Crawford [4]. The authors in [3, 55], introduced the integrity of a cubic graphs. For more details of integrity see [2, 7, 20, 25, 43]

In [18] Moazzami et al. compared the integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. To know more about integrity and edge-integrity one can see [7, 8, 10, 24].

In 2015, Sultan et al. [34] have introduced the concept of hub-integrity of a graph as a new measure of vulnerability which is defined as follows.

Definition 3.1.3. *[34] The hub-integrity of a graph G denoted by $HI(G)$ is defined by, $HI(G) = \{min|S| + m(G-S)\}, S$ is a hub set of G, where $m(G-S)$ is the order of a maximum component of $G-S$.*

Proposition 3.1.4. *[34] The hub-integrity of some specific classes of graphs are as below:*

1. *For any complete graph K_n, $HI(K_n) = n$.*
2. *For any path P_n with $n \geq 3$, $HI(P_n) = n-1$.*
3. *For any cycle C_n,*
$$HI(C_n) = \begin{cases} n-1, & \text{if } n = 4,5\ ; \\ n-2, & \text{if } n \geq 6. \end{cases}$$
4. *For the star $K_{1,n-1}$, $HI(K_{1,n-1}) = 2$.*
5. *For the double star $S_{p,q}$, $HI(S_{p,q}) = 3$.*
6. *For the complete bipartite graph $K_{p,q}, HI(K_{p,q}) = \min\{p,q\} + 1$.*
7. *For the wheel graph $W_{1,n-1}$, $HI(W_{1,n-1}) = \lceil 2\sqrt{n-1} \rceil$.*
8. *For the complete k-bipartite graph $K_{n_1,n_2,\ldots\ldots,n_k}$,*
$$HI(K_{n_1,n_2,\ldots\ldots,n_k}) = \sum_{i=1}^{k} n_i + 1 - \max_{1\leq i\leq k} n_i.$$

For more details on the hub-integrity see [35, 38].

In 2019, Sultan and Veena [40] have introduced the concept of Hub-integrity of line graphs and includes results on the hub-integrity of line graphs of some graphs.

Remark 3.1.1. [40] The hub-integrity of graph G and hub-integrity of line graph are not comparable. For this situation consider the graphs in the following cases:

- In the star $K_{1,n-1}$, $HI(L(K_{1,n-1})) > HI(K_{1,n-1})$.

- In the cycle C_n, $HI(L(C_n)) = HI(C_n)$.

- In the path P_n, $n \geq 4, HI(L(P_n)) < HI(P_n)$.

This motivated us to introduce a new measure of stability of a graph G and it is called hop hub-integrity.

The following results will be useful in the proof of our results

Theorem 3.1.5. *[49] Let T be a tree with n vertices and l levels, Then $h(T) = h_h(G) = n - l$.*

Theorem 3.1.6. *[34] For any graph G, $\gamma(G) \leq HI(G)$.*

Theorem 3.1.7. *[33] If T is a binary tree order n with l terminal vertices, then T has $l-1$ internal vertices.*

3.2 The hop hub-integrity of graphs

Definition 3.2.1. *[49] A hub set S is a hop hub set of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop hub set of G is called a hop hub number and is denoted by $h_h(G)$.*

Definition 3.2.2. *The hop hub-integrity of a graph G is denoted as $H_hI(G) = min\{|S| + m(G - S)\}$, S is a hop hub set, where $m(G-S)$ is the order of a maximum component of $G - S$.*

A H_hI-set of G is any subset S of $V(G)$ for which $H_hI(G) = min\{|S| + m(G - S)\}$. For any disconnected graph G having k components $G_1, G_2, ..., G_k$ of orders $p_1, p_2, ..., p_{k-1}, p_k$, respectively such that $p_1 \leq p_2 \leq ... \leq p_{k-1} \leq p_k$. We have $H_hI(G) = p_1 + p_2 + ... + p_{k-1} + H_hI(G_k)$. Also, by the definition of hop hub-integrity we obtain the obvious bound $H_hI(G) \geq HI(G) \geq I(G)$.

Proposition 3.2.3. *The hop hub-integrity of some specific classes of graphs are as below:*

1. *For any complete graph K_n, $H_hI(K_n) = n$.*
2. *For any path P_n with $n \geq 4$, $H_hI(P_n) = n - 1$.*
3. *For the wheel graph $W_{1,n-1}$, $H_hI(W_{1,n-1}) = \lceil 2\sqrt{n-1} \rceil$.*

4. For the complete bipartite graph $K_{p,q}$, $H_hI(K_{p,q}) = 2 + min\{p-1, q-1\}$.

5. For the double star $S_{p,q}$, $H_hI(S_{p,q}) = 3$.

6. For any cycle C_n,

$$H_hI(C_n) = \begin{cases} n, & \text{if } n = 3, 4. \\ n-1, & \text{if } n \geq 5. \end{cases}$$

Remark 3.2.1. In general, the inequality $H_hI(G') \leq H_hI(G)$ is not true for a subgraph G' of G, for the graph G and a subgraph G' shown in Figure 3.1, we have $H_hI(G) = 4$, while $H_hI(G') = 5$.

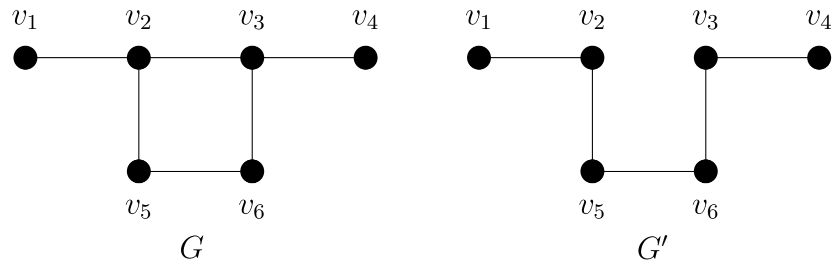

Figure 3.1: G and subgraph of G'

Proposition 3.2.4. *For any counted graph G, $2 \leq H_hI(G) \leq n$.*

The lower bound and the upper bound attains for a complete graph K_n, $n \geq 2$.

Theorem 3.2.5. *Let T be a tree with n vertices and l terminals vertices, such that internal vertices $i \geq 2$. Then $H_hI(G) = n - l + 1$.*

Proof. Let $H_hI(T) = |S| + m(S - T)$. The set $n - l$ of all internals vertices in T forms a hop hub set by Theorem 3.1.5, since the unique path between any two terminals never passes through another terminal. Note that any proper subset of $n - l$ cannot be a hop hub set. So $|S| = n - l$, since every internal vertex is a cut-vertex. If we delete of all $n - l$ vertices, we get one competent or more than two competent of order 1. So, $H_hI(T) = |S| + m(T - S) = n - l + 1$. □

Theorem 3.2.6. *For any tree T, $H_hI(T) \geq \alpha(T) + 1$.*

Proof. Let S' be a minimum covering set of T. Then

$$\begin{aligned} H_hI(T) &= |S| + m(T - S) \\ &\geq |S'| + m(T - S')(Because \quad S \geq S') \\ &\doteq |S'| + 1 \\ &\geq \alpha(T) + 1. \end{aligned}$$

□

Corollary 3.2.7. *For any graph G, $H_hI(G) \geq \gamma(G) + 1$.*

Proof. By using Theorem 3.1.6 and from $H_hI(G) \geq HI(G)$ we get result. □

Corollary 3.2.8. *For any graph G, $H_hI(G) \geq h_h(G)$, and if G is complete then is equality .*

Definition 3.2.9. *[33] A tree is called a binary tree if it has one vertex of degree 2 and each of the remaining vertices of degree 1 or 3. Clearly, P_3 is the smallest binary tree .*

Theorem 3.2.10. *If a tree T is a binary tree of order n. Then $H_hI(G) = \lceil n/2 \rceil$.*

Proof. Let $H_hI(T) = |S| + m(T - S)$. Since the hop hub set in any binary tree is i such that i is internal vertices, and by Theorem 3.1.7, we have $|S| = i = l - 1$, where l is the set of its number of terminal vertices of T. If we remove $l-1$ internal vertices from binary tree T we get a totally disconnected graph. So, $m(T-S) = 1$. Therefore, $H_hI(T) = l - 1 + 1 = l$. Since the number of terminal vertices in any binary tree equal $\lceil n/2 \rceil$, it follows that $l = \lceil n/2 \rceil$. Therefore $H_hI(T) = l = \lceil n/2 \rceil$. □

Theorem 3.2.11. *Let $G \cong K_n - e$, $e \in E(G)$. Then $H_hI(\overline{G}) = n$.*

Proof. If $G \cong K_n - e$, then $\overline{G} \cong K_2 \cup (n-2)K_1$. By definition a hop hub-integrity of disconnected graph, we have

$$\begin{aligned} H_hI(\overline{G}) &= n - 2 + H_hI(K_2) \\ &= n - 2 + 2 \\ &= n. \end{aligned}$$

□

3.3 Some properties of hop hub-integrity of line graphs

Definition 3.3.1. *[28] The line graph $L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G.*

Proposition 3.3.2.

- *In the star $K_{1,n-1}$, $H_hI(L(K_{1,n-1}) = n-1$.*

- *In the cycle C_n, $H_hI(L(C_n)) = H_hI(C_n)$.*

- *In the path P_n, $n \geq 4, H_hI(L(P_n) = H_hI(P_{n-1})$.*

- *In the double star $S_{p,q}$, $p, q \geq 2$, $H_hI(L(S_{p,q}) = min\{p-1, q-1\} + 3$.*

Remark 3.3.1. The hop hub-integrity of a graph G and hop hub-integrity of line graph are not comparable. For this situation consider the graphs in the following cases:

- In the star $K_{1,n-1}$, $H_hI(L(K_{1,n-1})) > H_hI(K_{1,n-1})$.

- In the cycle C_n, $H_hI(L(C_n)) = H_hI(C_n)$.

- In the path P_n, $n \geq 4, H_hI(L(P_n)) < H_hI(P_n)$.

Proposition 3.3.3. *For any path P_n, $n \geq 5$, $H_hI(L(P_n)) + H_hI(\overline{L(P_n)}) = 2n-4$.*

Theorem 3.3.4. *Let $G \cong K_n - e$, $e \in E(G)$. Then $H_hI(L(\overline{G})) = 1$.*

Proof. Since $G \cong K_n - e$, then $\overline{G} \cong K_2 \cup (n-2)K_1$, and $L(\overline{G}) \cong K_1$. Thus $H_hI(L(\overline{G})) = 1$. □

Proposition 3.3.5. *If G is regular graph of degree 2, then $H_hI(G) = H_hI(L(G))$.*

Proof. G is regular of degree 2, hence $G \cong C_n$, and $H_hI(C_n) = H_hI(L(C_n))$, so the result. □

Corollary 3.3.6. *Let G be a connected graph and let $\alpha(G) = 1$. Then*

$$H_hI(L(G)) = n - 1$$

.

Proof. Suppose $\alpha(G) = 1$, then $G \cong K_{1,n-1}$. Then $L(G) = K_{n-1}$, so proof follows from Proposition 3.3.2. □

Proposition 3.3.7. *If $H_hI(L(G)) = |E(G)|$, then $G \cong K_{1,n-1}$ or $G \cong C_3$.*

Theorem 3.3.8. *For any subset D of vertices in a graph $L(G)$, $H_hI(L(G) - D) \geq H_hI(L(G)) - |D|$.*

Proof. Let S be a H_hI- set of $L(G)$, let $D \leq V(L(G))$ and S^* be a H_hI-set of $L(G)-D$ such that $S^{**} = S^* \cup D$. Then $|S^{**}| = |S^*|+|D|$ and $L(G)-S^{**} = L(G)-(S^* \cup D) = (L(G) - D) - S^*$. Therefore

$$\begin{aligned}
H_hI(L(G)) &= |S| + m(L(G) - S) \\
&\leq |S^{**}| + m(L(G) - S^{**}) \\
&= |S^*| + |D| + m[(L(G) - D) - S^*] \\
&= HI(L(G) - D) + |D|.
\end{aligned}$$

□

Corollary 3.3.9. $H_hI(L(K_{1,n-1})) + H_hI(\overline{L(K_{1,n-1})}) = 2n - 2$.

Proof. Since $L(K_{1,n-1}) \cong K_{n-1}$, it follows from Proposition 3.3.2 such that $H_hI(K_{n-1}) = n - 1$, and $\overline{L(K_{1,n-1})} \cong \overline{K_{n-1}}$, so $H_hI(\overline{K_{n-1}}) = n - 1$, hence the result. □

Remark 3.3.2. If G is a connected graph, and $|E(L(G))| < |E(G)|$, then $H_hI(L(G)) < H_hI(G)$. We note that $|E(L(G))| < |E(G)|$ obtained only in a path graph, hence the result. But the converse is not true, for example, the graphs shown in Figure 3.2.

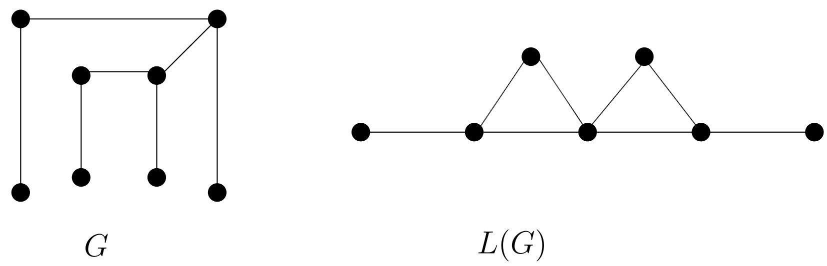

Figure 3.2: Graph G and $L(G)$

Note that $H_hI(G) = 5$ and $H_hI(L(G)) = 4$, while $|E(L(G))| > |E(G)|$.

CHAPTER 4

New results on hop hub number and hop hub-integrity of splitting graph

4.1 Introduction

Splitting graph $S'(G)$ was introduced by Sampath Kumar and Walikar [48]. For each vertex v of a graph G, take a new vertex v' and join v' to all vertices of G adjacent to v. The graph $S'(G)$ thus obtained is called the splitting graph of G.

Vaidya and Kothari [52] have discussed domination integrity of a graph obtained by duplication of an edge by vertex and duplication of vertex by an edge in path and cycle. Also Vaidya and Kothari [53] have discussed domination integrity of splitting graph of path and cycle.

In 2016, Sultan and Veena [38] have discussed hub-integrity of splitting graph and duplication of an edge by vertex and duplication of vertex by an edge of some graphs.

Theorem 4.1.1. *[38] For $n \geq 2$,*

$$HI(S'(P_n)) = \begin{cases} 2, & \text{if } n = 2, 3, \\ n-2, & \text{if } n \geq 4. \end{cases}$$

The following results will be useful in the proof of our results

Theorem 4.1.2. *[38] For $n \geq 2$, $h(S'(P_n)) = n - 2$, if $n \geq 4$.*

Theorem 4.1.3. *[38] For all $n \geq 3$,*

$$h(S'(C_n)) = \begin{cases} 2, & \text{if } n = 3, \\ n-2, & \text{if } n \geq 4. \end{cases}$$

Theorem 4.1.4. *[38] For all $n \geq 4$, $h(S'(K_{1,n-1})) = 2$.*

Theorem 4.1.5. *[38] For all $p, q \geq 2$, $h(S'(S_{p,q})) = 2$.*

4.2 The hop hub number of splitting graph

Definition 4.2.1. *[53] For a graph G the splitting graph $S'(G)$ of graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$ where $N(v)$ and $N(v')$ are the neighborhood sets of v and v', respectively.*

Theorem 4.2.2. *For $n \geq 2$,*

$$h_h(S'(P_n)) = \begin{cases} 2, & \text{if } n = 2, 3, \\ n-2, & \text{if } n \geq 4. \end{cases}$$

Proof. Let $\{u_1, u_2, ..., u_n\}$ be the vertices of path P_n and $\{v_1, v_2, ..., v_n\}$ be the new vertices corresponding to $\{u_1, u_2, ..., u_n\}$ which are added to obtain $S'(P_n)$. As $N(v_1) = \{u_2\}$, $N(v_n) = \{u_{n-1}\}$, $N(u_2) = \{u_1, u_3, v_1, v_3\}$ and $N(u_{n-1}) = \{u_{n-2}, u_n, v_n, v_{n-2}\}$, we have the three cases:

Case 1: For $n = 2$, $S'(P_2) \cong P_4$, as shown in Figure 4.1, and by Proposition 2.2.1. Hence $h_h(S'(P_2)) = 2$.

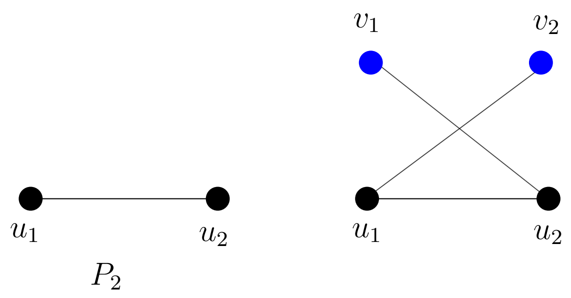

Figure 4.1: P_2 and $S'(P_2)$

Case 2: For $n = 3$, consider $S = \{u_1, u_2\}$ is a hop hub set for $S'(P_3)$ that is shown in Figure 4.2, it is clear the set S is a minimum hop hub set. Hence $h(S'(P_3)) = 2$.

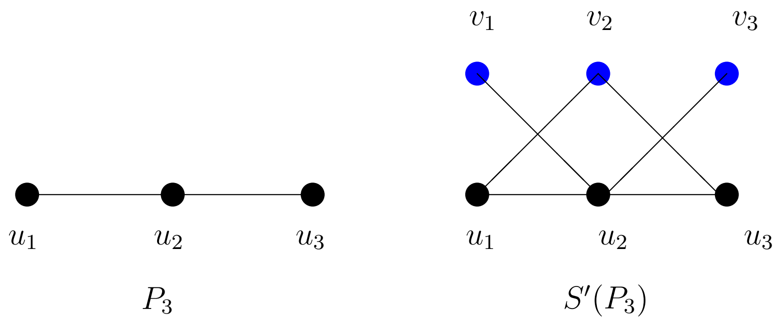

Figure 4.2: P_3 and $S'(P_3)$

Case 3: For $n \geq 4$, consider $S = \{u_2, u_3, ..., u_{n-1}\}$ is a hop hub set for $S'(P_n)$, as shown in Figure 4.3, and $|S| = n - 2$. As v_1 is adjacent to u_2 and v_n is adjacent to u_{n-1} and any vertex $x \in V(S'(P_n)) - S$ then there exist S-path between them, and any vertex $v_i \in V(S'(P_n)) - S$, $1 \leq i \leq n$ there exists $u_j \in S$, $2 \leq j \leq n-1$ such that

$d(v_i, u_j) = 2$. Then S is a hop hub set. Now we claim that set $S = \{u_2, ..., u_{n-1}\}$ is a minimum hop hub set. By Theorem 4.1.2 and $h_h(G) \geq h(G)$. Thus S is minimum hop hub set. Hence $S'(P_n) = n - 2$.

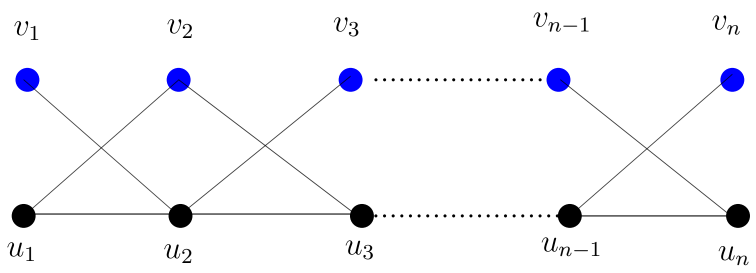

Figure 4.3: Splitting graph of path P_n

□

Theorem 4.2.3. *For all $n \geq 3$,*

$$h_h(S'(C_n)) = \begin{cases} 3, & \text{if } n = 3, \\ n-2, & \text{if } n \geq 4. \end{cases}$$

Proof. Let $\{u_1, u_2, ..., u_n\}$ be the vertices of cycle C_n and $\{v_1, v_2, ..., v_n\}$ be the new vertices corresponding to $\{u_1, u_2, ..., u_n\}$ which are added to obtain $S'(C_n)$. We have the two following cases:

Case 1: $n = 3$. Consider $S = \{u_1, u_2, u_3\}$ a hop hub set of $S'(C_3)$, as shown in Figure 4.4, and any vertices $v_i \in (S'(C_3)) - S$, $1 \leq i \leq 3$ there exist $u_j \in S$, such that

$1 \leq j \leq 3$, $d(v_i, u_j) = 2$, then S is hop hub set. We claim that set $S = \{u_1, u_2, u_3\}$ is a minimum hop hub set. If we remove any vertex $u_j \in S$, such that $1 \leq j \leq 3$ there exist $v_i \in (S'(C_3)) - S$ such that $d(v_i, u_j) \neq 2$. Thus S is minimum hop hub set. Hence $h_h(S'(C_3)) = 3$.

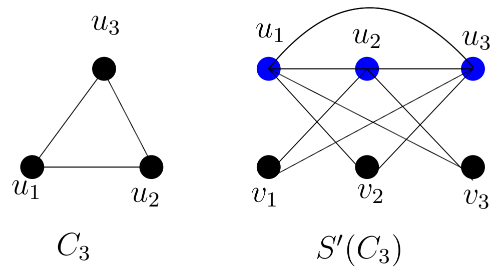

Figure 4.4: C_3 and $S'(C_3)$

Case 2: $n \geq 4$. Consider $S = \{u_1, u_2, ..., u_{n-2}\}$, a hop hub set of $S'(C_n)$, as shown in Figure 4.5, and $|S| = n - 2$. As $N(u_1) = \{u_2, v_2, u_n, v_n\}$ and $N(u_{n-2}) = \{u_{n-3}, v_{n-3}, u_{n-1}, v_{n-1}\}$, then any vertex in $S'(C_n)$ there exist path between them, and any vertex $v_i \in V(S'(C_n)) - S$ such that $1 \leq i \leq n$ exists $u_j \in S$ such that $d(v_i, u_j) = 2$ and any vertex $u_i \in V(S'(C_n)) - S$ such that $n - 1 \leq i \leq n$ there exist $u_j \in S$ such that $d(u_i, u_j) = 2$, then S is a hop hub set of $S'(C_n)$. Now we claim that set $S = \{u_1, u_2, ..., u_{n-2}\}$ is a minimum hop hub set. By Theorem 4.1.3 $h(S'(C_n)) = n - 2$ and $h_h(G) \geq h(G)$. Thus S is the minimum hop hub set. Hence $h_h(S'(C_n)) = n - 2$. $\square$

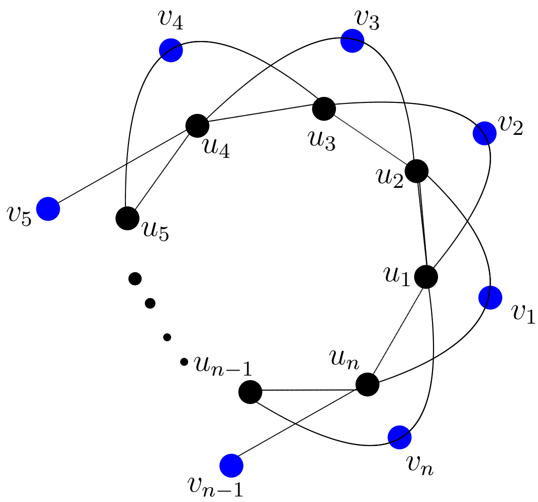

Figure 4.5: Splitting graph of cycle C_n

Theorem 4.2.4. *For all $n \geq 2$, $h_h(S'(K_{1,n-1})) = 2$.*

Proof. Let $\{u, u_1, ..., u_{n-1}\}$ be the vertices of $K_{1,n-1}$ and $\{v, v_1, ..., v_{n-1}\}$ be the new vertices corresponding to $\{u, u_1, ..., u_{n-1}\}$ show in Figure 4.6, which are added to obtain $S'(K_{1,n-1})$. Consider $S = \{u, u_1\}$ a hop hub set of $S'(K_{1,n-1})$. Since $N(u) = \{u_1, u_2, ..., u_{n-1}, v_1, v_2, ..., v_{n-1}\}$, and $N(u_1) = \{v, u\}$, then there exist S-path between them and any $v_i, 1 \leq i \leq n-1$ there exists $u_1 \in S$ such that $d(v_i, u_1) = 2$ and any $u_i, 2 \leq i \leq n-1$ exists $u_1 \in S$ such that $d(u_i, u_1) = 2$ and the vertex v also there exists $u \in S$ such that $d(v, u) = 2$, then S is hop hub set of $S'(K_{1,n-1})$. We claim that S is a minimum hop hub set of $S'(K_{1,n-1})$. By Theorem 4.1.4, and since $h_h(G) \geq h(G)$ then S is a minimum hop hub set of $S'(K_{1,n-1})$. Hence $h_h(S'(K_{1,n-1})) = 2$. $\square$

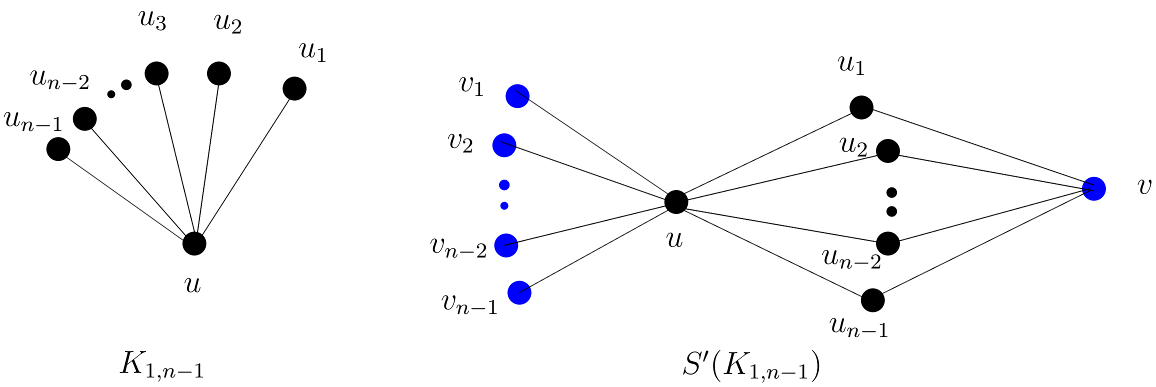

Figure 4.6: $K_{1,n-1}$ and splitting graph of star $S'(K_{1,n-1})$

Theorem 4.2.5. *For all $p, q \geq 2$, $h_h(S'(S_{p,q})) = 2$.*

Proof. Let $\{u, u_1, u_2, ..., u_{p-1}, v, v_1, v_2, ..., v_{q-1}\}$ be the vertex set of double star $S_{p,q}$ and $\{u', u'_1, u'_2, ..., u'_{p-1}, v', v'_1, v'_2, ..., v'_{q-1}\}$ be the new vertices corresponding to $\{u, u_1, u_2, ..., u_{p-1}, v, v_1, v_2, ..., v_{q-1}\}$ which are added to obtain $S'(S_{p,q})$ show in Figure 4.7. Consider $S = \{u, v\}$ is a hop hub set of $S'(S_{p,q})$ and $|S| = 2$. Since $N(u) = \{v, v', u_1, u_2, ..., u_{p-1}, u'_1, u'_2, ..., u'_{p-1}\}$ and $N(v) = \{u, u', v_1, v_2, ..., v_{q-1}, v'_1, v'_2, ..., v_{q-1}\}$, Then there exist S-path between them, and any vertex $x \in \{v', u_1, u_2, ..., u_{p-1}, u'_1, u'_2, ..., u'_{p-1}\} \subset V(S'(S_{p,q})) - S$, there exists $v \in S$ such that $d(v, x) = 2$, and any vertex $y \in \{u', v_1, v_2, ..., v_{q-1}, v'_1, v'_2, ..., v_{q-1}\} \subset V(S'(S_{p,q})) - S$ there exists $u \in S$ such that $d(u, y) = 2$, then S is a hop hub set. We claim that S is a minimum hop hub set of $S'(S_{p,q})$. Let us claim that S is a minimum hop hub set of $S'(S_{p,q})$. By Theorem 4.1.5, Since $h_h(G) \geq h(G)$ then S is a minimum hop hub set of $S'(S_{p,q})$. Hence

$h_h(S'(S_{p,q})) = 2.$

$\square$

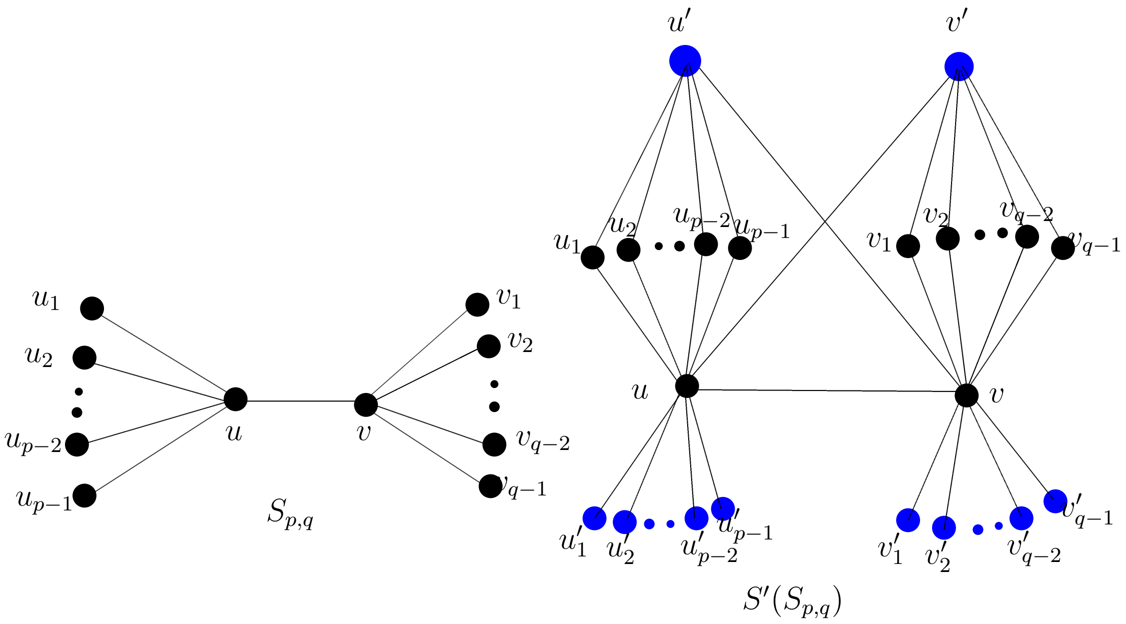

Figure 4.7: $S_{p,q}$ and splitting graph of double star $S'(S_{p,q})$

4.3 The hop hub-integrity of splitting graph

Theorem 4.3.1. *For $n \geq 2$,*

$$H_hI(S'(P_n)) = \begin{cases} 3, & \text{if } n = 2, \\ 4, & \text{if } n = 3, \\ n, & \text{if } n \geq 4. \end{cases}$$

Proof. Let $\{u_1, u_2, ..., u_n\}$ be the vertices of path P_n and $\{v_1, v_2, ..., v_n\}$ be the new vertices corresponding to $\{u_1, u_2, ..., u_n\}$ which are added to obtain $S'(P_n)$. We have the following cases:

Case 1: For $n = 2$. From Theorem 4.2.2, we have $h_h(S'(P_2)) = 2$ and $H = \{u_1, u_2\}$ is a hop hub set of $S'(P_2)$. Then $m(S'(P_2) - H) = 1$. This implies that $H_hI(S'(P_2)) = h_h(S'(P_2)) + m(S'(P_2) - H) = 2 + 1 = 3$. Clearly there does not exist any hop hub set S_1 of $S'(P_2)$ such that $|S_1| + m(S'(P_2) - S_1) \leq h_h(S'(P_2)) + m(S'(P_2) - H)$. Hence, $H_hI(S'(P_2)) = 3$.

Case 2: For $n = 3$. From Theorem 4.2.2, we have $h_h(S'(P_3)) = 2$ and $H = \{u_1, u_2\}$ is a hop hub-set of $S'(P_3)$. Then $m(S'(P_3) - H) = 2$. This implies that $H_hI(S'(P_3)) = h_h(S'(P_3)) + m(S'(P_3) - H) = 2 + 2 = 4$. Moreover, for any hop hub set S of $S'(P_3)$ we have, $|S| + m(S'(P_3) - S) \geq |H| + m(S'(P_3) - H)$. Hence $H_hI(S'(P_3)) = 4$.

Case 3: For $n \geq 4$. From Theorem 4.2.2, we have $h_h(S'(P_n)) = n - 2$. Let $H = \{u_2, u_3, ..., u_{n-1}\}$ be a hop hub-set of graph $S'(P_n)$. Then $m(S'(P_n) - H) = 2$. Therefore,

$$H_hI(S'(P_n)) \leq h_h(S'(P_n)) + m(S'(P_n) - H) = n - 2 + 2 = n. \tag{4.3.1}$$

For showing that the number $|H| + m(S'(P_n) - H)$ is minimum. The minimality of both $|H|$ and $m(S'(P_n) - H)$ is taken into consideration. The minimality of $|H|$ is guaranteed as H is hop hub-set. It remains to show that if S is any hop hub set other then H, $|S| + m(S'(P_n) - S) \geq n$. If $m(S'(P_n) - S) = 1$, then $|S| \geq n > n - 1$, consequently $|S| + m(S'(P_n) - S) \geq n + 1$. If $m(S'(P_n) - S) \geq 2$, then trivially

$|S| + m(S'(P_n) - S) \geq n$. Hence for any hop hub set S,

$$|S| + m(S'(P_n) - S) \geq n. \tag{4.3.2}$$

From (4.3.1) and (4.3.2), $H_hI(S'(P_n)) = n$.

□

Theorem 4.3.2. *For all $n \geq 3$,*

$$H_hI(S'(C_n)) = \begin{cases} 4, & \text{if } n = 3, \\ n+1, & \text{if } n \geq 4. \end{cases}$$

Proof. Let $\{u_1, u_2, ..., u_n\}$ be the vertices of cycle C_n and $\{v_1, v_2, ..., v_n\}$ be the new vertices corresponding to$\{u_1, u_2, ..., u_n\}$ which are added to obtain $S'(C_n)$. We have the three following cases:

Case 1: For $n = 3$. From Theorem 4.2.3, we have $h_h(S'(C_3)) = 3$, and $H = \{u_1, u_2, u_3\}$ is a hop hub-set of $S'(C_3)$. Then $m(S'(C_3) - H) = 1$.

This implies that $H_hI(S'(C_3)) = h_h(S'(C_3)) + m(S'(C_3) - H) = 3 + 1 = 4$. Clearly there does not exist any hop hub set S_1 of $S'(C_3)$ such that $|S_1| + m(S'(C_3) - S_1) \leq h_h(S'(C_3)) + m(S'(C_3) - H)$. Hence, $H_hI(S'(C_3)) = 4$.

Case 2: $n \geq 4$. From Theorem 4.2.3, we have $h_h(S'(C_n)) = n - 2$ and $H = \{u_1, u_2, ..., u_{n-2}\}$ is a hop hub-set of $S'(C_n)$. Then $m(S'(C_n) - H) = 6$. Therefore

$$H_hI(S'(C_n)) \leq h_h(S'(C_n)) + m(S'(C_n) - H) = n - 2 + 6 = n + 4. \tag{4.3.3}$$

If S_1 is any hop hub set of $S'(C_n)$ other than H with $m(S'(C_n) - S_1) = 4$ or 5, then $|S_1| \geq h_h(S'(C_n)) = n - 2$. This implies that

$$|S_1| + m(S'(C_n) - S_1) \geq h_h(S'(C_n)) + 4 = n - 2 + 4 = n + 2. \tag{4.3.4}$$

If S_2 is any hop hub set of $S'(C_n)$ other than H with $m(S'(C_n) - S_2) = 2$ or 3, then $|S_2| \geq n - 1$. This implies that

$$|S_2| + m(S'(C_n) - S_2) \geq n - 1 + 3 = n + 2. \tag{4.3.5}$$

Let $S_3 = \{u_1, u_2, ..., u_n\}$, a hop hub set of $S'(C_n)$, then $m(S'(C_n) - S_3) = 1$. This implies that

$$|S_3| + m(S'(C_n) - S_3) = n + 1. \tag{4.3.6}$$

Hence from (4.3.3), (4.3.4), (4.3.5) and (4.3.6), $H_hI(S'(C_n)) = n + 1$.

□

Theorem 4.3.3. *For all $n \geq 4$, $H_hI(S'(K_{1,n-1})) = 4$.*

Proof. Let $\{u, u_1, ..., u_{n-1}\}$ be the vertices of star $K_{1,n-1}$ and $\{v, v_1, ..., v_{n-1}\}$ be the new vertices corresponding to $\{u, u_1, ..., u_{n-1}\}$ which are added to obtain $S'(K_{1,n-1})$. From Theorem 4.2.4, we have $h_h(S'(K_{1,n-1})) = 2$ and $m(S'(K_{1,n-1}) - H) = n$, then

$$H_hI(S'(K_{1,n-1})) \leq h_h(S'(K_{1,n-1})) + m(S'(K_{1,n-1}) - H) = n + 2. \tag{4.3.7}$$

If $H = \{u, v, u_1\}$ is a hop hub-set of $S'(K_{1,n-1})$. Then $m(S'(K_{1,n-1}) - H) = 1$. Therefore,

$$H_hI(S'(K_{1,n-1})) = |H| + m(S'(K_{1,n-1}) - H) = 3 + 1 = 4. \tag{4.3.8}$$

To show that the number $|H| + m(S'(K_{1,n-1}) - H)$ is minimum, it is assumed that S is any hop hub set other than H and $m(S'(K_{1,n-1}) - S) > 1$, and $|S| \geq 3$, then $|S| + m(S'(K_{1,n-1}) - S) > 1 + 3 = 4$. Hence for any hop hub set S,

$$|S| + m(S'(K_{1,n-1}) - S) > h_h(S'(K_{1,n-1})) + 1. \tag{4.3.9}$$

From (4.3.8) and (4.3.9), we have $H_hI(S'(K_{1,n-1})) = 4$.

$\square$

Theorem 4.3.4. *For all* $p, q \geq 2$, $H_hI(S'(S_{p,q})) = 5$.

Proof. Let $\{u, u_1, u_2, ..., u_{p-1}, v, v_1, v_2, ..., v_{q-1}\}$ be the vertex set of double star $S_{p,q}$ and $\{u', u'_1, u'_2, ..., u'_{p-1}, v', v'_1, v'_2, ..., v'_{q-1}\}$ be the new vertices corresponding to $\{u, u_1, u_2, ..., u_{p-1}, v, v_1, v_2, ..., v_{q-1}\}$ which are added to obtain $S'(S_{p,q})$. Consider $S = \{u, v\}$, a hop hub set of $S'(S_{p,q})$.

Case 1: For $p = q = 2$. From Theorem 4.2.5, we have $h_h(S'(S_{2,2})) = 2$ and $S = \{u, v\}$ is a hop hub-set of $S'(S_{2,2})$. Then $m(S'(S_{2,2}) - S) = 3$. Therefore

$$H_hI(S'(S_{2,2})) \leq h_h(S'(S_{2,2})) + m(S'(S_{2,2}) - S) = 5. \tag{4.3.10}$$

Consider S_1 is any hop hub set of $S'(S_{2,2})$ other than S with $m(S'(S_{2,2}) - S_1) = 2$, then $|S_1| \geq 4$. This implies that

$$|S_1| + m(S'(S_{2,2}) - S_1) \geq 2 + 4 = 6. \tag{4.3.11}$$

Let $S_2 = \{u, v, u', v'\}$ be a hop hub set of $S'(S_{2,2})$, then $m(S'(S_{2,2}) - S_2) = 1$. This implies that

$$|S_2| + m(S'(S_{2,2}) - S_2) = 4 + 1 = 5. \tag{4.3.12}$$

Hence from (4.3.10), (4.3.11) and (4.3.12), $H_hI(S'(S_{2,2})) = 5$.

Case 2: For $p \geq 2, q > 2$ or $p > 2, q \geq 2$.

From Theorem 4.2.5, $h_h(S'(S_{p,q})) = 2$, and $S = \{u, v\}$ is a hop hub set of $S'(S_{p,q})$. Then $m(S'(S_{p,q}) - S) = max\{p + 1, q + 1\}$. Therefore

$$H_hI(S'(S_{p,q})) \leq h_h(S'(S_{p,q})) + m(S'(S_{p,q}) - S) = 2 + max\{p + 1, q + 1\}. \tag{4.3.13}$$

Consider $S_1 = \{u, v, u', v'\}$ a hop hub set of $S'(S_{p,q})$, then $m(S'(S_{p,q}) - S_1) = 1$. This implies that

$$|S_1| + m(S'(S_{p,q}) - S_1) = 4 + 1 = 5. \tag{4.3.14}$$

We claim that S_1 is a minimum hop hub set. Since u is adjacent to $\{v, v', u_1, ..., u_p, u'_1, ..., u'_p\}$, and removal of u from S_1 leads to nonexistence of S_1-path between u_i and u'_i, it follows that S_1 is a minimum hop hub set. Hence from (4.3.13) and (4.3.14), $H_hI(S'(S_{p,q})) = 5$. □

Theorem 4.3.5. *For any wheel* $W_{1,n-1}$*,* $H_hI(S'(W_{1,n-1})) = n + 1$.

Proof. Since $S'(W_{1,n-1})$ contains a wheel graph $W_{1,n-1}$ as its subgraph. If we choose the set S as all vertices of $W_{1,n-1}$ of $S'(W_{1,n-1})$, then there exist n components each contains only one vertex. So $H_hI(S'(W_{1,n-1})) = n + 1$. □

CHAPTER 5

Hop hubtic number and hop hub polynomial of graphs

Reference [50] is based on this chapter.

5.1 Introduction

A set D of vertices in a graph G is called dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D, the minimum cardinality of a dominating set in G is called the domination number $\gamma(G)$ of a graph G [29].

In 1977, E. J. Cockayne and S. T. Hedetniemi introduced the concept of domatic number of graph G and defined by, a D-partition of G is a partition of $V(G)$ into dominating sets, the domatic number of G denoted by $d(G)$ is the maximum order of a D-partition of G [17].

Introduced by Shadi, Veena, and Sultan [31] the maximum order of partition of the vertex set $V(G)$ in to hub sets is called hubtic number of G, and denoted by $\xi(G)$. A H-partition of a graph G is a partition of $V(G)$ into hub sets.

Observation 5.1.1. *[31]*

(1) *For any complete graph K_n, $\xi(K_n) = n$.*

(2) *For any cycle C_n,*

$$\xi(C_n) = \begin{cases} 3, & \text{if } n = 3, \\ 4, & \text{if } n = 4, \\ 2, & \text{if } n = 5, 6, \\ 1, & \text{if } n \geq 7. \end{cases}$$

(3) *For any path* P_n,

$$\xi(P_n) = \begin{cases} 2, & \text{if } n = 2, \\ 3, & \text{if } n = 3, \\ 2, & \text{if } n = 4, \\ 1, & \text{if } n \geq 5. \end{cases}$$

(4) *For the wheel graph* $W_{1,n-1}$, $n \geq 4$,

$$\xi(W_{1,n-1}) = \begin{cases} 4, & \text{if } n = 4, \\ 5, & \text{if } n = 5, \\ 3, & \text{if } n = 6, 7, \\ 2, & \text{if } n \geq 8. \end{cases}$$

(5) *For the star* $K_{1,n-1}$, $n \geq 4$ $\xi(K_{1,n-1}) = 2$.

(6) *For the double star* $S_{p,q}$, $\xi(S_{p,q}) = 2$.

(7) *For the complete bipartite graph* $K_{p,q}$, $p, q \geq 3$, $\xi(K_{p,q}) = min\{p, q\}$

Using the concept of hop hub set of a graph G and the definition of the hubtic number of a graph G, motivated by this, we introduce the concept of hop hubtic number of a graph G as a new parameter of a graph.

The following results will be useful in the proof of our results

Proposition 5.1.2. *[17] For any graph* G, $daim(G) \leq \delta(G) + 1$.

Lemma 5.1.3. *[49] Let T be a tree with n vertices and l leaves and i internal vertices, then $h_h(T) = h(T) = n - l$ such that $i \geq 3$.*

Proposition 5.1.4. *[49] The hop hub numbers of some specific classes of graphs are as below:*

1. *For any path P_n,*

$$h_h(P_n) = \begin{cases} 2, & \text{if } n = 2, \\ 3, & \text{if } n = 3, \\ n-2, & \text{if } n \geq 4. \end{cases}$$

2. *For any complete graph K_n, $h_h(K_n) = n$.*

3. *For the wheel graph $W_{1,n-1}$,*

$$h_h(W_{1,n-1}) = \begin{cases} 4, & \text{if } n = 4, \\ 3, & \text{if } n \geq 5, \end{cases}$$

4. *For the complete bipartite graph $K_{p,q}$, $h_h(K_{p,q}) = 2$.*

5. *For the double star $S_{p,q}$, $h_h(S_{p,q}) = 2$.*

6. *For any cycle C_n,*

$$h_h(C_n) = \begin{cases} 2, & \text{if } n = 4, \\ 3, & \text{if } n = 3, \\ n-3, & \text{if } n \geq 5. \end{cases}$$

5.2 The hop hubtic number of graphs

Definition 5.2.1. *The maximum order of partition of the vertex set $V(G)$ in to hop hub sets is called hop hubtic number of G, and denoted by $h_\xi(G)$. A H_h-partition of a graph G is a partition of $V(G)$ in to hop hub sets.*

Example 5.2.2. *The Figure 5.1, shows a hop hubtic partition of a graph. The sets $S_1 = \{v_2, v_3\}$, and $S_2 = \{v_1, v_4, v_5, v_6\}$, are hop hub sets of G, so $h_\xi(G) = 2$.*

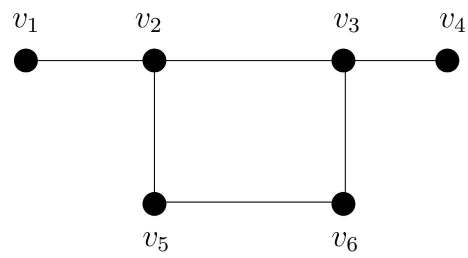

Figure 5.1: Graph (G)

Theorem 5.2.3. *For any connected graph G, $1 \leq h_\xi(G) \leq \lfloor \frac{n}{h_h(G)} \rfloor$.*

Proof. Let $H = \{H_1, H_2, H_3, ..., H_{h_\xi(G)}\}$ be the hop hubtic partition of graph G. Clearly, $|Hi| \geq h_h(G)$ for all $i = 1, 2, ..., h_\xi(G)$, so $h_\xi(G)|Hi| \geq h_\xi(G)h_h(G)$ for all $i = 1, 2, ..., h_\xi(G)$, then

$$n = \Sigma|Hi| \geq h_\xi(G)h_h(G).$$

Hence the assertion follows. □

By Theorem 5.2.3, we get the next result.

Proposition 5.2.4. *1. For any complete graph K_n, $h_\xi(K_n) = 1$.*

2. For any path P_n with $n \geq 5$, $h_\xi(P_n) = 1$.

3. For the wheel graph $W_{1,n-1}$, $h_\xi(W_{1,n-1}) = 1$.

4. For the complete bipartite graph $K_{p,q}$, $h_\xi(K_{p,q}) = min\{p, q\}$.

5. For the double star $S_{p,q}$, $h_\xi(S_{p,q}) = 2$.

6. For any cycle C_n,

$$h_\xi(C_n) = \begin{cases} 2, & \text{if } n = 4, 5, 6. \\ 1, & \text{if } n \geq 7. \end{cases}$$

Proof. 1- By Theorem 5.2.3, and Proposition 5.1.4 we have

$$\begin{aligned} h_\xi(K_n) &\leq \lfloor \frac{n}{h_h(K_n)} \rfloor \\ &= \lfloor \frac{n}{n} \rfloor \\ &= \lfloor 1 \rfloor \\ &= 1 \\ h_\xi(K_n) &\leq 1. \end{aligned}$$

And for any graph G, $h_\xi(K_n) \geq 1$, then $h_\xi(K_n) = 1$.

2- By Theorem 5.2.3, and Proposition 5.1.4,

$$
\begin{aligned}
h_\xi(P_n) &\leq \lfloor \frac{n}{h_h(P_n)} \rfloor \\
&= \lfloor \frac{n}{n-2} \rfloor \\
&= \lfloor \frac{n-2+2}{n-2} \rfloor \\
&= \lfloor \frac{n-2}{n-2} + \frac{2}{n-2} \rfloor \\
&= \lfloor 1 + \frac{2}{n-2} \rfloor \\
&= 1 + \lfloor \frac{2}{n-2} \rfloor, n \geq 5 \\
&= 1 + 0, \lfloor \frac{2}{n-2} \rfloor = 0, n \geq 5. \\
&= 1 \\
h_\xi(P_n) &\leq 1.
\end{aligned}
$$

And for any graph G, $h_\xi(P_n) \geq 1$, then $h_\xi(P_n) = 1$ if $n \geq 5$.

3- If $n = 4$, $W_{1,3} \cong K_4$ and $h_\xi(K_4) = 1$ form Proposition 5.2.4, part 1.

If $n \geq 5$, let $V(W_{1,n-1}) = \{v, v_1, v_2, \cdots, v_{n-1}\}$ and $h_h(W_{1,n-1}) = 3$, since v is center of $W_{1,n-1}$ adjacent any vertex in $W_{1,n-1}$, then $h_\xi(W_{1,n-1}) = 1$.

4- Let $V(K_{p,q}) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_q\}$. Consider $H_h = \{v_1, u_1\}$ is a hop hub set of $K_{p,q}$ such that $|H_h| = 2$, therefore any hop hub set H_h must contain $\{v_i, u_j\}$ such that $i \in p, j \in q$, therefore, the number of hop hub set depended on the minimum vertex i or j.

5- Let $V(S_{p,q}) = \{v, v_1, v_2, \ldots, v_p, u, u_1, u_2, \ldots, u_q\}$. There are two H_h-partition of

$V(S_{p,q})$ are $S_1 = \{v, u\}$ and $S_2 = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m\}$, then $h_\xi(S_{p,q}) = 2$.

6- Consider $V(C_n) = \{v_1, v_2, ..., v_n\}$ be the vertex of cycle, the following cases are discussed :

Case 1: When $n = 4$, there only two H_h sets of cardinality two, namely, $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_3, v_4\}$. Therefore, $h_\xi(C_4) = 2$.

Case 2: When $n = 5$, there only one H_h set of cardinality two, namely, $S_1 = \{v_1, v_2\}$ and one set of cardinality three, namely, $S_2 = \{v_3, v_4, v_5\}$. Then, $h_\xi(C_5) = 2$.

Case 3: When $n = 6$, there only two H_h sets of cardinality three, namely, $S_1 = \{v_1, v_2, v_3\}$ and $S_2 = \{v_4, v_5, v_6\}$. So, $h_\xi(C_6) = 2$.

Case 4: When $n \geq 7$. By Theorem 5.2.3, and Proposition5.1.4, we have

$$\begin{aligned}
h_\xi(C_n) &\leq \lfloor \frac{n}{h_h(C_n)} \rfloor \\
&= \lfloor \frac{n}{n-3} \rfloor \\
&= \lfloor \frac{n-3+3}{n-3} \rfloor \\
&= \lfloor \frac{n-3}{n-3} + \frac{3}{n-3} \rfloor \\
&= \lfloor 1 + \frac{3}{n-3} \rfloor \\
&= 1 + \lfloor \frac{3}{n-3} \rfloor \\
&= 1 + 0. \lfloor \frac{3}{n-3} \rfloor = 0, n \geq 7. \\
&= 1 \\
h_\xi(C_n) &\leq 1.
\end{aligned}$$

And for any graph G, $h_\xi(C_n) \geq 1$, so $h_\xi(C_n) = 1$.

$\square$

Theorem 5.2.5. *Let G be a tree with at least 3 non-leaf vertices. Then $h_\xi(G) = 1$.*

Proof. Suppose that G is a tree with at least 3 non-leaf vertices. We discuss the following cases:

Case 1. Let H_h be a set of all non-leaf vertices. Clearly, any path between two leaves does not pass through another leaf. So, H_h is a hop hub set of G, and by Lamma 5.1.3, it is a minimum hop hub set. Now, suppose that $D \subseteq V(G) - H_h$ is a hop hub set of G. Since G is a tree with at least 3 non-leaf vertices, take any two non-adjacent vertices $u, v \in H_h$. Since all vertices in D are leaves, then there is no path between u and v with all internal vertices in D. This is a contradiction. Hence $h_\xi(G) = 1$.

Case 2. Suppose that H_h is a hop hub set of G but not containing all non-leaf vertices. Since G has at least three non-leaf vertices, let $\{v_1, v_2, v_3\}$ be non-leaf vertices and $v_1v_3 \notin E(G)$, let l_1, l_3 be leaves adjacent to v_1 and v_3, respectively. Clearly, $G[\{l_1, v_1, v_2, v_3, l_3\}]$ is a path P_5. Since $h_h(P_5) = 3$, then H contains at least three vertices from P_5. Then any other hop hub set of G must intersects H since $|P_5| = 5$, therefore $h_\xi(G) = 1$. $\square$

Theorem 5.2.6. *For any graph G, $h_\xi(G) \leq \delta(G) + 1$.*

Proof. Suppose $h_\xi(G) > \delta(G) + 1$, We have the following cases :

Case 1: If $G \cong K_n$,

$$\begin{aligned} h_\xi(K_n) &> \delta(K_n) + 1 \\ 1 &> \delta(K_n) + 1 \\ 1 - 1 &> \delta(K_n) \\ 0 &> \delta(K_n). \end{aligned}$$

impossible

Case 2: If G is tree, hop hubtic of tree is 1 or 2 by Theorem 5.2.5.

If hop hubtic of tree is 1 and since $\delta(T) = 1$, then

$$\begin{aligned} h_\xi(T) &> 1 + 1 \\ h_\xi(T) &> 2 \\ 1 &> 2, \end{aligned}$$

this is impossible.

If hop hubtic of tree is 2, then

$$\begin{aligned} 2 &> 1 + 1 \\ 2 &> 2 \end{aligned}$$

this is impossible.

Case 3: Now If G is not tree, then there exist some graph such that $h_\xi(G) = \delta(G)$, so the relation $h_\xi(G) > \delta(G) + 1$ is not true. Therefore, $h_\xi(G) \leq \delta(G) + 1$. $\square$

Lemma 5.2.7. *For any graph G, $h_\xi(G) + daim(G) \leq 2\delta(G) + 2$.*

Proof. By Proposition 5.1.2, $daim(G) \leq \delta(G)+1$ and Theorem 5.2.6, $h_\xi(G) \leq \delta(G)+1$, then $h_\xi(G) + daim(G) \leq 2\delta(G) + 2$. □

5.3 Hop hub polynomial of graphs

In 2020, R. P. Veettil and T. V. Ramakrishnan [54] introduce hub polynomial of a connected graph G. The hub polynomial of a connected graph G of order n is the polynomial $H_G(x) = \sum_{i=h(G)}^{|V(G)|} h(G,i)x^i$ where hG, i denotes the number of hub sets of G of cardinality i and h is the hub number of G. And they obtain hub polynomial of some special classes of graphs and study hub roots of some graph G. Also obtain hub polynomial of join of two graphs.

Theorem 5.3.1. *[54] The hub polynomial of the path P_n is*

$$H_{P_n} = \binom{n}{2}x^{n-2} + \binom{n}{1}x^{n-1} + x^n$$

Theorem 5.3.2. *[54] The hub polynomial of the cycle C_n is*

$$H_{C_n} = \binom{n}{3}x^{n-3} + \binom{n}{2}x^{n-2} + \binom{n}{1}x^{n-1} + x^n$$

Theorem 5.3.3. *[54] The hub polynomial of the star graph $K_{1,n-1}, n \geq 3$ is*

$$H_{K_{1,n-1}}(x) = x(1+x)^n + nx^{n1} + x^n$$

5.4 Hop hub polynomial of standard graphs

Definition 5.4.1. *The hop hub polynomial of a connected graph G of order n is the polynomial*

$$H_h(G,x) = \sum_{i=h_h(G)}^{|V(G)|} h_h(G,i)x^i,$$

where $h_h(G,i)$ denotes the number of hop hub sets of G of cardinality i and $h_h(G)$ is the hop hub number of G.

To show this polynomial, we discuss this example.

Example 5.4.2. *Let G be a graph as shown in Figure 5.2.*

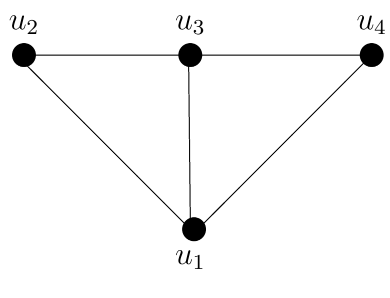

Figure 5.2: Graph (G)

We have $h_h(G) = 3$ such that $S_1 = \{u_1, u_2, u_3\}$, $S_2 = \{u_1, u_2, u_4\}$, $S_3 = \{u_1, u_3, u_4\}$ and $S_4 = \{u_1, u_2, u_3, u_4\}$, are H_h-sets of G.

Then, $H_h(G, x) = 3x^3 + x^4$.

Theorem 5.4.3. *Let G be a path P_n. Then*

$$H_h(P_n, x) = \begin{cases} x^2 & \text{if } n = 2, \\ 2x^2 + x^3 & \text{if } n = 3, \\ 4x^2 + 4x^3 + x^4 & \text{if } n = 4, \\ 7x^3 + 5x^4 + x^5 & \text{if } n = 5, \\ x^n + nx^{n-1} + (\binom{n}{2} - 2)x^{n-2} & \text{if } n \geq 6. \end{cases}$$

Proof. We have the following cases:

Case 1: When n = 2. Let $S = \{v_1, v_2\}$ is H_h set of P_2, hence $h_h(P_2) = 2$. By definition of hop hub polynomial,

$$H_h(P_2, x) = \sum_{i=2}^{2} h_h(P_2, i)x^i = x^2.$$

Case 2: When $n = 3$. Let Consider $V(P_3) = \{v_1, v_2, v_3\}$ and $h_h(P_3) = 2$. Also every subset S of vertex set of P_3 consisting of 2 elements, let it be $S_1 = \{v_1, v_2\}$ and $S_1 = \{v_2, v_3\}$, clearly, the number of sets that contain two elements are two sets. Also the number of sets that contain three elements is only one set. Then by definition of hop hub polynomial, we have

$$H_h(P_3, x) = \sum_{i=2}^{3} h_h(P_3, i)x^i = 2x^2 + x^3.$$

Case 3: When $n = 4$, consider $V(P_4) = \{v_1, v_2, v_3, v_4\}$ and $h_h(P_4) = 2$. Every subset S of vertex set of P_4 consisting of 2 elements, let it be $S_1 = \{v_1, v_2\}$, $S_2 = \{v_3, v_4\}$, $S_3 = \{v_1, v_4\}$ and $S_4 = \{v_2, v_3\}$, note that the number of sets that contain two elements are four sets. We also note that the number of sets that contain three elements is only four set $\binom{4}{3} = 4$. Note that the number of sets that contain four elements are one set. Then by definition of hop hub polynomial $H_h(P_4, x) = \sum_{i=2}^{4} h_h(G, i)x^i = 4x^2 + 4x^3 + x^4$.

Case 4: When $n = 5$, consider $V(P_5) = \{v_1, v_2, v_3, v_4, v_5\}$ and $h_h(P_5) = 3$. Also every subset S of vertex set of P_5 consisting of 3 elements is $\binom{5}{3} = 10$ but the sets $S_1 = \{v_2, v_4, v_5\}$, $S_2 = \{v_1, v_2, v_4\}$ and $S_3 = \{v_1, v_3, v_5\}$ are not hop hub set, then every subset S of vertex set of P_5 consisting of 3 elements is 7. We note that the number of sets that contain four elements are five sets such that $\binom{5}{4} = 5$. Note that the number of sets that contain five elements are one set. Then by definition of hop hub polynomial $H_h(P_5, x) = \sum_{i=2}^{5} h_h(G, i)x^i = 7x^3 + 5x^4 + x^5$.

Case 5: When $n \geq 6$. Let $P_n = \{v_1, v_2 ..., v_n\}$ be a path. Then we have $h_h(P_n) = n-2$ from Proposition 2.2.1. Also every subset of vertex set of P_n consisting of $n - 2$ elements and all its super sets form a hop hub set for the path P_n. Hence

$h_h(P_n, n-2) = \binom{n}{n-2} - 2 = \binom{n}{2} - 2.$

$h_h(P_n, n-1) = \binom{n}{n-1} = \binom{n}{1} = n.$

$h_h(P_n, n) = 1.$ $\square$

Theorem 5.4.4. *Let G be a cycle C_n. Then*

$$H_h(C_n, x) = \begin{cases} x^3 & \text{if } n = 3, \\ 4x^2 + 4x^3 + x^4 & \text{if } n = 4, \\ 18x^3 + 15x^4 + 6x^5 + x^6 & \text{if } n = 6, \\ -nx^{n-3} + \sum_{i=n-3}^{n} \binom{n}{n-3} x^{n-3} & \text{if } n{=}5 \text{ , } n \geq 7. \end{cases}$$

Proof. We have the following cases:

Case 1: When $n = 3$. Suppose that $V(C_3) = \{v_1, v_2, v_3\}$ and $h_h(C_3) = 3$. It is clear that, the number of sets that contain three elements is only one set. So by definition of hop hub polynomial, $H_h(C_3, x) = x^3$.

Case 2: When $n = 4$. Consider $V(C_4) = \{v_1, v_2, v_3, v_4\}$ and $h_h(C_4) = 2$. Also every hop hub subset S of vertex set of C_4 consisting of 2 elements. let it be $S_1 = \{v_1, v_2\}$, $S_2 = \{v_2, v_3\}$, $S_3 = \{v_3, v_4\}$ and $S_4 = \{v_1, v_4\}$. Also every hop hub sub set S of vertex set of C_4 consisting of 3 elements. let it be $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_2, v_3, v_4\}$, $S_3 = \{v_3, v_4, v_1\}$ and $S_4 = \{v_4, v_1, v_2\}$, so the number of sets that contain three element are four sets. Also the number of sets that contain four element is only one set. Then by definition of hop hub polynomial $H_h(P_4, x) = 4x^2 + 4x^3 + x^4$.

Case 3: When $n = 6$. Consider $V(C_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $h_h(C_6) = 3$. Also every subset of vertex set of C_6 consisting of 3 elements except subset S_1 and S_2 as in the following Figure 5.3, and all its super sets form a hop hub set for the cycle C_n. Hence

$h_h(C_6, 3) = \binom{6}{3} - 2 = 20 - 2 = 18.$

$h_h(C_6, 4) = \binom{6}{4} = 15.$

$h_h(C_6, 5) = \binom{6}{5} = 6.$

$h_h(C_6, 6) = \binom{6}{6} = 1.$

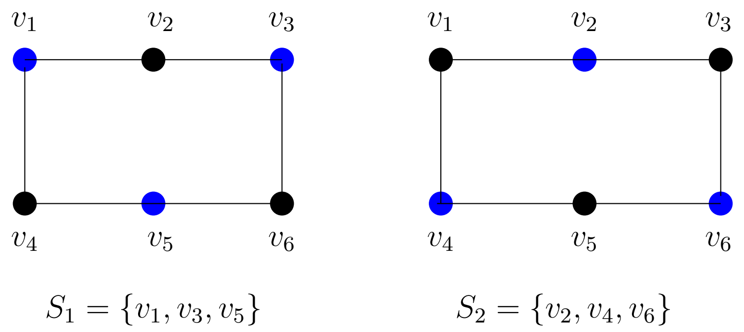

Figure 5.3: Cycle C_6

Case 4: When $n = 5$ and $n \geq 7$. Let $C_n = \{v_1, v_2, ..., v_n\}$ be a cycle. Then we have $h_h(C_n) = n - 3$. Also every subset of vertex set of C_n consisting of $n - 3$ elements and all its super sets form a hop hub set for the cycle C_n. Hence

$h_h(C_n, n-3) = \binom{n}{n-3} - n.$

$h_h(C_n, n-2) = \binom{n}{n-2}.$

$h_h(C_n, n-1) = \binom{n}{n-1}.$

$h_h(C_n, n) = \binom{n}{n} = 1.$ $\square$

Theorem 5.4.5. *For the complete graph K_n, $H_h(K_n, x) = x^n$.*

Proof. Since $h_h(K_n) = n$, and $h_h(K_n, n) = 1$. We have $H_h(K_n, x) = x^n$. $\square$

Proposition 5.4.6. *Let $G \cong \overline{K_n}$. Then $H_h(K_n, x) = x^n$.*

Theorem 5.4.7. *For the star graph $K_{1,n-1}$,*

$$H_h(K_{1,n-1}, x) = \sum_{i=1}^{n-1} \left[\binom{n}{i} - \binom{n-1}{i} \right] x^i + x^n.$$

Proof. Let $V(K_{1,n-1}) = \{v, v_1, v_1, ..., v_{n-1}\}$ is the vertices of $K_{1,n-1}$ and v is the central vertex of $K_{1,n-1}$, since $H_h = \{v, v_i\}$, $1 \leq i \leq n-1$, then every hop hub set of cardinality i must include the vertex v. The number of hop hub sets of cardinality 2 is $\binom{n}{2} - \binom{n-1}{2}$. The number of hop hub sets of cardinality 3 is $\binom{n}{3} - \binom{n-1}{3}$. So the number of hop hub sets of cardinality i is $\binom{n}{i} - \binom{n-1}{i}$, $i \leq n-1$. Therefore,

$$H_h(K_{1,n-1}, x) = \sum_{i=1}^{n-1} \left[\binom{n}{i} - \binom{n-1}{i} \right] x^i + x^n.$$

$\square$

Bibliography

[1] S. Akbari, S. Alikhani and Y. H. Peng, Characterization of graphs using domination polynomial, *European J.*, **31** (2010), 1714-1724.

[2] A. Atac and S. Celik, Vulnerability: integrity of a Middle graph, *J. Appl. Math.*, **9** (2008), 49-60.

[3] M. Atici, R. Crawford and C. Ernst, New upper bounds for the integrity of cubic graphs, *Int. J. Comput. Math.*, **81** (11) (2004), 1341-1348.

[4] M. Atici and R. Crawford, The integrity of small cage graphs, *Australas. J. Combin.*, **43** (2009), 39-55.

[5] S. K. Ayyaswamy and C. Natarajan, Hop domination in graphs II, *The Journal of 'Ovidius' University of Constant, Versite.*, **23 (2)** (2015), 187-199.

[6] S. K. Ayyaswamy, B. Krishnakumari, C. Natarajan and Y. B. Venkatakrishnan, Bounds on the Hop domination number of a tree, *Proc. Indian A Cad. Sci. (Math. Sci).*, **125(4)** (2015), 449-455.

[7] K. S. Bagga, L. W. Beineke, Wayne Goddard, M. J. Lipman and R. E. Pippert, A survey of integrity, *Discrete Applied Math.,* **37/38** (1992), 13-28.

[8] K. S. Bagga, L. W. Beineke, M. J. Lipman and R. E. Pippert, Edge-integrity: a survey, *Discrete Applied Math.,* **124** (1994), 3-12.

[9] C. A. Barefoot, R. Entringer and H. Swart, Vulnerability in graphs - A comparative survey, *J. Combin. Math. Combin. Comput.*, **1** (1987), 12-22.

[10] C. A. Barefoot, R. Entringer and H. Swart, Integrity of trees and powers of cycles, *Congr. Numer.*, **58** (1987), 103-114.

[11] M. Behzad, G. Chartrand, J. K, The cloring of copmlet graph, *Journal London Math. SOC.*, **42** (1967), 226-228.

[12] N. Biggs, Algebraic graph theory, *Second Ed., Cambridge University Press*, Cambridge, 1974.

[13] G. D. Birkhoff and D. C. Lewis, Chromatic polynomials, *Trans. Amer. Math. Soc.*, **60** (1946), 355-451.

[14] J. A. Bondy and U. S. R. Murthy, Graph theory, Springer, Berlin, 2008.

[15] G. Chartrand, S. F. Kapoor, T. A. McKee and O. R. Oellermann, The mean integrity of a graph, *in: Recent Studies in Graph Theory, (V.R. Kulli ed), Vishwa International Pub.*, (1989), 70-80.

[16] G. Chartrand and L. Lesniak, Graph and digraph, Chap. Hall/CRC, 1996.

[17] E. J. Cockayne, S. T. Hedetniemi, Towards a theory of domination in graphs, Networks 7 (1977), 247-261.

[18] M. Cozzens, D. Moazzami and S. Stueckle, *The tenacity of a graph*, Proc. Seventh International Conference on the Theory and Applications of Graphs, New York, USA, (1995), 1111-1122.

[19] E. C. Cuaresma. and R. N. Paluga, *On the hub number of some graphs*, Annals of Studies in Science and Humanities, **1 (1)** (2015), 17–24.

[20] P. Dundar and A. Aytac, Integrity of total graphs via certain parameters, *Math. Notes*, **76** (5) (2004), 665-672.

[21] E. J. Farrell, A note on the clique polynomial and its relation to other graph polynomials, *J. Math. Sci.*, (Calcutta), **8** (1997), 97-102.

[22] R. Frucht and F. Harary, On the corona of two graphs, *Aequat. Math.*, **4** (1970), 322-325.

[23] W. Goddard and H. C. Swart, On the integrity of combinations of graphs, *J. Combin. Math. Combin. Comput.*, **4** (1988), 3 -18.

[24] W. Goddard, *On the vulnerability of graphs*, Ph. D. thesis. University of Natal, Durban, S. A. (1989).

[25] W. Goddard and H. C. Swart, Integrity in graphs: bounds and basics, *J. Combin. Math. Combin. Comput.*, **7** (1990), 139-151.

[26] T. Grauman, S. Hartke, A. Jobson, B. Kinnersley, D. west, L. wiglesworth, P. Worah and H. Wu, The hub number of a graph, *Inform. Process. Lett.*, **108** (2008), 226-228.

[27] I. Gutman and N. Trinajstic. Graph theory and molecular orbitals, Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.*, **17** (1972), 535-538.

[28] F. Harary, *Graph theory*, *Narosa Publishing House*, New Delhi, 2001.

[29] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dckker, New York, 1998.

[30] P. Johnson, P. Slater and M. Walsh, *The connected hub number and the connected domination number*, Wiley Online Library, **3 (58)**(2011), 232-237.

[31] S. I. Khalaf, V. Mathad and S. S. Mahde, *Hubtic number in graphs*, Opuscula Math., **38 (6)** (2018), 841–847.

[32] S. I. Khalaf, V. Mathad and S. S. Mahde, *Edge hubtic number in graphs*, International J. Math. Combin., **3** (2018), 141–146.

[33] V. R. Kulli, *College graph theory*, Vishwa International Publications, Gulbarga, India, 2012.

[34] S. S. Mahde, V. Mathad, A. M. Sahal, *Hub-integrity of graphs*, Bulletin of International Mathematical Virtual Institute, **(5)** (2015), 57-64.

[35] S. S. Mahde and V. Mathad, Some operations in hub-integrity of graphs, *Asia Pac. J. Math.*, **2** (2015), 108-123.

[36] S. S. Mahde and V. Mathad, Domination integrity of line splitting graph and central graph of path, cycle and star graphs, *Appl. Appl. Math.(AAM)*, **11** (1) (2016), 428-443.

[37] S. S. Mahde and V. Mathad, Some results on the edge hub-integrity of graphs, *Asia Pac. J. Math.*, **3** (2016), 173-185.

[38] S. S. Mahde and V. Mathad, hub-integrity of splitting graph and duplication of graph elements, *TWMS J. App. Eng. Math.*, **2** (2016), 289-297.

[39] S. S. Mahde and V. Mathad, On the weak hub-integrity of graphs, *Gulf J. Math.*, **5** (2017), 71-86.

[40] S. S. Mahde and V. Mathad, Hub-integrity of line graphs, *Electron. J. Math. Anal. Appl.*, **7** (2019), 140-150.

[41] S. S. Mahde and V. Mathad, Hub-integrity polynomial of graphs, *TWMS J. App. Eng. Math.*, **10**(2) (2020), 434-442.

[42] S. S. Mahde and A. S. Sand, Hop hub-integrity of graphs, *Int. J. Math. And Appl.*, **9**(2) (2021), 91-100.

[43] A. Mamut and E. Vumar, A note on the integrity of middle graphs, *Lecture Notes in Comput. Sci.,* **4381** (2007), 130-134.

[44] V. Mathad, A. M. Sahal, S. Kiran, *The total hub number of graphs*, Bulletin of the International Mathematical Virtual Institute, **4** (2014), 61–67.

[45] A. Mowshowitz, The characteristic polynomial of a graph, *J. Combin. Theory Ser.*, B 12 (1972), 177-193.

[46] J. Mycielski, Sur le coloriage des graphs, *Colloq. Math.*, **3** (1955), 161-162.

[47] A. M. Sahal, The doubly connected hub number of graph, International J.Math. Combin., **1**(2021), 79-84.

[48] E. Sampathkumar and H. B. Walikar, *On the splitting graph of a graph*, *J. Karnatak Univ. Sci.*, 25 and 26(combined) (1980-81), 13-16.

[49] A. S. Sand, S. S. Mahde, *Hop hub number in graphs* , submitted.

[50] A. S. Sand, S. S. Mahde, Hop hubtic number and hop hub polynomial of graphs, *Int. J. Comput. Math. Appl.*, **183**(52) (2022), 1-6.

[51] W. T. Tutte, A contribution to the theory of chromatic polynomials, *Canad. J. Math.*, **6** (1954), 80-91.

[52] S. K. Vaidya, N. J. Kothari, *Some new results on domination integrity of graphs*, Open Journal of Discrete Mathematics, **2(3)** (2012), 96-98.

[53] S. K. Vaidya, N. Kothari, *Domination integrity of splitting graph of path and cycle*, Hindawi Publishing Corporation, ISRN Combinatorics, Article ID 795427, (2013), 7.

[54] R. P. Veettil and T. V. Ramakrishnan, Introduction to hub polynomial of graphs, *Malaya Journal of Matematik.*, **8**(4) (2020) 1592-1596.

[55] A. Vince, The integrity of a cubic graph, *Discrete Applied Math.*, **140** (2004), 223-239.

[56] M. Walsh, The hub number of graphs, *Int. J. Math. Comput. Sci.*, **1** (2006), 117-124.