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SOME NEW MATRIX DOMAINS IN SEQUENCE SPACES AND THEIR MATRIX OPERATORS

A THESIS SUBMITTED AS PARTIAL FULFILMENT OF THE REQUIREMENTS

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Al- Baydha University

BY

ESSAM SALEH SAAD AL YARI

Supervised by

Prof. Dr. ABDULLAH K. NOMAN

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CERTIFICATE

This is to certify that the work presented in this thesis entitled

" **SOME NEW MATRIX DOMAINS IN SEQUENCE SPACES AND THEIR MATRIX OPERATORS** "

is an authentic and original research work carried out by

Mr. ESSAM SALEH SAAD AL YARI

under my supervision and submitted to the Department of Mathematics, Faculty of Education and Science - Rada'a, Al Baydha University, as a partial fulfillment of the requirements for the award of the Master Degree in Mathematics.

To the best of my knowledge and belief, the present work has fulfilled the prescribed conditions given in the academic ordinances and regulations of Al Baydha University and it has not been submitted before to another university for the award of any degree.

(SUPERVISOR)

Dr. Abdullah K. Noman

(Ph.D., Functional Analysis)

(Al Baydha University)

(February, 2022)

EXAMINING COMMITTEE REPORT

After reading the thesis entitlded "**Some New Matrix Domains in Sequence Spaces and Their Matrix Operators**" and examining the researcher **Mr. Essam Saleh Saad Al Yari** in its contents, we find it is adequate as a thesis for award of the **Master Degree** in **Mathematics** with grade **Excellent**.

On Thursday, 24 March 2022

Examining Committee Members

Supervisor (Member)	Internal Examiner (Member)	External Examiner (Head)
Dr. Abdullah K. Noman	**Dr. Ali H. Al-aati**	**Prof. Fateh A. Allahabi**
Dept. Math.	Dept. Math.	Dept. Math.
Fac. Edu. Sci.	Fac. Edu. Sci.	Fac. Sci.
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Dr. Sultan Senan Mahde
Dept. Math.
Fac. Edu. Sci
Al-Baydha University

Dean of The College
Prof. Dr. Naif Al-Abrat
Fac. Edu. Sci
Al-Baydha University

DEDICATION

To my parents, for their all time support and prayers.
My wife and my Kids: Ammar, Malek, Reham, Mohammed and Mukhtar for their support and love.

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First and foremost, all praise be to Almighty Allah, the most beneficent and the most merciful, who bestowed upon me the courage, patience and strength to embark upon this work and carry it to its completion successfully.

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Also, I am immensely thankful to the academic staff members of the Department of Mathematics (Faculty of Education and Science) and all academic staff members of Albaydha University.

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ESSAM AL YARI

ABSTRACT

In the present thesis, we have introduced some new sequence spaces of λ-type by means of the classical sequence spaces of bounded and convergent series. Also, we have studied the algebraic and topological properties of our new spaces with their Schauder bases and isomorphic relations. Further, we have deduced some inclusion relations concerning our new spaces and obtained their dual spaces. Moreover, we have concluded some new results characterizing certain classes of matrix operators acting on our spaces and the matrix operators acting into and between those spaces. Furthermore, many important and new facts have been obtained and discussed as particular cases of our results.

PREFACE

In modern analysis, the subject matter of functional analysis includes the study of abstract spaces, operators and transformations of these spaces, which provides a general framework for finding solutions of various problems in applied mathematics and physics. Above all, a study of functional analysis in itself provides new insight and understanding into the processes and techniques of elementary analysis which we are accustomed to use in our everyday calculations.

One of the most general types of abstract spaces is that type of spaces with infinite-dimensions and the sequence spaces are the most important spaces of this type. So, many mathematicians have done a lot of work in this field of sequence spaces and studied their matrix transformations which have been applied in all other areas of mathematics. Thus, we have chosen this field for study and research.

In the present thesis, our contribution is to introduce some new sequence spaces and study their topological properties, Schauder bases, inclusion relations, dual spaces and certain classes of matrix operators on our new sequence spaces. For more utility, we hope for the reader's familiarity with the basic concepts of our subject. Thus, for further knowledge in our notions, we refer the reader to [51] for basic idea of sequences and series, to [15, 31] for elementary concepts of functional analysis, to [13, 32, 61] for the notions of sequence spaces and to [47] for the particular sequence spaces of λ-type.

My thesis is divided into five chapters and the main results in the last four chapters have been published in two research papers as mentioned at the beginning of each chapter which have been presented in the 2nd conference of Albaydha University (2021). The materials of this exposition are organized as follows:

Chapter 1 is an introductory chapter to display the historical and theoretical background of our subject concerning the theory of sequence spaces and their matrix transformations with a short survey on some basic definitions, notations and preliminary results which are already known in the literature of this field.

In **Chapter 2**, we have introduced the new λ-sequence spaces of bounded, convergent and null series, and studied their isomorphic, algebraic and topological properties with contracting their Schauder bases.

Chapter 3 is devoted to derive some interesting inclusion relations between our new spaces and the classical sequence spaces, and some particular cases of equalities and strict inclusions will be discussed with important examples.

In **Chapter 4**, we have concluded the Köthe-Toeplitz duals of our new λ-sequence spaces defined in terms of series.

Chapter 5 is devoted to characterize the related classes of matrix operators acting on our new spaces and the matrix operators acting into and between those spaces, and some known or new results will be deduced as particular cases.

The obtained facts are those remarks, examples, lemmas or theorems, which are presented throughout this thesis as paragraphs and every paragraph is associated with triple decimal numbering. The first number indicates the chapter, the second represents the section, and the third refers to the number of current paragraph. For example, the form 3.2.1 refers to the first paragraph (remark, example, lemma or theorem) appearing in Section 2 of Chapter 3.

At the end of this monograph, we have given an exhaustive list of relevant references to the literature presented in this thesis. All results stated without proof are cited and can be found in the references given either before or after the statements.

In this world, nothing is complete! So, I hope for the reader's forgiveness if there is any typing mistake which may appear here or there throughout this simple work. Despite all efforts to make this thesis free from such errors, there may be some still left and for which the researcher takes all the responsibilities personally.

Finally, I have had the privilege of completing my Master thesis under the esteemed supervision of Prof. A.K. Noman, Department of Mathematics, Faculty of Education and Science-Rada'a, Albaydha University. His stimulating discussions, judicious advice, useful suggestions and constructive comments have been an unfailing source of a great inspiration for me at every stage during the preparation of this work. So, I would like to take this opportunity to put on the record my profound thanks to him.

ESSAM S. AL YARI

(Januery, 2022)

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Chapter 1

INTRODUCTION

1 INTRODUCTION

The theory of sequence spaces and matrix transformations is an interesting area for research in summability theory as a part of functional analysis. In this first chapter, we display a historical and theoretical background of our subject concerning the theory of sequence spaces and their matrix transformations with a short survey on some basic definitions, notations and preliminary results which are already known in the literature of this field. This introductory chapter is divided into three sections, the first is devoted to the theoretical frame, the second is for the research methodology and the last is to present some preliminary results which will be needed in the sequel.

1.1 Theoretical Frame

In this section, we display a historical background for the theory of sequence spaces and matrix transformations, and we give a short survey on some basic concepts of this area with certain previous studies.

1.1.1 Historical Background

Von Neumann began the analysis of the frame work of quantum mechanics in the years following 1926 but there were few attempts to study the structure of specific quantum systems (exceptions would be some of the work of Frèchet and Rollick). This situation changed in the early 1950's when Kato proved the self adjointness of atomic Hamiltonians and Garding and Wightman formulated the axioms for quantum field theory. These events demonstrated the usefulness of functional analysis.

Functional analysis was founded by S. Banach, M. Fréchet, H. Hahn, F. Hausdorff, D. Hilbert, F. Riesz and others. These names have become synonymous with the tools of this subject. Such tools have turned out to be powerful and widely used in several areas of functional analysis, especially in summability theory which encompasses a variety of fields and has many applications in various subjects. For instance, in numerical analysis, approximation theory, operator theory and the theory of differential equations and orthogonal series with their special functions. The summability theory has been originated from the attempts made by the mathematicians to give limits to the divergent sequences and series [32].

In particular, the theory of sequence spaces and matrix transformations is a significant area of research in summability theory and so many mathematicians have done a lot of work in this field. In fact, the most important methods of summability are given by infinite matrices and their matrix transformations. So that, our concern is with those infinite matrices that map a sequence space into another one. Such matrices arise naturally from the infinite-dimensions of sequence spaces [34].

But, why we should study matrix operators and transformations between sequence spaces; why not study the general linear operators? The reason is that, in many important cases, the most general linear operators acting between sequence spaces are actually determined by infinite matrices. So, there is no loss of generality in such study. Moreover, there is often a gain in that specific conditions on the entries of an infinite matrix which may be easy to verify.

The interest in matrix transformations was stimulated by special results in the summability theory which were obtained by E. Cesàro, L. Euler, N. Nörlund, F. Riesz and others. The earliest idea of summability were perhaps contained in a letter written by Leibnitz to C. Wolf in 1713, the sum of the oscillatory series $1 - 1 + 1 - \cdots$ as given by Leibnitz was in 1880. After that, Frobenius introduced the method of summability by arithmetic mean which has later been generalized by Cesàro in 1890 as the (C, α) method of summability. With the emergence of functional analysis, sequence spaces were studied with greater insight and motivation and the earliest applications of functional analysis to summability was made by S. Banach, H. Hahn, S. Mazur, G. Köthe and O. Toeplitz. In 1911, the celebrated mathematician Toeplitz determined the necessary and sufficient conditions for an infinite matrix to be regular, that is, he characterized those conservative matrices that preserve the limits invariant. In fact, Toeplitz was the first person who studies the summability methods as a class of operators defined on sequences by infinite matrices. It was followed by the works done by I. Schur, W. Orlicz, K. Knopp, G. Petersen, H. Nakano, S. Simons, G. Lorentz, G. Hardy, A. Wilansky, I. Maddox, W. Sargent, C. Lascarides, S. Nanda, D. Rath, G. Das, Z. Ahmed, H. Kızmaz B. Kuttner and many others like Russell and Rhoade [14].

After many years, exactly in 1950, Robinson initiated the study of summability by infinite matrices of linear operators on normed linear spaces which enabled the workers on summability to extend their results. Also, in 1951, the famous mathematician K. Zeller introduced the concept of BK spaces* which have proved its useful in summability theory, especially in the characterizations of matrix transformations between sequence spaces, and the most important result is that matrix operators between BK spaces are continuous [61].

The sequence spaces were motivated by problems in Fourier series, power series and systems of equations with infinitely many variables, and the theory of sequence

*The letters **B** and **K** stand for **B**anach and the German word **K**oordinate which means 'coordinate' as in the Zeller's terminology.

spaces and infinite matrices occupies a very prominent position in several branches of analysis and plays an important role in various fields of Mathematics as a powerful and pervading tool in almost all these branches with several important applications. For example, in the structural theory of topological vector spaces, Schauder basis theory and theory of differential equations and special functions [8].

Recently, the sequence spaces have been generalized in several directions by many mathematicians and some of them introduce new sequence spaces and study their various properties. At the present time, a lot of work have been done by many researchers around the world, like Boos, Rakočević, Malkowsky, Savaş, Başar, Altay, Mursaleen, Noman, Karakaya, Kiriçi, Kara, Polat and many others, only a few was named (e.g., see [4, 5, 6, 10, 11, 16, 18, 20, 22, 23, 24, 25, 26, 27, 36, 47, 53, 55, 63, 64]).

1.1.2 Basic Concepts and Notations

Here, we give a short survey on the basic definitions, concepts and notions which are the elementary tools in the theory of sequence spaces and matrix transformations. Also, we will define our common notations which are usually used by all authors and researchers in this area. Thus, our terminologies, as given here, will have the same meanings throughout this thesis (unless stated otherwise).

Let $\mathbb{K}$ be the scalar field (of real or complex numbers), that is $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and so our scalars are either real or complex numbers (according to the case of our spaces). Also, we will use the symbols k and n to be positive integers while $p, q \geq 1$ are reals.

By the word "sequence", we mean an infinite sequence of real or complex terms, and if $x = (x_1, x_2, x_3, \cdots)$ is a real or complex sequence; then we denote it by $x = (x_k)_{k=1}^{\infty}$ or simply $x = (x_k)$, where x_k is called the k-th term of x. Further, we shall use the following conventions: the first is that any term with a non-positive subscript is assumed to be nothing (e.g., the terms x_0 and x_{-1} have no meaning and can be considered to be not exist). Next, we will frequently use the sequences $e = (1, 1, 1, \ldots)$ and e_k for each $k \geq 1$, where e_k is the sequence whose only one non-zero term which is 1 in the k-th place for each $k \geq 1$, that is $e_1 = (1, 0, 0, \cdots)$, $e_2 = (0, 1, 0, 0, \cdots), \cdots$ etc. Also, the absolute value of a sequence x and its positive power are defined by means of their meanings for its scalar-terms, that is $|x| = (|x_k|)$ and $|x|^r = (|x_k|^r)$ for any real $r > 0$. The last conventions are concerning with some algebraic operations defined on sequences, namely the coordinate-wise addition, scalar multiplication, product and division. More precisely, if $x = (x_k)$ and $y = (y_k)$ are sequences and $\alpha \in \mathbb{K}$ is a scalar; then $x \pm y = (x_k \pm y_k)$, $\alpha x = (\alpha x_k)$, $xy = (x_k y_k)$ and if $y_k \neq 0$ for all k; then $x/y = (x_k/y_k)$ and $1/y = (1/y_k)$.

Together with any sequence $x = (x_k)$, there always exist two sequences, namely the difference sequence $\Delta(x)$ and the sum sequence $\sigma(x)$, where

$$\Delta(x) = (x_1, x_2 - x_1, x_3 - x_2, \cdots) \text{ and } \sigma(x) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, \cdots).$$

That is $\Delta(x) = (\,\Delta(x_k)\,)_{k=1}^{\infty} = (\,x_k - x_{k-1}\,)_{k=1}^{\infty}$ and $\sigma(x) = (\,\sigma_k(x)\,)_{k=1}^{\infty} = (\,\sum_{j=1}^{k} x_j\,)_{k=1}^{\infty}$ which leads us to write their terms as follows:

$$\sigma_k(x) = \sum_{j=1}^{k} x_j \text{ and } \Delta(x_k) = x_k - x_{k-1} \text{ with } \Delta(x_1) = x_1 \qquad (k \geq 1). \tag{1.1.1}$$

The sequence $x = (x_k)$ is said to be *bounded* if there exists a positive real $M > 0$ such that $|x_k| \leq M$ for all $k \geq 1$, that is x is bounded if and only if $\sup_k |x_k| < \infty$, where the supremum of $|x_k|$ is taken over all positive integers k. Also, the sequence x is said to be *convergent* if its limit $\lim_{k\to\infty} x_k$ exists. In particular, by a *null sequence*, we mean a convergent sequence which converges to zero, i.e. $\lim_{k\to\infty} x_k = 0$.

Every sequence $x = (x_k)$ is associated with a series $\sum_{k=1}^{\infty} x_k$ whose terms are exactly those of x and so it has the same sequence of partial sum which is $\sigma(x)$. Thus, it seems to be quite natural to similarly say that $\sum_{k=1}^{\infty} x_k$ is a null, convergent or bounded series if its sequence of partial sum $\sigma(x)$ is a null, convergent or bounded sequence, respectively. That is, the series $\sum_{k=1}^{\infty} x_k$ is bounded if $\sup_n |\sum_{k=1}^{n} x_k| < \infty$, and it is convergent if $\lim_{n\to\infty} \sum_{k=1}^{n} x_k$ exists. Also, by a null series, we mean a series which converges to zero, i.e. $\lim_{n\to\infty} \sum_{k=1}^{n} x_k = 0$ which yields that $\sum_{k=1}^{\infty} x_k = 0$.

A series $\sum_{k=1}^{\infty} x_k$ is said to be *absolutely convergent* if the series $\sum_{k=1}^{\infty} |x_k|$ converges and we denote it by $\sum_{k=1}^{\infty} |x_k| < \infty$ (it is well-known that absolute convergence of series implies their convergent, but the converse is not). In general, for any real $p \geq 1$, the series $\sum_{k=1}^{\infty} x_k$ is said to be *p-absolutely convergent* if $\sum_{k=1}^{\infty} |x_k|^p < \infty$.

A sequence $x = (x_k)$ is said to be of *bounded variation* if $\sum_{k=1}^{\infty} |x_k - x_{k-1}| < \infty$ or equivalently $\sum_{k=1}^{\infty} |\Delta(x_k)| < \infty$. In general, for any real $p \geq 1$, we say that x is of *p-bounded variation* if $\sum_{k=1}^{\infty} |x_k - x_{k-1}|^p < \infty$ or $\sum_{k=1}^{\infty} |\Delta(x_k)|^p < \infty$ [9].

By w, we denote the linear space of all (real or complex) sequences over the scalar field $\mathbb{K}$ (with coordinate-wise addition and scalar multiplication) and any vector subspace of w is called a *sequence space*. Throughout, we shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively. Also, for each real $1 \leq p < \infty$, the sequence space ℓ_p is consisting of all sequences associated with p-absolutely convergent series. These sequence spaces are known as the *classical sequence spaces* [34]. Further, we write bs, cs and cs_0 for the spaces of all sequences associated with bounded, convergent and null series, respectively. Moreover, by $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, we stand for the difference spaces of bounded, convergent and null difference sequences, respectively. Furthermore, for each real $1 \leq p < \infty$, we denote the space of all sequences of p-bounded variation by bv_p. That is, we have:

$$c_0 = \{x = (x_k) \in w : \lim_{k\to\infty} x_k = 0\},$$

$$c = \{x = (x_k) \in w : \lim_{k\to\infty} x_k \text{ exists}\},$$

$$\ell_\infty = \{x = (x_k) \in w : \sup_k |x_k| < \infty\},$$

$$\ell_p = \{x = (x_k) \in w : \textstyle\sum_{k=1}^{\infty} |x_k|^p < \infty\} \quad (1 \le p < \infty),$$

$$cs_0 = \{x = (x_k) \in w : \lim_{n\to\infty} \textstyle\sum_{k=1}^{n} x_k = 0\},$$

$$cs = \{x = (x_k) \in w : \lim_{n\to\infty} \textstyle\sum_{k=1}^{n} x_k \text{ exists}\},$$

$$bs = \{x = (x_k) \in w : \sup_n \left|\textstyle\sum_{k=1}^{n} x_k\right| < \infty\},$$

$$c_0(\Delta) = \{x = (x_k) \in w : \lim_{k\to\infty} (x_k - x_{k-1}) = 0\},$$

$$c(\Delta) = \{x = (x_k) \in w : \lim_{k\to\infty} (x_k - x_{k-1}) \text{ exists}\},$$

$$\ell_\infty(\Delta) = \{x = (x_k) \in w : \sup_k |x_k - x_{k-1}| < \infty\},$$

$$bv_p = \{x = (x_k) \in w : \textstyle\sum_{k=1}^{\infty} |x_k - x_{k-1}|^p < \infty\} \quad (1 \le p < \infty),$$

and we define the sequence space bv_0 by $bv_0 = c_0 \cap bv_1$ [34].

A normed sequence space is of course a sequence space X equipped with a norm $\|\cdot\|$ defined on X as a mapping $\|\cdot\| : X \to \mathbb{R}$ such that $\|x\| \ge 0$, $x = 0$ whenever $\|x\| = 0$, $\|\alpha x\| = |\alpha|\|x\|$ and $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$ and every $\alpha \in \mathbb{K}$. A normed sequence space X is called a Banach sequence space if it is complete with the topology generated by its norm. Also, as it is the case for arbitrary normed spaces, if a normed sequence space X contains a sequence $(b_k)_{k=1}^{\infty}$ with the property that for every $x \in X$ there exists a unique sequence $(\alpha_k)_{k=1}^{\infty}$ of scalars such that $\lim_{n\to\infty} \|x - (\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n)\| = 0$; then the sequence $(b_k)_{k=1}^{\infty}$ is called a *Schauder basis* for X (or simply a basis for X) and the series $\sum_{k=1}^{\infty} \alpha_k b_k$ which has the sum x is then called the expansion of x, with respect to the given basis, and we then say that x has uniquely been represented in the form $x = \sum_{k=1}^{\infty} \alpha_k b_k$. Further, a normed sequence space X is said to be *separable* if it contains a countable dense subset, and it is well-known that every Banach space with Schauder basis must be separable. Furthermore, if X is a normed sequence space; then for each positive integer k, there exists a mapping $\pi_k : X \to \mathbb{K}$ defined by $x \mapsto \pi_k(x) = x_k$ for all $x \in X$, these mappings π_k's (for all k) are called the *coordinate-maps* of X or the *coordinates* of X, where $\mathbb{K}$ is the scalar field of X [8].

A normed sequence space X is called a *BK space* if it is complete and all its coordinate-maps are continuous. In other words, by a BK space, we mean a Banach sequence space with continuous coordinates. It is well-known that the above mentioned sequence spaces are all BK spaces with their natural norms. More precisely, the spaces ℓ_∞, c and c_0 are BK spaces with the sup-norm $\|\cdot\|_\infty$ given by $\|x\|_\infty = \sup_k |x_k|$. Also, for $1 \leq p < \infty$, the spaces ℓ_p are BK spaces with the p-norm $\|\cdot\|_p$ defined by $\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}$ and the spaces bv_p are BK spaces with their norm $\|\cdot\|_{bv_p}$ given by $\|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k - x_{k-1}|^p\right)^{1/p}$. Moreover, the spaces bs, cs and cs_0 are BK spaces with the series-norm $\|\cdot\|_s$ defined by $\|x\|_s = \sup_n \left|\sum_{k=1}^{n} x_k\right|$. Besides, the difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are BK spaces with the Δ-norm $\|\cdot\|_\Delta$ given by $\|x\|_\Delta = \sup_k |x_k - x_{k-1}|$ [14].

For any sequence space X, the concept of *Köthe-Toeplitz duality* of X, so-called as the *α-, β- and γ-duals* of X can simply be given by means of the spaces $\langle\alpha\rangle = \ell_1$, $\langle\beta\rangle = cs$ and $\langle\gamma\rangle = bs$. For this, let θ be any of the duality symbols α, β or γ, that is $\theta := \alpha$, β or γ. Then, the θ-dual of X is a sequence space denoted by X^θ which can be defined as follows:

$$X^{\theta} = \{a \in w : \, ax \in \langle\theta\rangle \ \text{ for all } x \in X\} \qquad (\theta = \alpha, \beta \text{ or } \gamma), \tag{1.1.2}$$

where $\langle\alpha\rangle = \ell_1$, $\langle\beta\rangle = cs$ and $\langle\gamma\rangle = bs$. In other words, the α-, β- and γ-duals of X are respectively denoted by X^{α}, X^{β} and X^{γ} which are sequence spaces defined as follows:

$$X^{\alpha} = \Big\{a = (a_k) \in w : \ ax = (a_k x_k) \in \ell_1 \ \text{ for all } \ x = (x_k) \in X\Big\},$$

$$X^{\beta} = \Big\{a = (a_k) \in w : \ ax = (a_k x_k) \in cs \ \text{ for all } \ x = (x_k) \in X\Big\},$$

$$X^{\gamma} = \Big\{a = (a_k) \in w : \ ax = (a_k x_k) \in bs \ \text{ for all } \ x = (x_k) \in X\Big\}.$$

Besides, it is well-known that $X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$, the inclusion $X \subset Y$ implies that $Y^{\theta} \subset X^{\theta}$, and we have $c_0^{\theta} = c^{\theta} = \ell_{\infty}^{\theta} = \ell_1$, $\ell_1^{\theta} = \ell_{\infty}$ and $\ell_p^{\theta} = \ell_q$ for $p > 1$ with $q = p/(p-1)$, where X and Y are sequence spaces. The basic properties of dual spaces can be found in [8, 21, 33].

Due to the infinite dimensions of sequence spaces in the general case, the notion of *matrix transformations* between sequence spaces has been arisen for study the linear operators between such spaces which can be given by infinite matrices. For an infinite matrix A with real or complex entries a_{nk} $(n, k \geq 1)$, we write $A = [a_{nk}]_{n,k=1}^{\infty}$ or simply $A = [a_{nk}]$, and we will write A_n for the n-th row sequence in A, that is $A_n = (a_{nk})_{k=1}^{\infty}$ for each $n \geq 1$. Also, for any sequence $x \in w$, the A-*transform* of x, denoted by $A(x)$, is defined to be the sequence $A(x) = (A_n(x))_{n=1}^{\infty}$ whose terms given by

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} \, x_k \qquad (n \geq 1) \tag{1.1.3}$$

provided the convergence of series for each $n \geq 1$ and we then say that $A(x)$ exists. Further, for any two sequence spaces X and Y, we say that A acts from X into Y if $A(x)$ exists and $A(x) \in Y$ for every $x \in X$ [47]. Furthermore, the *matrix class* (X, Y)

is define to be the collection of all infinite matrices acting from X into Y. In particular, an infinite matrix A is said to be *conservative* if $A \in (c, c)$ and a conservative matrix A is said to be *regular* if $\lim_{n\to\infty} A_n(x) = \lim_{n\to\infty} x_n$ for all $x \in c$ [32]. In fact, there may exists an infinite matrix A such that $A \notin (X, Y)$ and so the infinite matrices in the class (X, Y) must be characterized from those matrices which are not in (X, Y). That is, there must exist a list of necessary and sufficient conditions on the entries of a given infinite matrix A to be in the class (X, Y), where $A \in (X, Y)$ if and only if $A(x)$ exists as well as $A(x) \in Y$ for every $x \in X$. In other words, $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for every $n \geq 1$ and $A(x) \in Y$ for all $x \in X$, and so the β-duality is an important tool for characterizing matrix classes [21]. Obviously, if $A \in (X, Y)$; then A defines a linear operator $A : X \to Y$ by $x \mapsto A(x)$, and we may call it as a *matrix operator* (matrix mapping) and the same for every linear operator from X into Y which can be given by an infinite matrix. That is, a linear operator between sequence spaces $L : X \to Y$ is called a matrix operator if there exists an infinite matrix $A \in (X, Y)$ such that $L(x) = A(x)$ for all $x \in X$ and we then say that L is given by an infinite matrix, viz A. Moreover, it is worth mentioning that the most general forms of linear operators between sequence spaces can be given by infinite matrices [34]. This fact gives a special importance for the notion of matrix transformations between sequence spaces, which has been studied by several authors in many research papers (see [33, 35, 58]) and has recently been used to introduce new sequence spaces and characterize their matrix classes by means of the idea of matrix domains (see [27, 34, 43]). For an infinite matrix A and a sequence space X, the *matrix domain* of A in X is a sequence space denoted by X_A and defined as follows:

$$X_A = \{x \in w : \ A(x) \in X\}. \tag{1.1.4}$$

The most useful cases of matrix domains are those obtained from special types of infinite matrices called as triangles, where an infinite matrix $T = [t_{nk}]_{k,n=1}^{\infty}$ is called a *triangle* if $t_{nn} \neq 0$ for every $n \geq 1$ and $t_{nk} = 0$ for all $k > n$ $(n, k \geq 1)$. For example, the sum-matrix σ and the band-matrix Δ are infinite matrices which are triangles defining the partial sum and the difference operator, respectively. To see that, the triangles

$$\sigma = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

have the transforms $\sigma(x) = (\sigma_n(x))$ and $\Delta(x) = (\Delta(x_n))$ which can be obtained by using (1.1.3) to get $\sigma_n(x) = \sum_{k=1}^{n} x_k$ and $\Delta(x_n) = x_n - x_{n-1}$ for all $n \geq 1$ with $\Delta(x_1) = x_1$ which is the same result as given in (1.1.1), where $x \in w$. That is, the sum sequence $\sigma(x)$ and the difference sequence $\Delta(x)$ are respectively the σ- and Δ-transforms of x. This fact, together with (1.1.4), leads us to obtain the following:

$$cs_0 = (c_0)_\sigma = \{x \in w : \sigma(x) \in c_0\},$$

$$cs = (c)_\sigma = \{x \in w : \sigma(x) \in c\},$$

$$bs = (\ell_\infty)_\sigma = \{x \in w : \sigma(x) \in \ell_\infty\},$$

$$c_0(\Delta) = (c_0)_\Delta = \{x \in w : \Delta(x) \in c_0\},$$

$$c(\Delta) = (c)_\Delta = \{x \in w : \Delta(x) \in c\},$$

$$\ell_\infty(\Delta) = (\ell_\infty)_\Delta = \{x \in w : \Delta(x) \in \ell_\infty\},$$

$$bv_p = (\ell_p)_\Delta = \{x \in w : \Delta(x) \in \ell_p\} \qquad (1 \leq p < \infty)$$

which means that these spaces are the matrix domains of the triangles σ and Δ in the classical sequence spaces [8]. This idea has been applied by many authors in several interesting studies as presented in the next section.

1.1.3 Previous Studies

The approach constructing a new sequence space by means of the matrix domain of a particular infinite matrix has been employed by Maddox, Wang, Ng, Lee, Kızmaz, Rakočević, Malkowsky, Savaş, Başar, Altay, Mursaleen, Noman, Karakaya, Kiriçi, Kara, Polat, Aydın, Bektaş and many others (e.g., see [4, 5, 6, 10, 11, 16, 18, 20, 22, 23, 24, 25, 26, 27, 36, 47, 53, 55, 63, 64]). More recently, due to the various properties of the triangles as an important particular case of infinite matrices, (for instance, the matrix domains of triangles in BK spaces are also BK spaces), the idea of introducing a new sequence space by means of the matrix domain of a given triangle has largely been used by several authors in many research studies with different manners. For instance, we display here the following previous studies:

(1) The Cesàro sequence spaces have been constructed by Ng and Lee in 1978 [46] (see also Şengönül and Başar, 2005 [54]) as domains of the Cesàro matrix C^1 of arithmetic mean in the spaces ℓ_p for $1 \leq p \leq \infty$, that is

$$c_p^1 = \{x \in w : C^1(x) \in \ell_p\} \qquad (1 \leq p \leq \infty)$$

which are BK spaces with $\|x\|_{c_p^1} = \|C^1(x)\|_p$, where

$$C_n^1(x) = \frac{1}{n} \sum\nolimits_{k=1}^{n} x_k \qquad (n \geq 1).$$

(2) The difference sequence spaces have been studied by Kızmaz in 1981 [29, 30] as domains of the band matrix Δ of difference in the spaces c_0, c and ℓ_∞, that is

$$\mu(\Delta) = \{x \in w : \Delta(x) \in \mu\} \qquad (\mu = c_0, c \text{ or } \ell_\infty)$$

which are BK spaces with $\|x\|_\Delta = \|\Delta(x)\|_\infty$, where $\Delta(x_n) = x_n - x_{n-1}$ for all $n \geq 1$ and $\Delta(x_1) = x_1$.

(3) The sequence spaces of p-bounded variation have been studied by Başar and Altay in 2003 [9] as domains of the band matrix Δ of difference in the spaces ℓ_p for $1 \leq p \leq \infty$, that is $bv_p = \{x \in w : \Delta(x) \in \ell_p\}$ which are BK spaces with $\|x\|_{bv_p} = \|\Delta(x)\|_p$ for all $x \in bv_p$ $(1 \leq p \leq \infty)$.

(4) The Euler sequence spaces have been introduced by Altay and Başar in 2005 [2] (and together with Mursaleen, 2006 [3, 38]) as domains of the Euler matrix E^r in the spaces c_0, c, ℓ_∞ and ℓ_p for $1 < p < \infty$, that is

$$e_0^r = \{x \in w : E^r(x) \in c_0\}, \quad e_c^r = \{x \in w : E^r(x) \in c\},$$

$$e_\infty^r = \{x \in w : E^r(x) \in \ell_\infty\}, \quad e_p^r = \{x \in w : E^r(x) \in \ell_p\}.$$

Also e_0^r, e_c^r and e_∞^r are BK spaces with $\|x\|_{E^r} = \|E^r(x)\|_\infty$ and all e_p^r are BK spaces with $\|x\|_{E_p^r} = \|E^r(x)\|_p$ $(1 < p < \infty)$, where $0 < r < 1$ and

$$E_n^r(x) = \sum\nolimits_{k=1}^{n} \binom{n-1}{k-1} (1-r)^{n-k}\, r^{k-1}\, x_k \qquad (n \geq 1).$$

(5) The generalized Cesàro sequence spaces have been studied by Malkowsky and Rakočević in 2007 [34] as domains of the generalized Cesàro matrix (C, α) of order α in the spaces μ, where $\mu = c_0,\ c,\ \ell_\infty$ or ℓ_p $(1 \leq p < \infty)$, that is

$$\mu(C, \alpha) = \{x \in w : (C, \alpha)(x) \in \mu\}$$

which are BK spaces with $\|x\|_{\mu(C,\alpha)} = \|(C, \alpha)(x)\|_\mu$, where $\alpha > 0$ and

$$(C, \alpha)_n(x) = \frac{(n-1)!\,\alpha}{\Gamma(\alpha+n)} \sum\nolimits_{k=1}^{n} \frac{\Gamma(\alpha+n-k+1)}{(n-k+1)!}\, x_k \qquad (n \geq 1).$$

(6) The sequence spaces of weighted means have been constructed by Malkowsky and Savaş in 2008 [35] as domains of the matrix W_s^t of weighted means in the spaces

μ, where $\mu = c_0,\ c$ or ℓ_∞, that is $w_s^t(\mu) = \{x \in w : W_s^t(x) \in \mu\}$ which are BK spaces with $\|x\|_{w_s^t} = \|W_s^t(x)\|_\infty$, where s and t are sequences of non-zero scalars and

$$(W_s^t)_n(x) = \frac{1}{s_n}\sum\nolimits_{k=1}^{n} t_k x_k \qquad (n \geq 1).$$

(7) The sequence spaces of generalized means have been defined by Mursaleen and Noman in 2011 [43] as domains of the matrix $A(r,s,t)$ of generalized means in the spaces μ, where $\mu = c_0,\ c,\ \ell_\infty$ or ℓ_p $(1 \leq p < \infty)$, that is $\mu(r,s,t) = \{x \in w : A(r,s,t)(x) \in \mu\}$ which are BK spaces with $\|x\|_{\mu(r,s,t)} = \|A(r,s,t)(x)\|_\mu$, where r and t are sequences of non-zero scalars, s is a sequence with first term $s_1 \neq 0$ and

$$A(r,s,t)_n(x) = \frac{1}{r_n}\sum\nolimits_{k=1}^{n} s_{n-k+1} t_k\, x_k \qquad (n \geq 1).$$

(8) The $\boldsymbol{\lambda}$-sequence spaces have been introduced by Mursaleen and Noman in 2010 - 2011 [39, 41, 42] as domains of the λ-matrix Λ in the spaces $c_0,\ c,\ \ell_\infty$ and ℓ_p for $1 < p < \infty$, that is

$$c_0^\lambda = \{x \in w : \Lambda(x) \in c_0\}, \quad c^\lambda = \{x \in w : \Lambda(x) \in c\},$$

$$\ell_\infty^\lambda = \{x \in w : \Lambda(x) \in \ell_\infty\}, \quad \ell_p^\lambda = \{x \in w : \Lambda(x) \in \ell_p\}.$$

Also c_0^λ, c^λ and ℓ_∞^λ are BK spaces with $\|x\|_\Lambda = \|\Lambda(x)\|_\infty$ and all ℓ_p^λ are BK spaces with $\|x\|_{\Lambda_p} = \|\Lambda(x)\|_p$ $(1 < p < \infty)$, where $\lambda = (\lambda_k)$ is a strictly increasing sequence of positive reals and

$$\Lambda_n(x) = \frac{1}{\lambda_n}\sum\nolimits_{k=1}^{n} (\lambda_k - \lambda_{k-1})\, x_k \qquad (n \geq 1).$$

It is worth mentioning that the notions of λ-matrix and λ-sequence spaces have been taken away by researchers and authors upto so far limits. For instance, they have introduced the concept of almost convergence of double sequences by using the λ-matrix and λ-sequence spaces (e.g., by Ahmad and Ganie in 2013 [1] and by Raj with others in 2015 [52]) and they have studied the general difference forms of the λ-matrix and λ-sequence spaces (e.g., by Sönmez and others in 2012 [56] and by Bişgin with others in 2014 [11]). Further, we will display the following studies depending on the λ-matrix and λ-sequence spaces:

(9) The $\boldsymbol{A_\lambda}$-sequence spaces have been constructed by Braha and Başar in 2013 [14] as domains of the matrix A_λ in the spaces c_0, c and ℓ_∞, that is

$$A_\lambda(\mu) = \{x \in w : A_\lambda(x) \in \mu\} \qquad (\mu = c_0, c \text{ or } \ell_\infty)$$

which are BK spaces with $\|x\|_{A_\lambda} = \|A_\lambda(x)\|_\infty$, where A_λ is the same matrix Λ with the sequence $\Delta(\lambda)$ instead of λ provided that $\Delta(\lambda)$ is increasing, that is

$$(A_\lambda)_n(x) = \frac{1}{\Delta(\lambda_n)} \sum\nolimits_{k=1}^{n} \left(\Delta(\lambda_k) - \Delta(\lambda_{k-1})\right) x_k \qquad (n \geq 1).$$

(10) The $\boldsymbol{\Delta_u^\lambda}$-sequence spaces have been studied by Ganie and Sheikh in 2013 [19] as domains of the matrix Δ_u^λ in the spaces c_0, c and ℓ_∞, that is

$$\mu(\Delta_u^\lambda) = \{x \in w : \Delta_u^\lambda(x) \in \mu\} \qquad (\mu = c_0, c \text{ or } \ell_\infty)$$

which are BK spaces with $\|x\|_{\Delta_u^\lambda} = \|\Delta_u^\lambda(x)\|_\infty$, where $u = (u_k)$ is a real or complex sequence of non-zero terms and

$$(\Delta_u^\lambda)_n(x) = \frac{1}{\lambda_n} \sum\nolimits_{k=1}^{n} u_k (\lambda_k - \lambda_{k-1}) (x_k - x_{k-1}) \qquad (n \geq 1).$$

(11) The $\boldsymbol{\Delta_v^\lambda}$-sequence spaces have been introduced by Ercan and Bektaş in 2014 [17] as domains of the matrix Δ_v^λ in the spaces c_0, c and ℓ_∞, that is

$$\mu^\lambda(\Delta_v) = \{x \in w : \Delta_v^\lambda(x) \in \mu\} \qquad (\mu = c_0, c \text{ or } \ell_\infty)$$

which are BK spaces with $\|x\|_{\Delta_v^\lambda} = \|\Delta_v^\lambda(x)\|_\infty$, where $v = (v_k)$ is a real or complex sequence of non-zero terms and

$$(\Delta_v^\lambda)_n(x) = \frac{1}{\lambda_n} \sum\nolimits_{k=1}^{n} (\lambda_k - \lambda_{k-1}) (v_k x_k - v_{k-1} x_{k-1}) \qquad (n \geq 1).$$

(12) The $\boldsymbol{U^{\lambda}}$-sequence spaces have been constructed by Zeren and Bektaş in 2014 [67] as domains of the matrix U^{λ} in the spaces c_0, c and ℓ_∞, that is

$$\mu^{\lambda}(u) = \{x \in w : U^{\lambda}(x) \in \mu\} \qquad (\mu = c_0, c \text{ or } \ell_\infty)$$

which are BK spaces with $\|x\|_{U^{\lambda}} = \|U^{\lambda}(x)\|_\infty$, where $u = (u_k)$ is a real or complex sequence of non-zero terms and

$$U_n^{\lambda}(x) = \frac{u_n}{\lambda_n}\sum\nolimits_{k=1}^{n}(\lambda_k - \lambda_{k-1})\, x_k \qquad (n \geq 1).$$

(13) The binomial sequence spaces have been studied by Bişgin in 2016 [12] as domains of the binomial matrix $B^{r,s}$ in the spaces ℓ_p for $1 \leq p \leq \infty$, that is

$$b_p^{r,s} = \{x \in w : B^{r,s}(x) \in \ell_p\} \qquad (1 \leq p \leq \infty)$$

which are BK spaces with $\|x\|_{B_p^{r,s}} = \|B^{r,s}(x)\|_p$ $(1 \leq p \leq \infty)$, where r and s are non-zero reals such that $r + s \neq 0$ and

$$B_n^{r,s}(x) = \frac{1}{(r+s)^{n-1}}\sum\nolimits_{k=1}^{n}\binom{n-1}{k-1} s^{n-k}\, r^{k-1}\, x_k \qquad (n \geq 1).$$

(14) The Taylor sequence spaces have been introduced by Talebi in 2017 [59] as domains of the Taylor matrix T^{θ} in the spaces ℓ_p for $1 \leq p \leq \infty$, that is

$$t_p^{\theta} = \{x \in w : T^{\theta}(x) \in \ell_p\} \qquad (1 \leq p \leq \infty)$$

which are BK spaces with $\|x\|_{T_p^{\theta}} = \|T^{\theta}(x)\|_p$ $(1 \leq p \leq \infty)$, where $0 \leq \theta < 1$ and

$$T_n^{\theta}(x) = \sum\nolimits_{k=n}^{\infty}\binom{k}{n}(1-\theta)^{n+1}\, \theta^{k-n}\, x_k \qquad (n \geq 1).$$

(15) The Pascal sequence spaces have been constructed by Aydin and Polat in 2018 [7] as domains of the Pascal matrix P in the spaces c_0, c and ℓ_∞, that is

$$P_0 = \{x \in w : P(x) \in c_0\}, \quad P_c = \{x \in w : P(x) \in c\}, \quad P_\infty = \{x \in w : P(x) \in \ell_\infty\}$$

which are BK spaces with $\|x\|_P = \|P(x)\|_\infty$, where

$$P_n(x) = \sum\nolimits_{k=1}^{n} \binom{n-1}{n-k} x_k \qquad (n \geq 1).$$

(16) The Pascal difference spaces have been studied by Aydin and Polat in 2019 [50] as domains of the generalized band matrix $\Delta^{(m)}$ of order m in the Pascal sequence spaces P_0, P_c and P_∞, that is

$$\eta(\Delta^{(m)}) = \{x \in w : \Delta^{(m)}(x) \in \eta\} \qquad (\eta = P_0, P_c \text{ or } P_\infty)$$

which are BK spaces with $\|x\|_{P(\Delta^{(m)})} = \|P(\Delta^{(m)}(x))\|_\infty$, where P is Pascal matrix and

$$\Delta_n^{(m)}(x) = \sum\nolimits_{k=\max\{1,\, n-m\}}^{n} (-1)^{n-k} \binom{m}{n-k} x_k \qquad (n \geq 1).$$

(17) The tribonacci sequence spaces have been introduced by Yaying and Kara in 2021 [65] as domains of the tribonacci matrix T in the spaces c_0 and c, that is

$$c_0(T) = \{x \in w : T(x) \in c_0\} \quad \text{and} \quad c(T) = \{x \in w : T(x) \in c\}$$

which are BK spaces with $\|x\|_T = \|T(x))\|_\infty$, where

$$T_n(x) = \frac{2}{t_{n+2} + t_n - 1} \sum\nolimits_{k=1}^{n} t_k\, x_k \qquad (n \geq 1)$$

and $t = (t_k)$ is the sequence of tribonacci numbers [65].

Furthermore, we refer the reader to [4, 5, 6, 10, 11, 16, 18, 20, 22, 23, 24, 25, 26, 27, 36, 47, 53, 55, 63, 64] for additional similar studies constructing new sequence spaces by means of the concept of matrix domains.

1.2 Research Methodology

In this section, we display the research methodology used in our investigation.

1.2.1 Research Problem

By going on through the previous studies in the last section, it maybe noted that there is a gap in the knowledge which was left by researchers in the literature of the modern theory of sequence spaces. More precisely, many new sequence spaces of the λ-type have been introduced and studied, but the λ-sequence spaces defined by series have not. Thus, we are going to fill up that gap in the literature by introducing and study the λ-sequence spaces of bounded, convergent and null series.

1.2.2 Research Objectives

In this study, our aim is to add the following contributions:

- Introducing some new λ-sequence spaces of bounded, convergent and null series.
- Study some algebraic and topological properties of our new λ-sequence spaces.
- Constructing the Schauder bases for our new λ-sequence spaces.
- Deducing some new inclusion relations between these new spaces.
- Concluding the Köthe-Toeplitz duals of our new λ-sequence spaces.
- Characterizing some new classes of matrix operators between our spaces.

1.2.3 Research Tools

In the present thesis, our study and investigation will be based on the usual mathematical tools as the proof and conclusion, and the usual mathematical methodology as the mathematical induction and investigation. Also, many mathematical concepts will be used as main tools in our thesis, and the most important tools among them are sequence, series, matrix and space.

1.3 Preliminaries

In this section, we give a list of the preliminary results which will be used in proving the main results in this thesis. These preliminaries are already known in the literature of the theory of sequence spaces and matrix transformations.

Lemma 1.3.1 (Boos [13]) *If $p < p'$ $(1 \leq p < \infty)$; then the inclusions $\ell_p \subset \ell_{p'}$ and $bv_p \subset bv_{p'}$ are strictly satisfied. Further, we have the following strict inclusions:*

$$c_0 \subset c \subset \ell_\infty, \quad \ell_p \subset c_0, \quad \ell_p \subset bv_p \subset c_0(\Delta), \quad cs_0 \subset cs \subset bs,$$

$$c_0(\Delta) \subset c(\Delta) \subset \ell_\infty(\Delta), \quad bs \subset \ell_\infty \subset \ell_\infty(\Delta), \quad bv_1 \subset c \subset c_0(\Delta), \quad \ell_1 \subset cs \subset c_0.$$

Lemma 1.3.2 (Maddox [32]) *We have the following facts:*

(1) *The spaces ℓ_∞, c and c_0 are BK spaces with the sup-norm $\|\cdot\|_\infty$ given by $\|x\|_\infty = \sup_k |x_k|$.*

(2) *The spaces bs, cs and cs_0 are BK spaces with the series-norm $\|\cdot\|_s$ defined by $\|x\|_s = \sup_n |\sum_{k=1}^{n} x_k|$.*

Lemma 1.3.3 (Malkowsky and others [34]) *We have the following facts:*

(1) *The sequence $(e_1, e_2, e_3, \cdots)$ is a Schauder basis for the space c_0 and every $x \in c_0$ has the unique representation $x = \sum_{k=1}^{\infty} x_k\, e_k$.*

(2) *The sequence $(e, e_1, e_2, \cdots)$ is a Schauder basis for the space c and every $x \in c$ has the unique representation $x = Le + \sum_{k=1}^{\infty}(x_k - L)\, e_k$, where $L = \lim_{k\to\infty} x_k$.*

(3) *The spaces c_0 and c are separable while the space ℓ_∞ is not separable and has no Schauder basis (in general, if X is a Banach sequence space with Schauder basis; then it must be separable).*

Lemma 1.3.4 (Wilansky [61]) *Let X and Y be sequence spaces. Then, we have the following:*

(1) $X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$.

(2) *If $X \subset Y$; then $Y^{\theta} \subset X^{\theta}$, where $\theta = \alpha$, β or γ.*

(3) $c_0^{\theta} = c^{\theta} = \ell_{\infty}^{\theta} = \ell_1$, $\ell_1^{\theta} = \ell_{\infty}$ *and* $\ell_p^{\theta} = \ell_q$ *for* $p > 1$ *with* $q = p/(p-1)$.

Lemma 1.3.5 (Darling [21]) *We have the following:*

(1) ${cs_0}^{\alpha} = \ell_1$, $cs^{\alpha} = \ell_1$ *and* $bs^{\alpha} = \ell_1$.

(2) ${cs_0}^{\beta} = bv_1$, $cs^{\beta} = bv_1$ *and* $bs^{\beta} = bv_0$.

(3) ${cs_0}^{\gamma} = bv_1$, $cs^{\gamma} = bv_1$ *and* $bs^{\gamma} = bv_1$.

Lemma 1.3.6 (Banaś and others [8]) *Let X, Y and Z be sequence spaces, and A an infinite matrix. Then:*

(1) $A \in (X, Y) \Longleftrightarrow A_n \in X^{\beta}$ *for every* $n \geq 1$ *and* $A(x) \in Y$ *for all* $x \in X$.

(2) *If $X \subset Y$; then $(Y, Z) \subset (X, Z)$.*

(3) *If $Y \subset Z$; then $(X, Z) \subset (X, Y)$.*

Lemma 1.3.7 (Malkowsky [33], Wilansky [61]) *Let X and Y be sequence spaces, A an infinite matrix and T a triangle. Then, we have the following facts:*

(1) $T \in (X, Y) \Longleftrightarrow T(x) \in Y$ *for all* $x \in X$ *(note that: $T(x)$ exists for all $x \in w$).*

(2) *If X is a BK space with a norm $\|\cdot\|$; then X_T ia a BK space with the norm $\|\cdot\|_T$ defined by $\|x\|_T = \|T(x)\|$ for all $x \in X_T$.*

(3) $A \in (X, Y_T) \Longleftrightarrow TA \in (X, Y)$.

Further, it seems to be quite natural, in view of the fact that matrix operators between BK spaces are continuous, to find necessary and sufficient conditions for the entries of an infinite matrix to define a linear operator between BK spaces which means the characterization of matrix classes concerning sequence spaces. The following familiar results can be found in the paper of Stieglitz and Tietz [58, pp. 2–9] and will be needed to prove our main results in the next chapters. In the following results, we will use the symbol μ to be any one of the spaces c_0, c or ℓ_∞, and $\mathcal{K}$ stands for the collection of all non-empty finite subsets of positive integers.

Lemma 1.3.8 *Let $1 \leq p < \infty$. Then, we have $(c_0, \ell_p) = (c, \ell_p) = (\ell_\infty, \ell_p)$, and $A \in (\mu, \ell_p)$ if and only if the following condition holds:*

$$\sup_{K \in \mathcal{K}} \sum_{n=1}^{\infty} \left| \sum_{k \in K} a_{nk} \right|^p < \infty ,$$

where $\mathcal{K}$ stands for the collection of all non-empty finite subsets of positive integers.

Lemma 1.3.9 *We have $(c_0, \ell_\infty) = (c, \ell_\infty) = (\ell_\infty, \ell_\infty)$, and $A \in (\mu, \ell_\infty)$ if and only if the following condition holds:*

$$\sup_{n} \sum_{k=1}^{\infty} |a_{nk}| < \infty . \tag{1.3.1}$$

Lemma 1.3.10 *We have the following:*

(1) *$A \in (\ell_\infty, c)$ if and only if (1.3.1) and the following conditions hold:*

$$\lim_{n\to\infty} a_{nk} = a_k \ \textit{ exists for every } k \geq 1 , \tag{1.3.2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk} - a_k| = 0 .$$

(2) *$A \in (c, c)$ if and only if (1.3.1), (1.3.2) and the following condition hold:*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = a \ \textit{ exists.}$$

(3) *$A \in (c_0, c)$ if and only if (1.3.1) and (1.3.2) hold.*

Lemma 1.3.11 *We have the following:*

(1) $A \in (\ell_\infty, c_0)$ *if and only if the following condition holds:*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = 0.$$

(2) $A \in (c, c_0)$ *if and only if* (1.3.1) *and the following conditions hold:*

$$\lim_{n\to\infty} a_{nk} = 0 \text{ for every } k \geq 1, \tag{1.3.3}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 0.$$

(3) $A \in (c_0, c_0)$ *if and only if* (1.3.1) *and* (1.3.3) *hold.*

Lemma 1.3.12 *We have the following:*

(1) $A \in (\ell_1, \ell_\infty)$ *if and only if the following condition holds:*

$$\sup_{n,\,k} |a_{nk}| < \infty. \tag{1.3.4}$$

(2) $A \in (\ell_1, c)$ *if and only if* (1.3.2) *and* (1.3.4) *hold.*

(3) $A \in (\ell_1, c_0)$ *if and only if* (1.3.3) *and* (1.3.4) *hold.*

Lemma 1.3.13 *Let* $1 < p < \infty$ *and* $q = p/(p-1)$*. Then, we have the following:*

(1) $A \in (\ell_p, \ell_\infty)$ *if and only if the following condition holds:*

$$\sup_{n} \sum_{k=1}^{\infty} |a_{nk}|^q < \infty. \tag{1.3.5}$$

(2) $A \in (\ell_p, c)$ *if and only if* (1.3.2) *and* (1.3.5) *hold.*

(3) $A \in (\ell_p, c_0)$ *if and only if* (1.3.3) *and* (1.3.5) *hold.*

Lemma 1.3.14 *We have the following:*

(1) $A \in (cs, c)$ *if and only if* (1.3.2) *and the following condition hold:*

$$\sup_n \sum_{k=1}^{\infty} |a_{nk} - a_{n,\,k+1}| < \infty. \tag{1.3.6}$$

(2) $A \in (bs, \ell_\infty)$ *if and only if both* (1.3.6) *and the following condition hold:*

$$\lim_{k\to\infty} a_{nk} = 0 \quad \textit{for every} \quad n \geq 1. \tag{1.3.7}$$

Chapter 2

NEW λ-SEQUENCE SPACES

2 NEW λ-SEQUENCE SPACES

The approach constructing a new sequence space by means of the matrix domain of a particular triangle has recently been employed by several authors in many research papers (see for example [19, 28, 33, 40, 42, 50, 60, 66]). In the present chapter, we study some additional properties of the well-known spaces bs, cs and cs_0 of bounded, convergent and null series, respectively. After that, we introduce the new λ-sequence spaces bs^λ, cs^λ and cs_0^λ of bounded, convergent and null series, respectively. Further, we study some algebraic and topological properties of our new spaces. Finally, we construct the Schauder basis for the spaces cs^λ and cs_0^λ with concluding their separability. This chapter is divided into three sections, the first is devoted to study the sequence spaces defined by series, the second is for introducing our new spaces with study their properties and the last is to construct their Schauder bases. The materials of this chapter are part of our research paper [48] which has been published in the Albaydha Univ. J., and presented in the 2nd conference of Albaydha University on 2021.

2.1 Sequence Spaces Via Series

In this section, we study some additional properties of the famous spaces bs, cs and cs_0 of bounded, convergent and null series, respectively. These spaces have been defined as the domains of the triangle σ, so-called the sum-matrix, in the spaces ℓ_∞, c and c_0, respectively. That is $bs = (\ell_\infty)_\sigma$, $cs = (c)_\sigma$ and $cs_0 = (c_0)_\sigma$ which can be written as $bs = \{x \in w : \sigma(x) \in \ell_\infty\}$, $cs = \{x \in w : \sigma(x) \in c\}$ and $cs_0 = \{x \in w : \sigma(x) \in c_0\}$, where $\sigma(x) = (\sigma_n(x))$ with $\sigma_n(x) = \sum_{k=1}^{n} x_k$ for all $n \geq 1$. Further, since ℓ_∞, c and c_0

are BK spaces with $\|\cdot\|_\infty$ and σ is a triangle; it follows that bs, cs and cs_0 are BK spaces with the norm $\|\cdot\|_s$ given by $\|x\|_s = \sup_n |\sum_{k=1}^{n} x_k|$ (see (2) of Lemma 1.3.2). Also, we may begin with proving the following results:

Lemma 2.1.1 *The spaces bs, cs and cs_0 are isometrically linear-isomorphic to the spaces ℓ_∞, c and c_0, respectively. That is $bs \cong \ell_\infty$, $cs \cong c$ and $cs_0 \cong c_0$.*

Proof. Let μ be standing for any one of the spaces ℓ_∞, c and c_0, and let $\bar{\mu}$ be the respective one of the spaces bs, cs and cs_0. Then, it follows by definition that the spaces $\bar{\mu}$ are the domains of the sum-matrix σ in the spaces μ, that is $\bar{\mu} = \mu_\sigma$ and so we have the linear operator $\sigma : \bar{\mu} \to \mu$. Also, since σ is a triangle and so invertible with $\sigma^{-1} = \Delta$ [34]; we deduce that σ is a linear bijection preserving the norm, where $\|\sigma(x)\|_\infty = \|x\|_s$ for all $x \in \bar{\mu}$. Hence, the spaces $\bar{\mu}$ are isometrically linear-isomorphic to the spaces μ, that is $\bar{\mu} \cong \mu$ and this completes the proof. $\square$

Lemma 2.1.2 *Suppose that $\hat{e}_1 = (1, -1, 0, 0, \cdots)$, $\hat{e}_2 = (0, 1, -1, 0, 0, \cdots)$, $\cdots$ etc. Then, the sequence $(\hat{e}_1, \hat{e}_2, \hat{e}_3 \cdots)$ is a Schauder basis for the space cs_0 and every $x \in cs_0$ has the unique representation $x = \sum_{k=1}^{\infty} \sigma_k(x)\, \hat{e}_k$. Also, the sequence $(\hat{e}, \hat{e}_1, \hat{e}_2, \hat{e}_3 \cdots)$ is a Schauder basis for the space cs and every $x \in cs$ has the unique representation $x = L\hat{e} + \sum_{k=1}^{\infty} (\sigma_k(x) - L)\, \hat{e}_k$, where $\hat{e} = e_1$ and $L = \lim_{n\to\infty} \sigma_n(x)$.*

Proof. This result is immediate by Lemma 1.3.3, since the spaces $\bar{\mu}$ are isometrically linear-isomorphic to the spaces μ (Lemma 2.1.1). $\square$

Lemma 2.1.3 *The spaces cs_0 and cs are separable while the space bs is not separable and has no Schauder basis.*

Proof. Since cs_0 and cs are BK spaces and so Banach spaces having Schauder bases (by Lemma 2.1.2); this result follows from (3) of Lemma 1.3.3. $\square$

Moreover, the α-, β- and γ-duals of the spaces bs, cs and cs_0 are given in Lemma 1.3.5 and some inclusion relations concerning these spaces can be found in Lemma 1.3.1. For example, we have the strict inclusions $cs_0 \subset cs \subset bs$, $\ell_1 \subset cs \subset c_0$ and $bs \subset \ell_\infty$. Furthermore, we prove the following result:

Lemma 2.1.4 *We have the following:*

(1) *The inclusion $c \cap bs \subset c_0$ holds.*

(2) *The equality $c \cap bs = c_0 \cap bs$ holds.*

(3) *The inclusion $cs \subset c_0 \cap bs$ strictly holds.*

Proof. For (1), take any $x \in c \cap bs$. Then $x \in c$ as well as $x \in bs$ and so $\sigma(x) \in \ell_\infty$. Also, since $x \in c$; the limit $\lim_{k\to\infty} x_k$ exists and we must have $\lim_{k\to\infty} \sigma_k(x)/k = \lim_{k\to\infty} x_k$ by the regularity of the Cesàro matrix C^1 of arithmetic mean [46, 54], where $C^1(x) = (\sigma_k(x)/k)$ (see the 1st study in section 1.1.3, p.12). But $\sigma(x) \in \ell_\infty$ (by assumption) and so $\lim_{k\to\infty} \sigma_k(x)/k = 0$ which implies that $\lim_{k\to\infty} x_k = 0$ (as $\lim_{k\to\infty} x_k = \lim_{k\to\infty} \sigma_k(x)/k$). Thus $x \in c_0$ and it follows that the inclusion $c \cap bs \subset c_0$ holds. To prove (2), we have $c_0 \subset c$ and so $c_0 \cap bs \subset c \cap bs$. Also, for the converse inclusion, it is clear that $c \cap bs \subset bs$ and we have $c \cap bs \subset c_0$ by part (1) which together imply that $c \cap bs \subset c_0 \cap bs$. Therefore, we deduce the equality $c \cap bs = c_0 \cap bs$. For the final part (3), it is obvious that the inclusion $cs \subset c_0 \cap bs$ holds by Lemma 1.3.1. To show that this inclusion is strict, consider the sequence $x = (x_k)$ defined by $x_k = (-1)^{n-1}/n$ for $(n^2 - n)/2 < k \leq (n^2 + n)/2$, where n is any positive integer, i.e.

$$x = \Big(1, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \cdots\Big).$$

Then, obviously $x \in c_0$ and we obtain that

$$\sigma(x) = \Big(1, \frac{1}{2}, 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{3}{4}, \frac{2}{4}, \frac{1}{4}, 0, \cdots\Big)$$

which can be written as follows

$$\sigma_k(x) = \frac{1}{2} + (-1)^n \left(\frac{2k - (n+1)^2}{2(n+1)} \right) \text{ for } \frac{n(n+1)}{2} \leq k \leq \frac{(n+1)(n+2)}{2} \qquad (n \geq 1).$$

Thus, we deduce that $-1/2 \leq [2k - (n+1)^2]/[2(n+1)] \leq 1/2$ whenever $n(n+1)/2 \leq k \leq (n+1)(n+2)/2$ and so $0 \leq \sigma_k(x) \leq 1$ for all $k \geq 1$ which means that $\sigma(x) \in \ell_\infty$ and hence $x \in c_0 \cap bs$. But, it is obvious that $\sigma(x) \notin c$ (as $\sigma_{n(n+1)/2}(x) = (1 - (-1)^n)/2$ for all n) and so $x \notin cs$. That is, there exists a null sequence whose a bounded sum but not convergent. Therefore, the inclusion $cs \subset c_0 \cap bs$ is strict. □

Finally, we end this section with the following basic example:

Example 2.1.5 Consider the sequences x, y and z given by $x_k = \Delta(k/(k+1)^2)$, $y_k = k/(k+1)!$ and $z_k = (-1)^{k-1}$ for all $k \geq 1$. Then, it can easily be seen that $\sigma(x) = (n/(n+1)^2) \in c_0$, $\sigma(y) = (1 - 1/(n+1)!) \in c$ and $\sigma(z) = (\,(1-(-1)^n)/2\,) \in \ell_\infty$. Thus, we find that $x \in cs_0$, $y \in cs$ and $z \in bs$.

2.2 $\boldsymbol{\lambda}$-Sequence Spaces

In this section, we present the idea of λ-sequence spaces and introduce the new λ-sequence spaces bs^λ, cs^λ and cs_0^λ of bounded, convergent and null series, respectively. Also, we show that our new spaces are BK spaces and conclude their isomorphic relations with the spaces ℓ_∞, c and c_0, and with the spaces bs, cs and cs_0.

Throughout this study, we assume that $\lambda = (\lambda_j)_{j=1}^\infty$ is a strictly increasing sequence of positive reals tending to ∞. That is $0 < \lambda_1 < \lambda_2 < \cdots$ and $\lambda_j \to \infty$ as $j \to \infty$. Also, we define the λ-triangle $\Lambda = [\lambda_{nk}]_{k,n=1}^\infty$ for every $n, k \geq 1$ by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n}\,; & (1 \leq k \leq n), \\ 0\,; & (k > n \geq 1), \end{cases} \tag{2.2.1}$$

where $\lambda_0 = 0$. Then, for every $x \in w$, the Λ-transform of x is the sequence $\Lambda(x) = (\Lambda_n(x))_{n=1}^{\infty}$ given by

$$\Lambda_k(x) = \frac{1}{\lambda_k} \sum_{j=1}^{k} (\lambda_j - \lambda_{j-1})\, x_j \qquad (k \geq 1). \tag{2.2.2}$$

The λ-sequence spaces c_0^λ, c^λ, ℓ_∞^λ and ℓ_p^λ $(1 \leq p < \infty)$ have been introduced by Mursaleen and Noman [39, 41] as the matrix domains of Λ in the spaces c_0, c, ℓ_∞ and ℓ_p, respectively. That is

$$c_0^\lambda = (c_0)_\Lambda = \{x \in w : \ \Lambda(x) \in c_0\},$$

$$c^\lambda = (c)_\Lambda = \{x \in w : \ \Lambda(x) \in c\},$$

$$\ell_\infty^\lambda = (\ell_\infty)_\Lambda = \{x \in w : \ \Lambda(x) \in \ell_\infty\},$$

$$\ell_p^\lambda = (\ell_p)_\Lambda = \{x \in w : \ \Lambda(x) \in \ell_p\} \qquad (1 \leq p < \infty).$$

Also, it has been shown that the spaces c_0^λ, c^λ and ℓ_∞^λ are BK spaces with the norm $\|x\|_{\Lambda_\infty} = \sup_k |\Lambda_k(x)|$ and the spaces ℓ_p^λ $(1 \leq p < \infty)$ are BK spaces with the norm $\|x\|_{\Lambda p} = \left(\sum_{k=1}^{\infty} |\Lambda_k(x)|^p\right)^{1/p}$. Further, the following important results will be used in the sequel which can be found in [39] and [41].

Lemma 2.2.1 *We have the following:*

(1) *The matrix Λ is regular, that is $\lim_{k\to\infty} \Lambda_k(x) = \lim_{k\to\infty} x_k$ for every $x \in c$.*

(2) *The following equality satisfies for every $x \in w$:*

$$x_k - \Lambda_{k-1}(x) = \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} [\Lambda_k(x) - \Lambda_{k-1}(x)] \qquad (k \geq 1). \tag{2.2.3}$$

Lemma 2.2.2 *Let $1 \leq p < \infty$. Then, we have the following:*

(1) *The inclusions $\ell_p^\lambda \subset c_0^\lambda \subset c^\lambda \subset \ell_\infty^\lambda$ strictly hold.*

(2) *The inclusions $c_0 \subset c_0^\lambda$, $c \subset c^\lambda$ and $\ell_\infty \subset \ell_\infty^\lambda$ hold.*

(3) *The inclusion* $\ell_p \subset \ell_p^\lambda$ *holds if and only if* $1/\lambda \in \ell_p$, *where* $1/\lambda = (1/\lambda_j)_{j=1}^\infty$.

(4) *The equalities* $\ell_p^\lambda = \ell_p$, $c_0^\lambda = c_0$, $c^\lambda = c$ *and* $\ell_\infty^\lambda = \ell_\infty$ *hold if and only if* $\lambda/\Delta(\lambda) \in \ell_\infty$, *where* $\lambda/\Delta(\lambda) = (\,\lambda_j/\Delta(\lambda_j)\,)_{j=1}^\infty$.

We refer the reader to [39] and [41] for additional knowledge concerning the λ-sequence spaces. Now, as a natural continuation of above work, we introduce, in the next definition, the new λ-sequence spaces of bounded, convergent and null series.

Definition 2.2.3 *The λ-sequence spaces bs^λ, cs^λ and cs_0^λ are defined as the matrix domains of the triangle Λ in the spaces bs, cs and cs_0, respectively. That is*

$$bs^\lambda = (bs)_\Lambda = \{x \in w : \Lambda(x) \in bs\},$$

$$cs^\lambda = (cs)_\Lambda = \{x \in w : \Lambda(x) \in cs\},$$

$$cs_0^\lambda = (cs_0)_\Lambda = \{x \in w : \Lambda(x) \in cs_0\}.$$

So that, our contribution is the following new spaces:

$$bs^\lambda = \left\{ x \in w \,:\, \sup_n \left| \sum_{k=1}^n \Lambda_k(x) \right| < \infty \right\},$$

$$cs^\lambda = \left\{ x \in w \,:\, \lim_{n\to\infty} \sum_{k=1}^n \Lambda_k(x) \text{ exists} \right\},$$

$$cs_0^\lambda = \left\{ x \in w \,:\, \lim_{n\to\infty} \sum_{k=1}^n \Lambda_k(x) = 0 \right\}.$$

Besides, we define the triangle $\hat{\Lambda} = [\hat{\lambda}_{nk}]_{k,n=1}^\infty$ for every $n, k \geq 1$ by

$$\hat{\lambda}_{nk} = \begin{cases} (\lambda_k - \lambda_{k-1}) \displaystyle\sum_{j=k}^n \frac{1}{\lambda_j}\,; & (1 \leq k \leq n), \\ 0\,; & (k > n \geq 1). \end{cases} \tag{2.2.4}$$

Then, for every sequence $x \in w$, we have

$$\hat{\Lambda}_n(x) = \sum_{k=1}^n \left(\sum_{j=k}^n \frac{1}{\lambda_j} \right) (\lambda_k - \lambda_{k-1}) x_k \qquad (n \geq 1). \tag{2.2.5}$$

Thus, it can easily be seen that $\hat{\Lambda}(x) = \sigma(\,\Lambda(x)\,)$ for all $x \in w$ which can be written as

$$\hat{\Lambda}_n(x) = \sigma_n(\Lambda(x)) = \sum_{k=1}^{n} \Lambda_k(x) \qquad (n \geq 1). \tag{2.2.6}$$

This means that $\hat{\Lambda} = \sigma\Lambda$ and it follows that bs^{λ}, cs^{λ} and cs_0^{λ} are sequence spaces which can be redefined as follows:

Definition 2.2.4 *The λ-sequence spaces bs^{λ}, cs^{λ} and cs_0^{λ} are defined as the matrix domains of the triangle $\hat{\Lambda}$ in the spaces ℓ_∞, c and c_0, respectively. That is*

$$bs^{\lambda} = (\ell_\infty)_{\hat{\Lambda}}\,, \quad cs^{\lambda} = (c)_{\hat{\Lambda}} \quad \text{and} \quad cs_0^{\lambda} = (c_0)_{\hat{\Lambda}}. \tag{2.2.7}$$

Thus, from the definition, it follows that

$$bs^{\lambda} = \{x \in w : \hat{\Lambda}(x) \in \ell_\infty\},$$

$$cs^{\lambda} = \{x \in w : \hat{\Lambda}(x) \in c\},$$

$$cs_0^{\lambda} = \{x \in w : \hat{\Lambda}(x) \in c_0\}.$$

Now, we may begin with the following results which are essential for our study.

Theorem 2.2.5 *The λ-sequence spaces bs^{λ}, cs^{λ} and cs_0^{λ} are BK spaces with the norm $\|\cdot\|_{s^{\lambda}}$ defined, for every sequence x in these spaces, by*

$$\|x\|_{s^{\lambda}} = \|\hat{\Lambda}(x)\|_\infty = \sup_n \left|\hat{\Lambda}_n(x)\right| = \sup_n \left|\sum_{k=1}^{n} \Lambda_k(x)\right|.$$

Proof. Since ℓ_∞, c and c_0 are BK spaces with their natural norm $\|\cdot\|_\infty$ by (1) of Lemma 1.3.2 and $\hat{\Lambda}$ is a triangle; this result is immediate by (2.2.7) with help (2) of Lemma 1.3.7 (this result can also be proved by using (2) of Lemma 1.3.2 and (2) of Lemma 1.3.7 with help of Definition 2.2.3). □

Theorem 2.2.6 *The λ-sequence spaces bs^{λ}, cs^{λ} and cs_0^{λ} are isometrically linear-isomorphic to the spaces ℓ_∞, c and c_0, respectively. That is $bs^{\lambda} \cong \ell_\infty$, $cs^{\lambda} \cong c$, and $cs_0^{\lambda} \cong c_0$.*

Proof. To prove this result, we will show that there exists a linear bijection between the spaces bs^λ and ℓ_∞ which preserves the norm. For this, we can use (2.2.7) from Definition 2.2.4 of the space bs^λ to define the matrix operator $\hat{\Lambda} : bs^\lambda \to \ell_\infty$ by $x \mapsto \hat{\Lambda}(x)$ for all $x \in bs^\lambda$, which is a linear operator. Then, it is obvious that $\hat{\Lambda}(x) = 0$ implies $x = 0$, and so $\hat{\Lambda}$ is injective. Also, let $y \in \ell_\infty$ be given and define a sequence $x = (x_j)$ in terms of the sequence y by

$$x_j = \frac{\Delta(\,\lambda_j\,\Delta(y_j)\,)}{\Delta(\lambda_j)} = \frac{\lambda_j\,\Delta(y_j) - \lambda_{j-1}\,\Delta(y_{j-1})}{\lambda_j - \lambda_{j-1}} \qquad (j \geq 1), \tag{2.2.8}$$

where $y_0 = \lambda_0 = 0$. Then, it follows by (2.2.2) that

$$\Lambda_k(x) = \frac{1}{\lambda_k}\sum_{j=1}^{k}[\lambda_j\,\Delta(y_j) - \lambda_{j-1}\,\Delta(y_{j-1})] = \ \Delta(y_k) \qquad (k \geq 1).$$

Thus, by using (2.2.6), we find that $\hat{\Lambda}_n(x) = \sum_{k=1}^{n}\Delta(y_k) = \ y_n$ for all n, which means that $\hat{\Lambda}(x) = y$, but $y \in \ell_\infty$ and so $\hat{\Lambda}(x) \in \ell_\infty$. Thus $x \in bs^\lambda$ and this means the existence of $x \in bs^\lambda$ such that $\hat{\Lambda}(x) = y$ and hence $\hat{\Lambda}$ is surjective. Further, it is clear by Theorem 2.2.5 that $\hat{\Lambda}$ is norm preserving, since $\|\hat{\Lambda}(x)\|_\infty = \|x\|_{s^\lambda}$ for every $x \in bs^\lambda$. Therefore, the operator $\hat{\Lambda} : bs^\lambda \to \ell_\infty$ is a linear bijection preserving the norm. That is, our $\hat{\Lambda}$ is an isometry isomorphism between bs^λ and ℓ_∞ which means that $bs^\lambda \cong \ell_\infty$. Similarly, it can be shown that $cs^\lambda \cong c$ and $cs_0^\lambda \cong c_0$. □

Corollary 2.2.7 *The λ-sequence spaces bs^λ, cs^λ and cs_0^λ are isometrically linear-isomorphic to the spaces bs, cs and cs_0, respectively. That is $bs^\lambda \cong bs$, $cs^\lambda \cong cs$, and $cs_0^\lambda \cong cs_0$.*

Proof. It is immediate by combining Lemma 2.1.1 with Theorem 2.2.6. □

Remark 2.2.8 We have already shown in the proof of Theorem 2.2.6 that the matrix $\hat{\Lambda}$ defines a linear operator from any of the spaces bs^λ, cs^λ or cs_0^λ into the respective one of the spaces ℓ_∞, c or c_0, which is an isometry isomorphism, and this implies the continuity of the matrix operator $\hat{\Lambda}$ which will be used in the sequel.

At the end of this section, we give an example to show that our new spaces bs^λ, cs^λ and cs_0^λ are totally different from the spaces ℓ_∞, c, c_0, bs, cs and cs_0. But before that, and for simplicity in notations, we will use our terminologies as in the proof of Lemma 2.1.1. That is, we will use the symbol μ to denote any of the spaces ℓ_∞, c or c_0 and so $\bar{\mu}$ stands for the respective one of the spaces bs, cs or cs_0 while $\bar{\mu}^\lambda$ is the corresponding one of the spaces bs^λ, cs^λ or cs_0^λ, respectively.

Example 2.2.9 In this example, our aim is to show that our spaces $\bar{\mu}^\lambda$ are different from all the sequence spaces μ and $\bar{\mu}$. For this, consider the sequence $\lambda = (\lambda_k)$ defined by $\lambda_k = k$ and so $\Delta(\lambda_k) = 1$ for all $k \geq 1$. Then, for any $x \in w$, we have $\Lambda_k(x) = (1/k)\sum_{j=1}^{k} x_j = \sigma_k(x)/k$ and $\hat{\Lambda}_n(x) = \sum_{k=1}^{n} \Lambda_k(x)$ for all $k, n \geq 1$. Thus, our spaces can be defined as $\bar{\mu}^\lambda = \{x \in w : (\,\sigma_k(x)/k\,) \in \bar{\mu}\} = \{x \in w : (\,\sum_{k=1}^{n} \sigma_k(x)/k\,) \in \mu\}$. Also, define the unbounded sequence $z = (z_k)$ by $z_1 = 1$ and for $k > 1$ by

$$z_k = \begin{cases} k\sqrt{2/(k+1)} + (k-1)\sqrt{2/(k-1)}\,; & (k \text{ is odd}), \\ -(2k-1)\sqrt{2/k}\,; & (k \text{ is even}). \end{cases}$$

Then, we have $z \notin \ell_\infty$ and so $z \notin \mu$ which also implies that $z \notin bs$ and hence $z \notin \bar{\mu}$ which can independently be obtained from its sum sequence $\sigma(z)$, where

$$\sigma_k(z) = \begin{cases} k\sqrt{2/(k+1)}\,; & (k \text{ is odd}), \\ -k\sqrt{2/k}\,; & (k \text{ is even}). \end{cases}$$

Further, by using (2.2.2) and (2.2.6) we respectively obtain that

$$\Lambda_k(z) = \begin{cases} \sqrt{2/(k+1)}\,; & (k \text{ is odd}), \\ -\sqrt{2/k}\,; & (k \text{ is even}), \end{cases}$$

$$\hat{\Lambda}_n(z) = \begin{cases} \sqrt{2/(n+1)}\,; & (n \text{ is odd}), \\ 0\,; & (n \text{ is even}). \end{cases}$$

This implies that $\hat{\Lambda}(z) \in c_0$ and so $z \in cs_0^\lambda$ which leads us to $z \in \bar{\mu}^\lambda$. Hence, we have shown that $z \in \bar{\mu}^\lambda$ while $z \notin \mu$ as well as $z \notin \bar{\mu}$. Therefore, we deduce that $\bar{\mu}^\lambda \not\subset \mu$ and $\bar{\mu}^\lambda \not\subset \bar{\mu}$. On other side, consider the sequence $z' = (z'_k)$ defined by $z'_k = \Delta\left(1/\log(1+k)\right)$ for all $k \geq 1$ with noting that $z'_1 = 1/\log 2$. Then, we get $\sigma(z') = \left(1/\log(1+k)\right) \in c_0$ and so $z' \in cs_0$ which implies both $z' \in \mu$ and $z' \in \bar{\mu}$. Besides, we find that $\Lambda(z') = \left(1/(k\log(1+k))\right)$ and so $\hat{\Lambda}_n(z') = \sum_{k=1}^{n} 1/(k\log(1+k))$ which diverges to ∞ as $n \to \infty$ and this means that $z' \notin bs^\lambda$ and so $z' \notin \bar{\mu}^\lambda$. Hence, we have shown that $z' \notin \bar{\mu}^\lambda$ while $z' \in \mu$ and $z' \in \bar{\mu}$. Therefore, we deduce that $\mu \not\subset \bar{\mu}^\lambda$ as well as $\bar{\mu} \not\subset \bar{\mu}^\lambda$. Consequently, we conclude that all the spaces $\bar{\mu}^\lambda$ are totally different from any of the spaces μ and any of the spaces $\bar{\mu}$, that is $\bar{\mu}^\lambda \neq \ell_\infty$, c, c_0, bs, cs or cs_0.

2.3 Schauder Basis

In the last section, we construct two sequences which form the Schauder bases for the λ-sequence spaces cs_0^λ and cs^λ, and we conclude their separability while the space bs^λ is not separable and has no Schauder basis.

At the beginning, the Schauder bases for the spaces c_0 and c can be found in Lemma 1.3.3 and for the spaces cs_0 and cs are given in Lemma 2.1.2. Thus, these spaces are separable while the spaces ℓ_∞ and bs are not separable and so they have no Schauder bases.

Now, we may begin this section with constructing the Schauder basis for the sequence space cs_0^λ of λ-null series and we will deduce the unique representation of every $x \in cs_0^\lambda$.

Theorem 2.3.1 *For each $k \geq 1$, define the sequence $e_k^\lambda = (e_{nk}^\lambda)_{n=1}^\infty$ for every $n \geq 1$ by*

$$e_{nk}^\lambda = \begin{cases} \dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}}\,; & (n = k), \\ -\left(\dfrac{\lambda_{k+1} + \lambda_k}{\lambda_{k+1} - \lambda_k}\right); & (n = k+1), \\ \dfrac{\lambda_{k+1}}{\lambda_{k+2} - \lambda_{k+1}}\,; & (n = k+2), \\ 0\,; & (\text{otherwise}). \end{cases}$$

Then, the sequence $\left(e_k^\lambda\right)_{k=1}^\infty$ is a Schauder basis for the space cs_0^λ and every $x \in cs_0^\lambda$ has a unique representation of the form

$$x = \sum_{k=1}^\infty \hat{\Lambda}_k(x)\, e_k^\lambda\,. \tag{2.3.1}$$

Proof. For each $k \geq 1$, it can easily be seen that

$$e_k^\lambda = \frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\, e_k - \left(\frac{\lambda_{k+1} + \lambda_k}{\lambda_{k+1} - \lambda_k}\right) e_{k+1} + \frac{\lambda_{k+1}}{\lambda_{k+2} - \lambda_{k+1}}\, e_{k+2}\,.$$

Thus, by using (2.2.2), we find that $\Lambda(e_k^\lambda) = e_k - e_{k+1}$ and so $\hat{\Lambda}(e_k^\lambda) = e_k$. This implies that $\hat{\Lambda}(e_k^\lambda) \in c_0$ and hence $e_k^\lambda \in cs_0^\lambda$ for all $k \geq 1$ which means that $\left(e_k^\lambda\right)_{k=1}^\infty$ is a sequence in cs_0^λ. Further, let $x \in cs_0^\lambda$ be given and for every positive integer m, we put

$$x^{(m)} = \sum_{k=1}^m \hat{\Lambda}_k(x)\, e_k^\lambda\,.$$

Then, by operating $\hat{\Lambda}$ on both sides, we find that

$$\hat{\Lambda}(x^{(m)}) = \sum_{k=1}^m \hat{\Lambda}_k(x)\, \hat{\Lambda}(e_k^\lambda) = \sum_{k=1}^m \hat{\Lambda}_k(x)\, e_k$$

and hence

$$\hat{\Lambda}_n(x - x^{(m)}) = \begin{cases} 0\,; & (1 \leq n \leq m), \\ \hat{\Lambda}_n(x)\,; & (n > m). \end{cases}$$

Now, since $\hat{\Lambda}(x) \in c_0$; for any positive real $\epsilon > 0$, there is a positive integer m_0 such that $|\hat{\Lambda}_m(x)| < \epsilon$ for every $m \geq m_0$. Thus, for any $m \geq m_0$, we have

$$\left\|x - x^{(m)}\right\|_{s^\lambda} = \sup_{n>m} \left|\hat{\Lambda}_n(x)\right| \leq \sup_{n>m_0} \left|\hat{\Lambda}_n(x)\right| \leq \epsilon\,.$$

We therefore deduce that $\lim_{m\to\infty}\|x - x^{(m)}\|_{s^\lambda} = 0$ which means that x is represented as in (2.3.1). Thus, it is remaining to show the uniqueness of the representation (2.3.1) of x. For this, suppose that $x = \sum_{k=1}^{\infty} \alpha_k\, e_k^\lambda$. Then, we have to show that $\alpha_n = \hat{\Lambda}_n(x)$ for all n, which is immediate by operating $\hat{\Lambda}_n$ on both sides of (2.3.1) for each $n \geq 1$, where the continuity of $\hat{\Lambda}$ (as we have seen in Remark 2.2.8) allows us to obtain that

$$\hat{\Lambda}_n(x) = \sum_{k=1}^{\infty} \alpha_k\, \hat{\Lambda}_n(e_k^\lambda) = \sum_{k=1}^{\infty} \alpha_k\, \delta_{nk} = \alpha_n$$

for all $n \geq 1$ and hence the representation (2.3.1) of x is unique. $\square$

Further, we have the following result constructing Schauder basis for the sequence space cs^λ of λ-convergent series.

Theorem 2.3.2 *The sequence* $\left(e^\lambda, e_1^\lambda, e_2^\lambda, \cdots\right)$ *is a Schauder basis for the space* cs^λ *and every* $x \in cs^\lambda$ *has a unique representation of the form*

$$x = L\, e^\lambda + \sum_{k=1}^{\infty} \left(\hat{\Lambda}_k(x) - L\right) e_k^\lambda\,, \tag{2.3.2}$$

where $L = \lim_{n\to\infty} \hat{\Lambda}_n(x)$*, the sequence* $\left(e_k^\lambda\right)_{k=1}^{\infty}$ *is as in Theorem* 2.3.1 *and* e^λ *is the following sequence:*

$$e^\lambda = e_1 - \left(\frac{\lambda_1}{\lambda_2 - \lambda_1}\right) e_2 = \left(1, -\frac{\lambda_1}{\lambda_2 - \lambda_1},\, 0,\, 0, 0, \cdots\right).$$

Proof. It can easily be shown that $\Lambda(e^\lambda) = e_1$ and so $\hat{\Lambda}(e^\lambda) = e \in c$ which means that $e^\lambda \in cs^\lambda$. This together with $e_k^\lambda \in cs_0^\lambda \subset cs^\lambda$ imply that $\left(e^\lambda, e_1^\lambda, e_2^\lambda, \cdots\right)$ is a sequence in cs^λ. Also, let $x \in cs^\lambda$ be given. Then $\hat{\Lambda}(x) \in c$ which yields the convergence of the sequence $\hat{\Lambda}(x)$ to a unique limit, say $L = \lim_{n\to\infty} \hat{\Lambda}_n(x)$. Thus, by taking $y = x - L\, e^\lambda$, we get $\hat{\Lambda}(y) = \hat{\Lambda}(x) - L\, e \in c_0$ and so $y \in cs_0^\lambda$. Hence, it follows by Theorem 2.3.1 that y can uniquely be represented in the following form:

$$y = \sum_{k=1}^{\infty} \hat{\Lambda}_k(y)\, e_k^\lambda = \sum_{k=1}^{\infty} \left(\hat{\Lambda}_k(x) - L\, \hat{\Lambda}_k(e^\lambda)\right) e_k^\lambda = \sum_{k=1}^{\infty} \left(\hat{\Lambda}_k(x) - L\right) e_k^\lambda.$$

Consequently, our x can also be uniquely written as

$$x = L\,e^{\lambda} + y = L\,e^{\lambda} + \sum_{k=1}^{\infty} \left(\hat{\Lambda}_k(x) - L \right) e_k^{\lambda}$$

which proves the unique representation (2.3.2) of x and this step ends the proof. □

Moreover, the following result concerning the topological property of separability.

Corollary 2.3.3 *We have the following facts:*

(1) *The spaces cs_0^{λ} and cs^{λ} are separable BK spaces.*

(2) *The space bs^{λ} is a non-separable BK space and has no a Schauder basis.*

Proof. Since cs_0 and cs are BK spaces and so Banach spaces having Schauder bases; this result is immediate by (3) of Lemma 1.3.3. □

Finally, we conclude this chapter with the following example on the Schauder basis for the space cs_0^{λ}.

Example 2.3.4 By returning back to Example 2.2.9, we have studied the particular case of the sequence $\lambda = (\lambda_k)$ given by $\lambda_k = k$ for all k and obtained that

$$cs_0^{\lambda} = \left\{ x \in w : \left(\frac{1}{k}\sigma_k(x) \right) \in cs_0 \right\} = \left\{ x \in w : \left(\textstyle\sum_{k=1}^{n} \frac{\sigma_k(x)}{k} \right) \in c_0 \right\}.$$

Also, we have shown that $z \in cs_0^{\lambda}$, where $z = (z_k)$ is the sequence given by $z_1 = 1$ and

$$z_k = \begin{cases} k\sqrt{2/(k+1)} + (k-1)\sqrt{2/(k-1)}\,; & (k \text{ is odd}), \\ -(2k-1)\sqrt{2/k}\,; & (k \text{ is even}) \end{cases}$$

for $k > 1$, where $\hat{\Lambda}(z) \in c_0$ is given by

$$\hat{\Lambda}_n(z) = \begin{cases} \sqrt{2/(n+1)}\,; & (n \text{ is odd}), \\ 0\,; & (n \text{ is even}). \end{cases}$$

On other hand, it follows by Theorem 2.3.1 that the sequence $\left(e_1^\lambda,\, e_2^\lambda,\, e_3^\lambda, \cdots\right)$ is the Schauder basis for the space cs_0^λ, where

$$e_1^\lambda = (1, -3, 2, 0, 0, 0, \cdots)\,, \quad e_2^\lambda = (0, 2, -5, 3, 0, 0, \cdots)\,, \quad e_3^\lambda = (0, 0, 3, -7, 4, 0, \cdots)\,, \quad \cdots$$

Thus, by applying Theorem 2.3.1 to z, the sequence z has the unique representation

$$z = \sum_{k=1}^{\infty} \hat{\Lambda}_{2k-1}(z)\, e_{2k-1}^\lambda = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}\, e_{2k-1}^\lambda\,.$$

Chapter 3

INCLUSION RELATIONS

3 INCLUSION RELATIONS

In the present chapter, we establish some interesting inclusion relations between our new λ-sequence spaces and derive other inclusion relations between our spaces and the classical sequence spaces. This chapter is divided into three sections, the first is devoted to derive some basic inclusion relation, the second is for proving some preliminary results to be used in deducing the main results in the last section. The materials of this chapter are part of our research paper [48] which has been published in the Albaydha Univ. J., and presented in the 2^{nd} conference of Albaydha University on 2021.

3.1 Basic Results

In this section, we establish some basic inclusion relations concerning with the new λ-sequence spaces bs^λ, cs^λ and cs_0^λ.

Lemma 3.1.1 *We have the following facts:*

(1) *The inclusions $cs_0^\lambda \subset cs^\lambda \subset bs^\lambda$ strictly hold.*

(2) *The inclusions $\ell_1^\lambda \subset cs^\lambda \subset c_0^\lambda$ and $\ell_1^\lambda \subset bs^\lambda \subset \ell_\infty^\lambda$ strictly hold.*

(3) *The inclusion $cs_0^\lambda \subset c_0^\lambda$ strictly holds.*

Proof. For (1), the inclusions $cs_0^\lambda \subset cs^\lambda \subset bs^\lambda$ are obviously satisfied (by the well-known inclusions $cs_0 \subset cs \subset bs$, see Lemma 1.3.1). To show that these inclusions are strict, define a sequence $x = (x_j)$ by

$$x_j = \frac{2^{-j}\,\lambda_j - 2^{-(j-1)}\,\lambda_{j-1}}{\lambda_j - \lambda_{j-1}} \qquad (j \geq 1).$$

Then, by using (2.2.2), we find that $\Lambda_k(x) = 2^{-k}$ for every $k \geq 1$ and so $\hat{\Lambda}(x) = (1 - 2^{-n}) \in c \setminus c_0$. This means that $x \in cs^\lambda \setminus cs_0^\lambda$ and so the inclusion $cs_0^\lambda \subset cs^\lambda$ is strict. Also, define the sequence $y = (y_j)$ by

$$y_j = (-1)^j \left(\frac{\lambda_j + \lambda_{j-1}}{\lambda_j - \lambda_{j-1}} \right) \qquad (j \geq 1).$$

Then, for every $k \geq 1$, we find that

$$\Lambda_k(y) = \frac{1}{\lambda_k} \sum_{j=1}^{k} (-1)^j (\lambda_j + \lambda_{j-1}) = (-1)^k \qquad (k \geq 1)$$

and hence $\hat{\Lambda}_n(y) = -1$ when n is odd or $\hat{\Lambda}_n(y) = 0$ when n is even. Thus, we deduce that $\hat{\Lambda}(y) \in \ell_\infty \setminus c$ which means that $y \in bs^\lambda \setminus cs^\lambda$ and hence the inclusion $cs^\lambda \subset bs^\lambda$ is also strict, and part (1) has been proved. To prove part (2), let $x \in \ell_1^\lambda$. Then, the series $\sum_{k=1}^\infty \Lambda_k(x)$ is absolutely convergent and so it converges which means that $x \in cs^\lambda$ and hence the inclusion $\ell_1^\lambda \subset cs^\lambda$ holds which implies the inclusion $\ell_1^\lambda \subset bs^\lambda$. Also, if $x \in cs^\lambda$; then it follows, from the convergence of the series $\sum_{k=1}^\infty \Lambda_k(x)$, that $\Lambda(x) \in c_0$ and hence $x \in c_0^\lambda$ which means that the inclusion $cs^\lambda \subset c_0^\lambda$ holds. Similarly, we can show that $bs^\lambda \subset \ell_\infty^\lambda$ holds. To show that these inclusions are strict, define the sequence $x = (x_j)$ by

$$x_j = (-1)^j \left(\frac{\lambda_j/(j+1) + (\lambda_{j-1}/j)}{\lambda_j - \lambda_{j-1}} \right) \qquad (j \geq 1).$$

Then, it can easily be seen that $\Lambda(x) = (\, (-1)^k/(k+1) \,) \in cs \setminus \ell_1$ and so $x \in cs^\lambda \setminus \ell_1^\lambda$ which means that the inclusion $\ell_1^\lambda \subset cs^\lambda$ is strict, and so is the inclusion $\ell_1^\lambda \subset bs^\lambda$. Further, define the sequence $y = (y_j)$ by

$$y_j = \frac{\Delta(\, \lambda_j/(j+1) \,)}{\lambda_j - \lambda_{j-1}} \qquad (j \geq 1).$$

Then, it is easy to show that $\Lambda(y) = (\, 1/(k+1) \,) \in c_0 \setminus cs$ which means that $y \in c_0^\lambda \setminus cs^\lambda$ and so the inclusion $cs^\lambda \subset c_0^\lambda$ is strict. Also, it is clear that $\Lambda(e) = e \in \ell_\infty \setminus bs$ which implies that $e \in \ell_\infty^\lambda \setminus bs^\lambda$ and hence the inclusion $bs^\lambda \subset \ell_\infty^\lambda$ is also strict which ends the proof of part(2). Finally, part (3) is clear by combining parts (1) and (2). $\square$

Lemma 3.1.2 *We have the following facts:*

(1) *If $1/\lambda \in \ell_1$; then the inclusion $\ell_1 \subset cs^\lambda$ strictly holds, where $1/\lambda = (1/\lambda_j)_{j=1}^\infty$.*

(2) *The space ℓ_1 cannot be included in cs_0^λ.*

Proof. For (1), suppose $1/\lambda \in \ell_1$. Then, the inclusion $\ell_1 \subset \ell_1^\lambda$ holds by (3) of Lemma 2.2.2. Thus, the inclusion $\ell_1 \subset cs^\lambda$ is strict by (2) of above Lemma. For (2), consider the sequence $e_1 = (1, 0, 0, \cdots)$. Then, by (2.2.2), we get that $\Lambda_k(e_1) = \lambda_1/\lambda_k$ for all $k \geq 1$ and so $\hat{\Lambda}_n(e_1) = \lambda_1 \, \sigma_n(1/\lambda) \geq 1$ for all n (as $\lambda_k > 0$ for all k). Thus $\hat{\Lambda}(e_1) \notin c_0$ which means that $e_1 \notin cs_0^\lambda$ while $e_1 \in \ell_1$ and hence $\ell_1 \not\subset cs_0^\lambda$. This ends the proof. $\square$

Remark 3.1.3 As in part (1) of Lemma 3.1.2, we will use the convention $1/\lambda = (1/\lambda_j)_{j=1}^\infty$. Also, since λ is a sequence of positive reals; we deduce that $1/\lambda \notin cs_0$, but its sum sequence $\sigma(1/\lambda)$ is increasing whose positive terms and this leads us to the equivalences: $1/\lambda \in \ell_1 \iff 1/\lambda \in cs \iff 1/\lambda \in bs$.

Lemma 3.1.4 *We have the following:*

(1) *The inclusion $c^\lambda \cap bs^\lambda \subset c_0^\lambda$ holds.*

(2) *The equality $c^\lambda \cap bs^\lambda = c_0^\lambda \cap bs^\lambda$ holds.*

(3) *The inclusion $cs^\lambda \subset c_0^\lambda \cap bs^\lambda$ strictly holds.*

Proof. This result is immediate by Lemma 2.1.4. To see that, take any $x \in c^\lambda \cap bs^\lambda$. Then $x \in c^\lambda$ as well as $x \in bs^\lambda$. Thus $\Lambda(x) \in c$ and $\Lambda(x) \in bs$. This implies that $\Lambda(x) \in c \cap bs$ and so $\Lambda(x) \in c_0$ (as $c \cap bs \subset c_0$ by (1) of Lemma 2.1.4). Thus $x \in c_0^\lambda$ which proves (1). Similarly (2) is obtained from (2) of Lemma 2.1.4. Also, the inclusion $cs^\lambda \subset c_0^\lambda \cap bs^\lambda$ is immediate by (1) and (2) of Lemma 3.1.1. To show that this inclusion is strict, there must exist a sequence $z \in c_0 \cap bs$ such that $z \notin cs$ (as we have seen in proving (3) of Lemma 2.1.4, where $cs \subset c_0 \cap bs$ is strict). This implies that $\sigma(z) \in c_0(\Delta) \cap \ell_\infty$ while $\sigma(z) \notin c$. Now, let $y = \sigma(z)$. Then $y \in c_0(\Delta) \cap \ell_\infty$ while $y \notin c$. Also, define a sequence x in terms of y by using (2.2.8), that is $x = \Delta(\,\lambda\,\Delta(y)\,)/\Delta(\lambda)$. Then, as we have already shown in the proof of Theorem 2.2.6, we can show that $\Lambda(x) = \Delta(y) = z \in c_0$ as well as $\hat{\Lambda}(x) = y = \sigma(z) \in \ell_\infty \setminus c$. Thus, we deduce that $x \in c_0^\lambda$ and $x \in bs^\lambda \setminus cs^\lambda$, that is $x \in c_0^\lambda \cap bs^\lambda$ while $x \notin cs^\lambda$ and hence the inclusion $cs^\lambda \subset c_0^\lambda \cap bs^\lambda$ is strict. This completes the proof. $\square$

3.2 Preliminary Results

In this section, we prove some preliminaries which will be used to prove our main results in the next section and for this purpose, we are in need to quoting some additional conventions and terminologies.

In what follows and for simplicity in notations, we define the following two positive real terms for every positive integer n

$$s_k^n = \lambda_k \sum_{j=k}^{n} \frac{1}{\lambda_j} \quad \text{and} \quad t_k^n = \Delta(\lambda_k) \sum_{j=k}^{n} \frac{1}{\lambda_j}, \qquad (1 \le k \le n). \tag{3.2.1}$$

Further, if $1/\lambda \in \ell_1$; then the limits $s_k^n \to s_k$ and $t_k^n \to t_k$ (as $n \to \infty$) exist for each $k \ge 1$. Thus, we can define the following three positive real sequences $s = (s_k)$, $t = (t_k)$ and $u = (u_k)$ by

$$s_k = \lambda_k \sum_{j=k}^{\infty} \frac{1}{\lambda_j}\,, \qquad t_k = \Delta(\lambda_k) \sum_{j=k}^{\infty} \frac{1}{\lambda_j} \quad \text{and} \quad u_k = \frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\,, \qquad (k \ge 1). \tag{3.2.2}$$

Moreover, it can easily be deriving the following equalities:

$$s_k = t_k\, u_k \quad (k \ge 1) \qquad \text{and} \qquad s_k^n = t_k^n\, u_k \quad (1 \le k \le n), \tag{3.2.3}$$

$$t_k = 1 + \Delta(s_k) \quad (k > 1) \qquad \text{and} \qquad t_k^n = 1 + \Delta(s_k^n) \quad (1 < k \le n), \tag{3.2.4}$$

where the difference is taken over k, that is $\Delta(s_k^n) = s_k^n - s_{k-1}^n$ for every $k \le n$.

Lemma 3.2.1 *Let $1/\lambda \in \ell_1$ and assume that $\Delta(u) \in c$. Then, there must exist a positive integer k_0 satisfying all the following:*

(1) $1 < u_k < k$ *for all* $k > k_0$ *and so* $0 \le \lim_{k\to\infty} \Delta(u_k) < 1$.

(2) *There is a positive real* $\delta < 1/2$ *such that* $-\delta < \Delta(u_k) < 1 - \delta$ *for all* $k > k_0$.

(3) *The difference sequence* $(\Delta(\lambda_k))_{k=k_0}^{\infty}$ *is strictly increasing to* ∞.

Proof. Suppose that $1/\lambda \in \ell_1$ and $\Delta(u) \in c$ which means that $\lim_{k\to\infty} \Delta(u_k)$ exists. Then $\lim_{k\to\infty} u_k/k$ exists (due to the equality between these two limits [29, 30]). Thus $(u_k/k) \in c \subset \ell_\infty$. Also, we claim that there is a positive integer k_1 such that $u_k/k < 1$ for all $k > k_1$ or $u_{k+1}/(k+1) < 1$ for all $k \ge k_1$ which can equivalently be written as $\lambda_{k+1}/(\lambda_{k+1} - \lambda_k) < k + 1$ for all $k \ge k_1$. Otherwise, suppose on contrary that the sequence $\lambda = (\lambda_k)$ has a subsequence $(\lambda_{k_r})_{r=1}^{\infty}$ such that $\lambda_{k_{r+1}}/(\lambda_{k_{r+1}} - \lambda_{k_r}) \ge k_{r+1} \ge r+1$ for all $r \ge 1$. Then, it follows that $\lambda_{k_{r+1}} \le \lambda_{k_r}((r+1)/r)$ and so $\lambda_{k_{r+1}} \le \lambda_{k_1}(r+1)$ for all $r \ge 1$. Thus, we deduce that $1/(r+1) \le \lambda_{k_1}/\lambda_{k_{r+1}}$ for all $r \ge 1$ and so $(1/\lambda_{k_{r+1}}) \notin \ell_1$ which contradicts with our hypothesis ($1/\lambda \in \ell_1$). Hence, our claim is true (as $u_{k+1} > 1$ for all k). Further, since $\lim_{k\to\infty} \Delta(u_k) = \lim_{k\to\infty} u_k/k$; we find that $0 \le \lim_{k\to\infty} \Delta(u_k) \le 1$. Moreover, it can easily be shown that $\lim_{k\to\infty} \Delta(u_k) \ne 1$. For, if $\lim_{k\to\infty} \Delta(u_k) = 1$; we can similarly get $\lambda_k \le ak$ for some positive real $a > 0$ which is a contradiction with $1/\lambda \in \ell_1$. Therefore, we conclude that $0 \le \lim_{k\to\infty} \Delta(u_k) < 1$. To prove (2), assume that $a = \lim_{k\to\infty} \Delta(u_k)$, where $0 \le a < 1$. Then, for every positive real $\epsilon > 0$, there is a positive integer $k' = k'(\epsilon)$ such that $|\Delta(u_{k+1}) - a| < \epsilon$ and so $a - \epsilon < \Delta(u_{k+1}) < a + \epsilon$ for all $k \ge k'$. Now, choose a positive real $\delta < 1/2$ such that

$(1-a)/4 < \delta < (1-a)/2$ and so $\delta < (1-a)/2 < 2\delta$. Then, by taking $\epsilon = (1-a)/2 - \delta$ with its $k_2 = k'(\epsilon)$, we get $0 < \epsilon < 1/2$ and find that $a + \epsilon = (1+a)/2 - \delta < 1 - \delta$ and $a - \epsilon \geq -\epsilon = \delta - (1-a)/2 > \delta - 2\delta = -\delta$. Hence, we deduce that $-\delta < \Delta(u_{k+1}) < 1 - \delta$ for all $k \geq k_2$, that is $-\delta < \Delta(u_k) < 1 - \delta$ for all $k > k_2$. To prove (3), we obtain from (2) that $\Delta(u_{k+1}) < 1$ and so $1 + \lambda_k \Delta(\, 1/\Delta(\lambda_{k+1})\,) < 1$ for all $k \geq k_2$. This implies that $\Delta(\lambda_{k+1}) > \Delta(\lambda_k)$ for all $k \geq k_2$. Thus, the sequence $(\,\Delta(\lambda_k)\,)_{k=k_2}^{\infty}$ is strictly increasing and cannot be bounded (as $1/\lambda \in \ell_1$) but must tend to ∞ which proves (3). Finally, by taking $k_0 = \max\{k_1, k_2\}$, we get the common integer k_0 in parts (1), (2) and (3). $\square$

Lemma 3.2.2 *Let $1/\lambda \in \ell_1$ and assume that $\lim_{k\to\infty} \Delta(u_k) = a$ $(0 \leq a < 1)$. Then, we have $\lim_{k\to\infty} t_k = 1/(1-a)$ and $\lim_{k\to\infty} \Delta(s_k) = a/(1-a)$.*

Proof. Suppose that $\lim_{k\to\infty} \Delta(u_k) = a$, where $0 \leq a < 1$ by (1) of Lemma 3.2.1. Then, for every positive real $\epsilon > 0$, there is a positive integer k' such that $|\Delta(u_{k+1}) - a| < \epsilon$ and so $|1 + \lambda_k \Delta(\, 1/\Delta(\lambda_{k+1})\,) - a| < \epsilon$ for all $k \geq k'$. Thus, it follows that

$$\left| \frac{1-a}{\lambda_k} - \left(\frac{1}{\Delta(\lambda_k)} - \frac{1}{\Delta(\lambda_{k+1})} \right) \right| < \frac{\epsilon}{\lambda_k}$$

for all $k \geq k'$ and by taking the sum of both sides from $k = n$ to ∞ and noting that $\Delta(\lambda_k) \to \infty$ as $k \to \infty$ by (3) of Lemma 3.2.1, we get

$$\left| (1-a) \sum_{k=n}^{\infty} \frac{1}{\lambda_k} - \frac{1}{\Delta(\lambda_n)} \right| \leq \sum_{k=n}^{\infty} \left| \frac{1-a}{\lambda_k} - \left(\frac{1}{\Delta(\lambda_k)} - \frac{1}{\Delta(\lambda_{k+1})} \right) \right| \leq \epsilon \sum_{k=n}^{\infty} \frac{1}{\lambda_k}\,.$$

Dividing both sides by the positive number $\sum_{k=n}^{\infty} 1/\lambda_k$ (as $1/\lambda \in \ell_1$ and $\lambda_k > 0$ for all k); we obtain that $|(1-a) - 1/t_n| \leq \epsilon$ for all $n \geq k'$, and since ϵ was arbitrary; we deduce that $1/t_n \to 1 - a$ or $t_n \to 1/(1-a)$ as $n \to \infty$. The second limit is immediate by (3.2.4) and this completes the proof. $\square$

Lemma 3.2.3 *Let $1/\lambda \in \ell_1$ and suppose that $t \in c$. Then, all the following are true:*

(1) $\lim_{k\to\infty} t_k \geq 1$ *and* $\Delta(s) \in c$ *such that* $\lim_{k\to\infty} \Delta(s_k) \geq 0$.

(2) *If* $\lim_{k\to\infty} t_k = b$; *then* $\lim_{k\to\infty} \Delta(u_k) = (b-1)/b$ *and* $\lim_{k\to\infty} \Delta(s_k) = b-1$.

(3) *There exists a positive integer* k_0 *such that the difference sequence* $(\,\Delta(\lambda_k)\,)_{k=k_0}^{\infty}$ *is strictly increasing to* ∞.

Proof. Suppose that $1/\lambda \in \ell_1$ and $t \in c$. Then, it follows by (3.2.4) that $\Delta(s) \in c$ and so $\lim_{k\to\infty} \Delta(s_k) = \lim_{k\to\infty} s_k/k \geq 0$ (since $s_k > 0$ for all k). Again, by (3.2.4) we get $\lim_{k\to\infty} t_k \geq 1$ which is (1). To prove (2), we first show that $t \in c$ implies $\Delta(u) \in c$. For this, it follows from $t \in c$ that $\Delta(s) \in c$ and $\Delta(t) \in c$ such that $\lim_{k\to\infty} \Delta(t_k) = \lim_{k\to\infty} t_k/k$. Thus, from (3.2.3), we find that $\Delta(s_k) = \Delta(t_k u_k) = t_k\Delta(u_k) + u_{k-1}\Delta(t_k)$ which implies that $\lim_{k\to\infty} \Delta(s_k) = \lim_{k\to\infty} t_k(\,\Delta(u_k) + u_{k-1}/k\,)$ exists. Thus $(\,\Delta(u_k) + u_{k-1}/k\,) \in c$. On other side, we have $\lim_{k\to\infty} \Delta(s_k) = \lim_{k\to\infty} \Delta(t_k u_k) = \lim_{k\to\infty}(t_k u_k)/k = \lim_{k\to\infty} t_k(u_k/k)$. Hence $(u_k/k) \in c$, and since $(k/(k+1)) \in c$; we find that $(u_{k-1}/k) \in c$. Therefore, we deduce that $\Delta(u) = (\Delta(u_k)) = (\Delta(u_k) + u_{k-1}/k) - (u_{k-1}/k) \in c$. Now, if $\lim_{k\to\infty} t_k = b$; then by Lemma 3.2.2 we get $\lim_{k\to\infty} \Delta(u_k) = (b-1)/b$ and the other limit is trivial by (3.2.4). Finally, part (3) is now immediate by (3) of Lemma 3.2.1 because $\Delta(u) \in c$. This ends the proof. □

Lemma 3.2.4 *Suppose that* $1/\lambda \in \ell_1$. *Then, we have the following equivalences:*

(1) $\Delta(u) \in c$ *if and only if* $t \in c$.

(2) $\Delta(u) \in bv_1$ *if and only if* $t \in bv_1$.

(3) $\sup_n \sum_{k=1}^{n} |\Delta(t_k^n)| < \infty$ *if and only if* $\Delta(u) \in bv_1$.

Proof. Suppose that $1/\lambda \in \ell_1$. Then, the equivalence in part (1) can be obtained by combining Lemma 3.2.2 and (2) of Lemma 3.2.3. To prove (2), let us first note that

$bv_1 \subset c$ by Lemma 1.3.1. Thus, if $\Delta(u) \in bv$ or $t \in bv$; then $\Delta(u) \in c$ as well as $t \in c$. Hence, in both direction of current equivalence, we have $\Delta(u) \in c$ and $t \in c$. Therefore, it follows by (2) of Lemma 3.2.1 that there are $\delta > 0$ (real) and $k_0 \geq 1$ (integer) such that $\delta < 1 - \Delta(u_k) < 1 + \delta$ for all $k > k_0$. Thus $(\,1 - \Delta(u_{k+1})\,)_{k=k_0}^{\infty}$ is a convergent sequence of positive reals with non-zero limit, that is $(\,1-\Delta(u_{k+1})\,) \in c \setminus c_0$. Also, it is obvious that t is a convergent sequence of positive reals with non-zero limit, that is $t \in c \setminus c_0$. Further, it follows by Lemma 3.2.2 and (2) of Lemma 3.2.3 that $\lim_{k\to\infty} t_k(1-\Delta(u_{k+1})) = 1$. Hence, if $\Delta(u) \in bv_1$ or $t \in bv_1$; then $(\,t_k(1-\Delta(u_{k+1}))\,) \in bv_1$ and so $(\,\Delta[t_k(1 - \Delta(u_{k+1}))]\,) \in \ell_1$. Therefore, we obtain that

$$(\,\Delta[t_k(1 - \Delta(u_{k+1}))]\,) = \Big(\,t_k\Delta(1 - \Delta(u_{k+1})) + (1 - \Delta(u_k))\Delta(t_k)\,\Big) \in \ell_1 \cdot$$

Now, if $t \in bv_1$; then $\Delta(t) \in \ell_1$ and so $(\,(1 - \Delta(u_k))\Delta(t_k)\,) \in \ell_1$ which implies that $(\,t_k\Delta(1 - \Delta(u_{k+1}))\,) \in \ell_1$ and hence $(\,\Delta(1 - \Delta(u_{k+1}))\,) \in \ell_1$ (as $t \in c \setminus c_0$) and this means that $(1 - \Delta(u_{k+1})) \in bv_1$ and so $\Delta(u) \in bv_1$. Similarly, if $\Delta(u) \in bv_1$; then we get $(1 - \Delta(u_{k+1})) \in bv_1$ and so $(\,\Delta(1 - \Delta(u_{k+1}))\,) \in \ell_1$ which implies that $(\,t_k\Delta(1 - \Delta(u_{k+1}))\,) \in \ell_1$ and hence $(\,(1 - \Delta(u_k))\Delta(t_k)\,) \in \ell_1$. Thus $(\,\Delta(t_k)\,) \in \ell_1$ (as $(1 - \Delta(u_k)) \in c \setminus c_0$), that is $\Delta(t) \in \ell_1$ and so $t \in bv_1$. Finally, to prove (3), let us first note that $t \in c$ in both direction of current equivalence (as we have already shown in proving (2)) and hence there is an integer $k_0 \geq 1$ such that $(\Delta(\lambda_k))_{k=k_0}^{\infty}$ is strictly increasing to ∞ by (3) of Lemma 3.2.1 (or Lemma 3.2.3). Now, let $n \geq 1$. Then, for every $k \leq n$, we have $t_k - t_k^n = (\,t_{n+1}/\Delta(\lambda_{n+1})\,)\,\Delta(\lambda_k)$ and so $\Delta(t_k - t_k^n) = (\,t_{n+1}/\Delta(\lambda_{n+1})\,)\,(\,\Delta(\lambda_k) - \Delta(\lambda_{k-1})\,)$, where $t_0 = t_0^n = 0$. Thus, it follows that

$$\Big|\,|\Delta(t_k)| - |\Delta(t_k^n)|\,\Big| \leq |\Delta(t_k) - \Delta(t_k^n)| = \frac{t_{n+1}}{\Delta(\lambda_{n+1})}\,|\Delta(\lambda_k) - \Delta(\lambda_{k-1})|$$

and by taking the summation from $k=1$ to n $(n \geq 1)$, we get

$$\left| \sum_{k=1}^{n} \left(|\Delta(t_k)| - |\Delta(t_k^n)| \right) \right| \leq \sum_{k=1}^{n} \Big| \, |\Delta(t_k)| - |\Delta(t_k^n)| \, \Big| \leq M \, t_{n+1} \frac{\Delta(\lambda_n)}{\Delta(\lambda_{n+1})}$$

for some $M > 0$. But, we have $\left(t_{n+1}\Delta(\lambda_n)/\Delta(\lambda_{n+1}) \right) \in \ell_\infty$ and it follows that $\left(\sum_{k=1}^{n} |\Delta(t_k)| - \sum_{k=1}^{n} |\Delta(t_k^n)| \right)_{n=1}^{\infty} \in \ell_\infty$. Thus, we deduce that $\left(\sum_{k=1}^{n} |\Delta(t_k^n)| \right) \in \ell_\infty \iff \left(\sum_{k=1}^{n} |\Delta(t_k)| \right) \in \ell_\infty$. That is $\sup_n \sum_{k=1}^{n} |\Delta(t_k^n)| < \infty \iff \sum_{k=1}^{\infty} |\Delta(t_k)| < \infty$ which can equivalently be written as $\sup_n \sum_{k=1}^{n} |\Delta(t_k^n)| < \infty \iff t \in bv_1$. This ends the proof, since $t \in bv_1 \iff \Delta(u) \in bv_1$ by part (2). $\square$

3.3 Main Results

In the last section, we prove the main inclusion relations between the old and new sequence spaces of series. We essentially characterize the case in which the inclusions $bs \subset bs^\lambda$, $cs \subset cs^\lambda$ and $cs_0 \subset cs_0^\lambda$ hold, and discuss their equalities.

Theorem 3.3.1 *Let $u = (u_k)$ be defined by $u_k = \lambda_k/(\lambda_k - \lambda_{k-1})$ for all $k \geq 1$. Then, we have the following facts:*

(1) *The inclusions $cs \subset cs^\lambda$ and $bs \subset bs^\lambda$ hold if and only if $1/\lambda \in \ell_1$ and $\Delta(u) \in bv_1$.*

(2) *The equalities $cs^\lambda = cs$ and $bs^\lambda = bs$ hold if and only if $u \in \ell_\infty$ and $\Delta(u) \in bv_0$.*

(3) *The inclusions $cs \subset cs^\lambda$ and $bs \subset bs^\lambda$ strictly hold if and only if $1/\lambda \in \ell_1$, $u \notin \ell_\infty$ and $\Delta(u) \in bv_1$.*

Proof. To prove (1), suppose that the inclusions $cs \subset cs^\lambda$ and $bs \subset bs^\lambda$ hold. Then, we have $e_1 \in cs$ and $e_1 \in bs$, where $e_1 = (1, 0, 0, \cdots)$. Thus, we must have $e_1 \in cs^\lambda$ as well as $e_1 \in bs^\lambda$. This implies that $\hat{\Lambda}(e_1) \in c$ and $\hat{\Lambda}(e_1) \in \ell_\infty$, respectively. Also, by using (2.2.5), we find that $\hat{\Lambda}_n(e_1) = \lambda_1 \, \sigma_n(1/\lambda) = \lambda_1 \sum_{k=1}^{n}(1/\lambda_k)$ for all $n \geq 1$. Thus, we conclude that $\sigma(1/\lambda) \in c$ and $\sigma(1/\lambda) \in \ell_\infty$ and hence $1/\lambda \in cs$ and $1/\lambda \in bs$, respectively. Therefore, in both cases, we get the same result which is $1/\lambda \in \ell_1$ (see Remark 3.1.3). That is $1/\lambda \in \ell_1$ is necessary condition for both given inclusions (if $1/\lambda \notin \ell_1$; then both inclusions cannot be satisfied, see Example 2.2.9). Thus, we assume that $1/\lambda \in \ell_1$ and then it can easily be seen that the inclusions $cs \subset cs^\lambda$ and $bs \subset bs^\lambda$ hold if and only if $\hat{\Lambda} \in (cs, c)$ and $\hat{\Lambda} \in (bs, \ell_\infty)$, respectively. To see that, we have the following equivalences:

$$cs \subset cs^\lambda \iff x \in cs^\lambda \text{ for all } x \in cs \iff \hat{\Lambda}(x) \in c \text{ for all } x \in cs \iff \hat{\Lambda} \in (cs, c),$$

$$bs \subset bs^\lambda \iff x \in bs^\lambda \text{ for all } x \in bs \iff \hat{\Lambda}(x) \in \ell_\infty \text{ for all } x \in bs \iff \hat{\Lambda} \in (bs, \ell_\infty).$$

Thus, to obtain the other necessary and sufficient conditions for these two inclusions, we have to find the required conditions for $\hat{\Lambda} \in (cs, c)$ and $\hat{\Lambda} \in (bs, \ell_\infty)$ by means of Lemma 1.3.14 with $\hat{\Lambda}$ instead of A. For this, it follows from (3.2.1) and the definition of our matrix $\hat{\Lambda}$ that $\hat{\lambda}_{nk} = t_k^n$ for $1 \leq k \leq n$ and $\hat{\lambda}_{nk} = 0$ for $k > n$, where $n, k \geq 1$. Thus, by using the entries of $\hat{\Lambda}$, we deduce from condition (1.3.2) that $\lim_{n\to\infty} \hat{\lambda}_{nk} = \lim_{n\to\infty} t_k^n$ exists for every $k \geq 1$. But these limits actually exist for all $k \geq 1$ (as $1/\lambda \in \ell_1$), where $\lim_{n\to\infty} t_k^n = t_k = \Delta(\lambda_k) \sum_{j=k}^{\infty} 1/\lambda_j$ for each k. Thus, condition (1.3.2) is already satisfied for $\hat{\Lambda}$. Also, condition (1.3.7) trivially holds, since $\hat{\Lambda}$ is a triangle and so $\hat{\lambda}_{nk} = 0$ when $k > n$ for each $n \geq 1$ and this implies that $\lim_{k\to\infty} \hat{\lambda}_{nk} = 0$ for every $n \geq 1$. Thus, the common condition (1.3.6) is left, and this condition together with $1/\lambda \in \ell_1$ are the necessary and sufficient conditions for both inclusions. Moreover, for every $n, k \geq 1$ we have

$$\hat{\lambda}_{nk} - \hat{\lambda}_{n,k+1} = \begin{cases} -\Delta(t_{k+1}^n)\,; & (k < n), \\ \Delta(\lambda_n)/\lambda_n\,; & (k = n), \\ 0\,; & (k > n), \end{cases}$$

$$\sum_{k=1}^{\infty}\left|\hat{\lambda}_{nk}-\hat{\lambda}_{n,k+1}\right| = \frac{\Delta(\lambda_n)}{\lambda_n}+\sum_{k=1}^{n-1}|\Delta(t_{k+1}^n)| = -t_1^n+\frac{\Delta(\lambda_n)}{\lambda_n}+\sum_{k=1}^{n}|\Delta(t_k^n)|,$$

and since $(-t_1^n+\Delta(\lambda_n)/\lambda_n)\in\ell_\infty$; we deduce that $\sup_n\sum_{k=1}^{\infty}|\hat{\lambda}_{nk}-\hat{\lambda}_{n,k+1}|<\infty$ if and only if $\sup_n\sum_{k=1}^{n}|\Delta(t_k^n)|<\infty$. Therefore, condition (1.3.6) is satisfied for $\hat{\Lambda}$ if and only if $\sup_n\sum_{k=1}^{n}|\Delta(t_k^n)|<\infty$ (or equivalently $\Delta(u)\in bv_1$ by (3) of Lemma 3.2.4). Consequently, the inclusions $cs\subset cs^\lambda$ and $bs\subset bs^\lambda$ hold if and only if $1/\lambda\in\ell_1$ and $\Delta(u)\in bv_1$. To prove (2), we can use the equality (2.2.3) mentioned in Lemma 2.2.1. That is, we have the equality $x_k-\Lambda_{k-1}(x)=u_k\,[\Lambda_k(x)-\Lambda_{k-1}(x)]$ which is satisfied for every $x\in w$ and all $k\geq 1$. Thus, by taking the summation of both sides from $k=1$ to $n\geq 1$, we get the following relation:

$$\sigma_n(x)-\hat{\Lambda}_{n-1}(x)=\sum_{k=1}^{n}u_k\,[\Lambda_k(x)-\Lambda_{k-1}(x)]\,, \qquad (n\geq 1)$$

which can be written as follows:

$$\sigma_n(x)-\hat{\Lambda}_{n-1}(x)=u_{n+1}\,\Lambda_n(x)-\sum_{k=1}^{n}\Delta(u_{k+1})\,\Lambda_k(x)\,, \qquad (n\geq 1). \tag{3.3.1}$$

Now, if the equalities $cs^\lambda=cs$ and $bs^\lambda=bs$ hold; we deduce from (3.3.1) that $u\in\ell_\infty$ and $\Delta(u)\in bv_1$. But $bv_1\subset c$ and so $\Delta(u)\in c$ such that $\lim_{k\to\infty}\Delta(u_k)=\lim_{k\to\infty}u_k/k=0$ (since u is bounded) which implies that $\Delta(u)\in bv_0$, where $bv_0=bv_1\cap c_0$. Conversely, if $u\in\ell_\infty$ and $\Delta(u)\in bv_0$; it follows from (3.3.1) that $x\in cs^\lambda\Leftrightarrow x\in cs$ as well as $x\in bs^\lambda\Leftrightarrow x\in bs$, which means that both equalities $cs^\lambda=cs$ and $bs^\lambda=bs$ are satisfied (we may note that: (i) $u\in\ell_\infty\Rightarrow 1/\lambda\in\ell_1$, (ii) $xy\in cs$ for all $x\in cs\Leftrightarrow y\in bv_1$, and (iii) $xy\in bs$ for all $x\in bs\Leftrightarrow y\in bv_0$). Finally, part (3) follows from (1) and (2). This completes the proof □

Corollary 3.3.2 *If the inclusion $cs\subset cs^\lambda$ holds; then for every $x\in cs$ we have*

$$\lim_{n\to\infty}\hat{\Lambda}_n(x)=\lim_{n\to\infty}\sum_{k=1}^{n}t_k^n\,x_k=\lim_{n\to\infty}\sum_{k=1}^{n}t_k\,x_k.$$

That is $\quad \lim_{n\to\infty} \sum_{k=1}^{n} \Big(\Delta(\lambda_k) \sum_{j=k}^{n} \frac{1}{\lambda_j}\Big) x_k = \lim_{n\to\infty} \sum_{k=1}^{n} \Big(\Delta(\lambda_k) \sum_{j=k}^{\infty} \frac{1}{\lambda_j}\Big) x_k \,.$

Proof. Let $x \in cs$. Then, for every $n \geq 1$, we have

$$\hat{\Lambda}_n(x) = \sum_{k=1}^{n} t_k^n\, x_k = \sum_{k=1}^{n} t_k\, x_k - \Big(\sum_{j=n+1}^{\infty} \frac{1}{\lambda_j}\Big)\Big(\sum_{k=1}^{n} \Delta(\lambda_k) x_k\Big)$$

and since $x_k = \sigma_k(x) - \sigma_{k-1}(x)$; we get

$$\sum_{k=1}^{n} \Delta(\lambda_k) x_k = \Delta(\lambda_{n+1})\sigma_n(x) - \sum_{k=1}^{n} (\, \Delta(\lambda_{k+1}) - \Delta(\lambda_k)\,)\, \sigma_k(x).$$

Thus, we obtain that

$$\hat{\Lambda}_n(x) = \sum_{k=1}^{n} t_k\, x_k + t_{n+1}\,(\, \ddot{\sigma}_n(x) - \sigma_n(x)\,) \qquad (n \geq 1), \tag{3.3.2}$$

where $\ddot{\sigma}_n(x)$ is given by

$$\ddot{\sigma}_n(x) = \frac{1}{\Delta(\lambda_{n+1})} \sum_{k=1}^{n} (\, \Delta(\lambda_{k+1}) - \Delta(\lambda_k)\,)\, \sigma_k(x) \qquad (n \geq 1).$$

That is $\ddot{\sigma}(x) = \ddot{\Lambda}(\sigma(x))$ and $\ddot{\Lambda}$ is the matrix Λ with the sequence $(\Delta(\lambda_{k+1}))$ instead of (λ_k), where $(\Delta(\lambda_{k+1}))_{k=k_0}^{\infty}$ is strictly increasing to ∞ (for some integer $k_0 \geq 1$ by Lemma 3.2.1). Hence, we conclude that $\lim_{n\to\infty} \ddot{\sigma}_n(x) = \lim_{n\to\infty} \sigma_n(x)$ by regularity of Λ and $\ddot{\Lambda}$ (see (1) of Lemma 2.2.1). Therefore, our result is now proved by going to the limits in both sides of (3.3.2) as $n \to \infty$. $\square$

Theorem 3.3.3 *The inclusion $cs_0 \subset cs_0^{\lambda}$ strictly holds if and only if there exists a positive real number $0 < a < 1$ such that $\Delta(u_{k+1}) = a$ for all $k \geq 1$ (equivalently: $cs_0 \subset cs_0^{\lambda}$ strictly holds if and only if there exists a positive real number $b > 1$ such that $t_k = b$ for all $k \geq 1$). Furthermore, the equality $cs_0^{\lambda} = cs_0$ cannot be held.*

Proof. Assume $\Delta(u_{k+1}) = a$ $(0 < a < 1)$ for all $k \geq 1$, i.e. $(\Delta(u_2), \Delta(u_3), \cdots)$ is constant. Then $1 + \lambda_k \Delta(1/\Delta(\lambda_{k+1})) = a$ and so $1/\Delta(\lambda_k) - 1/\Delta(\lambda_{k+1}) = (1-a)/\lambda_k$.

Thus $\Delta(\lambda)$ is increasing to ∞ and by taking the summation from $k = n$ to ∞ we get $t_n = 1/(1-a)$ for all $n \geq 1$ (t_n is constant). In such case, it is obvious that $1/\lambda \in \ell_1$ and $\Delta(u) \in bv_1$. Thus, it follows by (1) of Theorem 3.3.1 that the inclusion $cs \subset cs^\lambda$ holds. Also, for any $x \in cs_0$, we have $x \in cs^\lambda$ (since $cs_0 \subset cs \subset cs^\lambda$). Thus, we deduce from Corollary 3.3.2 that $\lim_{n\to\infty} \hat{\Lambda}_n(x) = (\,1/(1-a)\,) \lim_{n\to\infty} \sigma_n(x) = 0$ which means that $x \in cs_0^\lambda$. Hence, the inclusion $cs_0 \subset cs_0^\lambda$ holds. Conversely, if the inclusion $cs_0 \subset cs_0^\lambda$ holds; then for each $k \geq 1$, we have $\lim_{n\to\infty} \hat{\Lambda}_n(\hat{e}_k) = 0$, where $\hat{e}_k = e_k - e_{k+1} \in cs_0$ for all k. But $\lim_{n\to\infty} \hat{\Lambda}_n(\hat{e}_k) = -\Delta(t_{k+1})$ and so $\Delta(t_{k+1}) = 0$ for all $k \geq 1$. Thus, there exists a positive real $b > 1$ such that $t_k = b$ for all $k \geq 1$ (as $t_1 > 1$). Hence $t_k/\Delta(\lambda_k) - t_{k+1}/\Delta(\lambda_{k+1}) = b/\Delta(\lambda_k) - b/\Delta(\lambda_{k+1})$ and so $1 - 1/b = 1 + \lambda_k \Delta(1/\Delta(\lambda_{k+1}))$ which yields $\Delta(u_{k+1}) = (b-1)/b$ for all $k \geq 1$ and $0 < (b-1)/b < 1$ and this proves the given equivalence. Further, if the inclusion $cs_0 \subset cs_0^\lambda$ holds; then it must be strict, since the equality can only be held if $a = 0$ (as the equality implies that $cs_0 \subset cs_0^\lambda$ and so $\bar{\mu} \subset \bar{\mu}^\lambda$, see (2) of Theorem 3.3.1) which is impossible (as $\Delta(u_2) \neq 0$ for any λ). $\square$

At the end of this chapter, we give some examples to support our main results.

Example 3.3.4 To each non-negative integer $m \geq 0$, we will associate other spaces cs_0^λ, cs^λ and bs^λ depending on m (as particular cases of our spaces) such that the inclusions $cs_0 \subset cs_0^\lambda$, $cs \subset cs^\lambda$ and $bs \subset bs^\lambda$ strictly hold by Theorems 3.3.1 and 3.3.3. That is, it will be there an infinitely many number of these λ-sequence spaces according to m. For this, define the sequence $\lambda = (\lambda_k)$ by

$$\lambda_k = k(k+1)\cdots(k+m+1) = \frac{(k+m+1)!}{(k-1)!} \qquad (k \geq 1).$$

Then, it can easily be deriving the following ($k, n \geq 1$):

$$\Delta(\lambda_k) = (m+2)\,[(k+m)!/(k-1)!]$$

$$u_k = (k+m+1)/(m+2)\,, \qquad \Delta(u_{k+1}) = 1/(m+2) \ \ \text{(constant)}$$

$$\frac{1}{\lambda_j} = \frac{1}{j(j+1)\cdots(j+m+1)} = \frac{1}{(m+1)!}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\left[\frac{1}{j+i} - \frac{1}{j+i+1}\right]$$

$$\sum_{j=k}^{n}\frac{1}{\lambda_j} = \frac{1}{(m+1)!}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\left[\frac{1}{k+i} - \frac{1}{n+i+1}\right] = \frac{1}{m+1}\left[\frac{(k-1)!}{(k+m)!} - \frac{n!}{(n+m+1)!}\right]$$

$$t_k^n = \Delta(\lambda_k)\sum_{j=k}^{n}\frac{1}{\lambda_j} = \frac{m+2}{m+1}\left[1 - \binom{k+m}{k-1}\Big/\binom{n+m+1}{n}\right]$$

$$t_k = \Delta(\lambda_k)\sum_{j=k}^{\infty}\frac{1}{\lambda_j} = \frac{m+2}{m+1} \ \ \text{(constant)}$$

$$\hat{\Lambda}_n(x) = \frac{m+2}{m+1}\left[\sigma_n(x) - \sum_{k=1}^{n} x_k\binom{k+m}{k-1}\Big/\binom{n+m+1}{n}\right]$$

$$\hat{\Lambda}_n(x) = \frac{m+2}{m+1}\sum_{k=1}^{n}\sigma_k(x)\binom{k+m}{k}\Big/\binom{n+m+1}{n}\,.$$

Further, from the equality $t_k = (m+2)/(m+1)$; we deduce the following new or known formulae for summation ($m \geq 0$ and $k \geq 1$):

$$\sum_{n=k}^{\infty}\frac{m+1}{n(n+1)\cdots(n+m+1)} = \frac{1}{k(k+1)\cdots(k+m)}$$

$$\sum_{n=k}^{\infty}\frac{1}{n(n+1)\cdots(n+m+1)} = \frac{1}{(m+1)!}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\Big/(k+i)$$

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}\Big/(k+i) = \frac{m!}{k(k+1)\cdots(k+m)}$$

$$\sum_{n=k}^{\infty}\binom{k+m}{k-1}\Big/\binom{n+m+1}{n-1} = \frac{m+2}{m+1}\,.$$

Remark 3.3.5 We must note that the condition $0 < a < 1$ (or $b > 1$) is necessary in Theorem 3.3.3, see Example 2.2.9 for the case $a = 1$ in which $\bar{\mu} \not\subset \bar{\mu}^{\lambda}$ as well as $\bar{\mu}^{\lambda} \not\subset \bar{\mu}$, where $\bar{\mu}$ is any of the spaces bs, cs or cs_0.

Example 3.3.6 Consider the sequence $\lambda_k = \alpha^k$, where $\alpha > 1$. Then, we have the following:

$$\lambda = \big(\alpha,\, \alpha^2,\, \alpha^3,\, \alpha^4, \cdots\big),$$
$$\Delta(\lambda) = \Big(\alpha,\, \alpha(\alpha-1),\, \alpha^2(\alpha-1),\, \alpha^3(\alpha-1), \cdots\Big),$$
$$u = \Big(1,\, \frac{\alpha}{\alpha-1}\,,\, \frac{\alpha}{\alpha-1}\,,\, \frac{\alpha}{\alpha-1}\,, \cdots\Big) \in \ell_\infty,$$
$$\Delta(u) = \Big(1,\, \frac{1}{\alpha-1}\,,\, 0,\, 0,\, 0,\, \cdots\Big) \in bv_0,$$
$$t = \Big(\frac{\alpha}{\alpha-1},\, 1,\, 1,\, 1,\, 1,\, \cdots\Big) \in bv_1.$$

Thus, we note that $u \in \ell_\infty$, $\Delta(u) \in bv_0$ and $\Delta(u_2) \neq 0$ while $\Delta(u_k) = 0$ for all $k > 2$ (also $t_1 \neq 1$ while $t_k = 1$ for all $k > 1$). Hence, it follows by Theorems 3.3.1 and 3.3.3 that $bs^\lambda = bs$ and $cs^\lambda = cs$ while $cs_0 \not\subset cs_0^\lambda$ as well as $cs_0^\lambda \not\subset cs_0$ (as $\Delta(u_{k+1})$ is not constant for all $k \geq 1$).

Example 3.3.7 We have the following particular case:

$$\lambda = \big(2,\, 6,\, 12,\, 36,\, 72,\, 216,\, \cdots\big),$$
$$\Delta(\lambda) = \big(2,\, 4,\, 6,\, 24,\, 36,\, 144,\, \cdots\big),$$
$$u = \Big(1,\, \frac{3}{2},\, 2,\, \frac{3}{2},\, 2,\, \frac{3}{2},\, \cdots\Big) \in \ell_\infty,$$
$$\Delta(u) = \Big(1,\, \frac{1}{2},\, \frac{1}{2},\, -\frac{1}{2},\, \frac{1}{2},\, -\frac{1}{2},\, \cdots\Big) \notin bv_1,$$
$$t = \Big(\frac{8}{5},\, \frac{6}{5},\, \frac{4}{5},\, \frac{6}{5},\, \frac{4}{5},\, \frac{6}{5},\, \cdots\Big) \notin bv_1.$$

Thus, it follows by Theorems 3.3.1 and 3.3.3 that $\bar{\mu} \not\subset \bar{\mu}^\lambda$ as well as $\bar{\mu}^\lambda \not\subset \bar{\mu}$, where $\bar{\mu}$ is any of the spaces bs, cs or cs_0 (note that: $1/\lambda \in \ell_1$ and $u \in \ell_\infty$).

Chapter 4

KÖTHE-TOEPLITZ DUALITY

4 KÖTHE-TOEPLITZ DUALITY

In the present chapter, we conclude the α-, β- and γ-duals for the λ-sequence spaces of bounded, convergence and null series. Also, we study some additional properties of their duals. This chapter is divided into three sections, the first is devoted to study the α-duals, the second is for the β- and γ-duals and the last is to deduce some additional results. The materials of this chapter are part of our research paper [49] which has been published in the Global Sci. J. on 2022.

By μ, we denote any of the spaces c_0, c or ℓ_∞, and $\bar{\mu}$ stands for the respective one of the spaces cs_0, cs or bs, and so $\bar{\mu}^\lambda$ is the corresponding one of the λ-spaces cs_0^λ, cs^λ or bs^λ. By θ, we mean any one of the duality symbols α, β or γ, that is $\theta := \alpha$, β or γ. Thus, the θ-dual of a sequence space X is the α-, β- or γ-dual of X which was defined by (1.1.2) as $X^\theta = \{a \in w : ax \in \langle\theta\rangle \text{ for all } x \in X\}$, where $\langle\alpha\rangle = \ell_1$, $\langle\beta\rangle = cs$ and $\langle\gamma\rangle = bs$. For example, it is known that $\mu^\theta = \ell_1$ (Lemma 1.3.4), and the duals of $\bar{\mu}$ are given in Lemma 1.3.5, and we are going to find out the θ-duals of $\bar{\mu}^\lambda$.

4.1 The α-Duals

In the first section, we obtain the α-duals of the λ-sequence spaces $\bar{\mu}^\lambda$ of bounded, convergent and null series, where

$$\{\bar{\mu}^\lambda\}^\alpha = \{a \in w : \; ax \in \ell_1 \text{ for all } x \in \bar{\mu}^\lambda\}.$$

For this, we will use our usual notations and terminologies given in previous chapters. First, we define the sequence $u = (u_k)$ of positive reals as follows:

$$u_k = \frac{\lambda_k}{\Delta(\lambda_k)} = \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \qquad (k \geq 1). \tag{4.1.1}$$

Next, every sequence $x = (x_k) \in w$ will be connected with another sequence $y = (y_k)$ by the relation $y = \hat{\Lambda}(x)$, and we then say that y is the sequence connected with x by $y = \hat{\Lambda}(x)$ which together with (2.2.6) yields that

$$y_k = \hat{\Lambda}_k(x) = \sum\nolimits_{j=1}^{k} \Lambda_j(x) \quad \text{and} \quad \Delta(y_k) = \Lambda_k(x) \qquad (k \geq 1).$$

Then, by using (2.2.8), we have $x_k = \Delta(\lambda_k \Delta(y_k))/\Delta(\lambda_k)$ (for all $k \geq 1$) which can equivalently be written as follows:

$$x_k = \Delta(y_{k-1}) + u_k \Delta^2(y_k) = \Delta(y_{k-1}) + u_k(\, \Delta(y_k) - \Delta(y_{k-1})\,) \qquad (k \geq 1). \quad (4.1.2)$$

Thus, here and in what follows, we assume that x and y are connected by $y = \hat{\Lambda}(x)$ which implies the validity of (4.1.2) from which we obtain that

$$x_k = u_k y_k - (2u_k - 1) y_{k-1} + (u_k - 1) y_{k-2} \qquad (k \geq 1), \qquad (4.1.3)$$

where $x_1 = u_1 y_1 = y_1$ and $x_2 = u_2 y_2 - (2u_2 - 1) y_1$.

Remark 4.1.1 It is obvious that x and y are connected by $y = \hat{\Lambda}(x)$ if and only if (4.1.2) is satisfied. Also, it must be noted that $x \in \bar{\mu}^\lambda \iff \Delta(y) \in \bar{\mu} \iff y \in \mu$. In fact, it follows by Theorem 2.2.6 that for every $x \in \bar{\mu}^\lambda$ there exists a unique $y \in \mu$ given by $y = \hat{\Lambda}(x)$ and conversely, for every $y \in \mu$ there exists a unique $x \in \bar{\mu}^\lambda$ given by (4.1.2) and we have $\|x\|_{s^\lambda} = \|y\|_\infty$ by Theorem 2.2.5.

Now, we may begin with the following main result which shows that the spaces cs_0^λ, cs^λ and bs^λ have the same α-duals for the same sequence λ, i.e., their α-duals depend only upon λ.

Theorem 4.1.2 *The α-duals of the spaces $\bar{\mu}^\lambda$ are given by*

$$\{\bar{\mu}^\lambda\}^\alpha = \Big\{a \in w : \; au = (a_k u_k) \in \ell_1\Big\},$$

where $u = (u_k)$ is defined by $u_k = \lambda_k/\Delta(\lambda_k)$ for all $k \geq 1$.

Proof. For any $x \in \bar{\mu}^{\lambda}$, let $y = (y_k)$ be the sequence connected by $y = \hat{\Lambda}(x)$. Then $y \in \mu$ (see Remark 4.1.1) and for every $a = (a_n) \in w$, we can use (4.1.3) to get

$$a_n x_n = a_n u_n y_n - a_n(2u_n - 1)y_{n-1} + a_n(u_n - 1)y_{n-2} = A_n(y) \qquad (n \geq 1), \quad (4.1.4)$$

where $A = [a_{nk}]_{n,k=1}^{\infty}$ is a triangle defined by

$$a_{nk} = \begin{cases} a_n u_n\,; & (k = n), \\ -a_n(2u_n - 1)\,; & (k = n-1), \\ a_n(u_n - 1)\,; & (k = n-2), \\ 0\,; & (\text{otherwise}). \end{cases}$$

That is, our A is the following triangle:

$$A = \begin{bmatrix} a_1u_1 & 0 & 0 & 0 & \cdots \\ -a_2(2u_2-1) & a_2u_2 & 0 & 0 & \cdots \\ a_3(u_3-1) & -a_3(2u_3-1) & a_3u_3 & 0 & \cdots \\ 0 & a_4(u_4-1) & -a_4(2u_4-1) & a_4u_4 & \cdots \\ 0 & 0 & a_5(u_5-1) & -a_5(2u_5-1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Thus, it follows by (4.1.4) that $ax = A(y)$ for every $x \in \bar{\mu}^{\lambda}$ with $y = \hat{\Lambda}(x) \in \mu$. Also, by using (4.1.4), we deduce that

$$a \in \left\{\bar{\mu}^{\lambda}\right\}^{\alpha} \Longleftrightarrow ax \in \ell_1 \text{ for all } x \in \bar{\mu}^{\lambda} \Longleftrightarrow A(y) \in \ell_1 \text{ for all } y \in \mu \Longleftrightarrow A \in (\mu, \ell_1),$$

where $A(y)$ exists for every $y \in \mu$ (as A is a triangle by Lemma 1.3.7). This, together with Lemma 1.3.8 (when $p = 1$), leads us to conclude that

$$a \in \left\{\bar{\mu}^{\lambda}\right\}^{\alpha} \Longleftrightarrow \sup_{K \in \mathcal{K}} \sum_{n=1}^{\infty} \left| \sum_{k \in K} a_{nk} \right| < \infty\,,$$

where $\mathcal{K}$ stands for the collection of all non-empty finite subsets of positive integers. On other side, it must be noted that $u_1 = 1$ and so $u_1 = 2u_1 - 1$. Thus, for each $n \geq 1$, it follows by definition of A that $|\sum_{k \in K} a_{nk}| \leq |a_n|(2u_n - 1)$ for every $K \in \mathcal{K}$. Thus, we obtain that

$$\sum_{n=1}^{\infty} \left| \sum_{k \in K} a_{nk} \right| \leq \sum_{n=1}^{\infty} |a_n|(2u_n - 1)$$

for every $K \in \mathcal{K}$ which implies that

$$\sup_{K\in\mathcal{K}} \sum_{n=1}^{\infty} \left| \sum_{k\in K} a_{nk} \right| \leq \sum_{n=1}^{\infty} |a_n|(2u_n - 1). \tag{4.1.5}$$

Besides, by taking $K_m = \{1, 3, 5, \cdots, 2m-1\} \in \mathcal{K}$ for any positive integer m. Then, for each $n \geq 1$, it can easily be seen that

$$\left| \sum_{k\in K_m} a_{nk} \right| = \begin{cases} |a_n|(2u_n - 1)\,; & (n \leq 2m), \\ |a_{2m+1}|(u_{2m+1} - 1)\,; & (n = 2m+1), \\ 0\,; & (n \geq 2(m+1)). \end{cases}$$

Therefore, we obtain that

$$\sum_{n=1}^{2m} |a_n|(2u_n - 1) = \sum_{n=1}^{2m} \left| \sum_{k\in K_m} a_{nk} \right| = \sum_{n=1}^{\infty} \left| \sum_{k\in K_m} a_{nk} \right| - |a_{2m+1}|(u_{2m+1} - 1)$$

$$\Longrightarrow \sum_{n=1}^{2m} |a_n|(2u_n - 1) < \sum_{n=1}^{\infty} \left| \sum_{k\in K_m} a_{nk} \right| \leq \sup_{K\in\mathcal{K}} \sum_{n=1}^{\infty} \left| \sum_{k\in K} a_{nk} \right|$$

$$\Longrightarrow \sum_{n=1}^{2m} |a_n|(2u_n - 1) < \sup_{K\in\mathcal{K}} \sum_{n=1}^{\infty} \left| \sum_{k\in K} a_{nk} \right| \qquad (m \geq 1)$$

and by taking the supremum over all positive integer m, we get

$$\sum_{n=1}^{\infty} |a_n|(2u_n - 1) \leq \sup_{K\in\mathcal{K}} \sum_{n=1}^{\infty} \left| \sum_{k\in K} a_{nk} \right|. \tag{4.1.6}$$

Hence, by combining the inequalities (4.1.5) and (4.1.6), we deduce that

$$\sup_{K\in\mathcal{K}} \sum_{n=1}^{\infty} \left| \sum_{k\in K} a_{nk} \right| = \sum_{n=1}^{\infty} |a_n|(2u_n - 1)$$

which yields that

$$a \in \left\{\bar{\mu}^{\lambda}\right\}^{\alpha} \Longleftrightarrow \sum\nolimits_{n=1}^{\infty} |a_n|(2u_n - 1) < \infty\,.$$

Further, since $u_n \geq 1$ for all n; we find that $u_n \leq u_n + u_n - 1 \leq 2u_n$ and so $u_n \leq 2u_n - 1 \leq 2u_n$ for all n, which implies that $\sum_{n=1}^{\infty} |a_n|(2u_n - 1) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} |a_n u_n| < \infty$. Consequently, we deduce that $a \in \{\bar{\mu}^{\lambda}\}^{\alpha} \Longleftrightarrow \sum_{n=1}^{\infty} |a_n u_n| < \infty \Longleftrightarrow au \in \ell_1$ which proves our result and so we have done. $\square$

Corollary 4.1.3 *The α-duals of the spaces $\bar{\mu}^\lambda$ are given by*

$$\{\bar{\mu}^\lambda\}^\alpha = \Big\{(a_k/u_k) : \; a = (a_k) \in \ell_1\Big\},$$

where $u = (u_k)$ with $u_k = \lambda_k/\Delta(\lambda_k)$ for all $k \geq 1$.

Proof. This formula of $\{\bar{\mu}^\lambda\}^\alpha$ is immediate by Theorem 4.1.2. To see that, let's denote the new formula of $\{\bar{\mu}^\lambda\}^\alpha$ by D^α. Then, we have to show that $\{\bar{\mu}^\lambda\}^\alpha = D^\alpha$. For this, let $a \in \{\bar{\mu}^\lambda\}^\alpha$. Then $b = au \in \ell_1$ by Theorem 4.1.2. Now, since $b \in \ell_1$; we find $b/u \in D^\alpha$. But, from $b = au$, we get $b/u = a$ and so $a \in D^\alpha$. This implies that $\{\bar{\mu}^\lambda\}^\alpha \subset D^\alpha$. Conversely, let $a \in D^\alpha$. Then $a = (b_k/u_k)$ for some $b \in \ell_1$. Thus $au = (b_k) \in \ell_1$ and so $a \in \{\bar{\mu}^\lambda\}^\alpha$ by Theorem 4.1.2 which means that $D^\alpha \subset \{\bar{\mu}^\lambda\}^\alpha$. Hence $\{\bar{\mu}^\lambda\}^\alpha = D^\alpha$. $\square$

Remark 4.1.4 We have the inclusion $\{\bar{\mu}^\lambda\}^\alpha \subset \ell_1$. To see that, it follows by Theorem 4.1.2 that $a \in \{\bar{\mu}^\lambda\}^\alpha \Longrightarrow au \in \ell_1$. But $u_n \geq 1$ for all n and hence $a \in \ell_1$.

4.2 The β- and γ-Duals

In this section, we obtain the β- and γ-duals of our spaces $\bar{\mu}^\lambda$, where

$$\{\bar{\mu}^\lambda\}^\beta = \{a \in w : \; ax \in cs \text{ for all } x \in \bar{\mu}^\lambda\},$$

$$\{\bar{\mu}^\lambda\}^\gamma = \{a \in w : \; ax \in bs \text{ for all } x \in \bar{\mu}^\lambda\}$$

and we may begin with the following theorem which shows that β-dual of the space bs^λ is different from those of cs_0^λ and cs^λ. Thus, we will use the symbol $\bar{\eta}^\lambda$ to denote any of the spaces cs_0^λ or cs^λ, where $\bar{\eta}$ is the respective one of the spaces cs_0 or cs, and so η is the corresponding space c_0 or c, respectively. Also, for any sequence $a = (a_k)$, we define its associated sequence $\hat{a} = (\hat{a}_k)$, via the terms of a, as follows:

$$\hat{a}_k = \Delta(a_{k+1}u_{k+1}) - a_{k+1} = \lambda_k \Delta\left(\frac{a_{k+1}}{\Delta(\lambda_{k+1})}\right) \qquad (k \geq 1). \tag{4.2.1}$$

Theorem 4.2.1 *The β-duals of the spaces $\bar{\mu}^\lambda$ are given by*

$$\{bs^\lambda\}^\beta = \Big\{a \in w : \ au = (a_k u_k) \in c_0 \ \text{ and } \ \hat{a} = (\hat{a}_k) \in bv_1\Big\},$$

$$\{\bar{\eta}^\lambda\}^\beta = \Big\{a \in w : \ au = (a_k u_k) \in \ell_\infty \ \text{ and } \ \hat{a} = (\hat{a}_k) \in bv_1\Big\},$$

where $\bar{\eta}^\lambda$ stands for any one of the spaces cs_0^λ or cs^λ, and the sequences u and $\hat{a}$ are given by (4.1.1) *and* (4.2.1), *respectively.*

Proof. For every $x \in \bar{\mu}^\lambda$, let $y \in \mu$ be the sequence connected by $y = \hat{\Lambda}(x)$. Then, for any $a = (a_k) \in w$, we may use (4.1.2) and (4.2.1) with help of Abel's formula of summation by parts, to derive the following relation:

$$\begin{aligned}
\sum\nolimits_{k=1}^{n} a_k x_k &= \sum\nolimits_{k=1}^{n} a_k \Delta(y_{k-1}) + \sum\nolimits_{k=1}^{n} a_k u_k \Delta^2(y_k) \\
&= \sum\nolimits_{k=1}^{n-1} a_{k+1}\Delta(y_k) + a_n u_n \Delta(y_n) - \sum\nolimits_{k=1}^{n-1} \Delta(a_{k+1}u_{k+1})\Delta(y_k) \\
&= a_n u_n \Delta(y_n) - \sum\nolimits_{k=1}^{n-1} \hat{a}_k \Delta(y_k) \\
&= a_n u_n \Delta(y_n) - \hat{a}_{n-1} y_{n-1} + \sum\nolimits_{k=1}^{n-2} \Delta(\hat{a}_{k+1}) y_k,
\end{aligned}$$

where $\hat{a}_0 = y_0 = 0$ and the sum on right-hand side is zero for $n = 1, 2$. Thus, we have

$$\sum_{k=1}^{n} a_k x_k = a_n u_n \Delta(y_n) - \hat{a}_{n-1} y_{n-1} + \sum_{k=1}^{n-2} \Delta(\hat{a}_{k+1}) y_k \qquad (n \geq 1) \tag{4.2.2}$$

which can equivalently be written as follows:

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n-2} \Delta(\hat{a}_{k+1}) y_k - (a_n u_n + \hat{a}_{n-1}) y_{n-1} + a_n u_n y_n = A_n(y) \qquad (n \geq 1), \tag{4.2.3}$$

where $A = [a_{nk}]_{n,k=1}^{\infty}$ is a triangle defined by

$$a_{nk} = \begin{cases} a_n u_n\,; & (k = n), \\ -(a_n u_n + \hat{a}_{n-1})\,; & (k = n-1), \\ \Delta(\hat{a}_{k+1})\,; & (k \leq n-2), \\ 0\,; & (k > n). \end{cases}$$

That is, the infinite matrix A is the following triangle:

$$A=\begin{bmatrix} a_1u_1 & 0 & 0 & 0 & \cdots \\ -(a_2u_2+\hat{a}_1) & a_2u_2 & 0 & 0 & \cdots \\ \Delta(\hat{a}_2) & -(a_3u_3+\hat{a}_2) & a_3u_3 & 0 & \cdots \\ \Delta(\hat{a}_2) & \Delta(\hat{a}_3) & -(a_4u_4+\hat{a}_3) & a_4u_4 & \cdots \\ \Delta(\hat{a}_2) & \Delta(\hat{a}_3) & \Delta(\hat{a}_4) & -(a_5u_5+\hat{a}_4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Thus, it follows by (4.2.3) that $\sum_{k=1}^{n} a_kx_k = A_n(y)$ for all n and so $\left(\sum_{k=1}^{n} a_kx_k\right) = A(y)$ for every $x \in \bar{\mu}^{\lambda}$ with $y = \hat{\Lambda}(x) \in \mu$. Hence, we immediately deduce that

$$a \in \left\{\bar{\mu}^{\lambda}\right\}^{\beta} \iff ax \in cs \text{ for all } x \in \bar{\mu}^{\lambda} \iff A(y) \in c \text{ for all } y \in \mu \iff A \in (\mu, c).$$

This leads us to conclude the following three equivalences:

$$a \in \{bs^{\lambda}\}^{\beta} \iff A \in (\ell_{\infty}, c),$$

$$a \in \{cs^{\lambda}\}^{\beta} \iff A \in (c, c),$$

$$a \in \{cs_0^{\lambda}\}^{\beta} \iff A \in (c_0, c).$$

Thus, in the first case, it follows by (1) of Lemma 1.3.10 that $a \in \{bs^{\lambda}\}^{\beta} \iff$ the following conditions hold:

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty\,, \tag{4.2.4}$$

$$\lim_{n\to\infty} a_{nk} = a_k \text{ exists for every } k \geq 1\,, \tag{4.2.5}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk} - a_k| = 0\,. \tag{4.2.6}$$

On other side, for every $n > 1$, it follows by definition of A that

$$\sum_{k=1}^{\infty} |a_{nk}| = \sum_{k=1}^{n-2} |\Delta(\hat{a}_{k+1})| + |a_nu_n + \hat{a}_{n-1}| + |a_nu_n|$$

which implies that

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty \iff au \in \ell_{\infty} \text{ and } \hat{a} \in bv_1\,,$$

where $\hat{a} \in bv_1 \Longrightarrow \hat{a} \in c \subset \ell_\infty$ and so $(a_n u_n + \hat{a}_{n-1}) \in \ell_\infty$ whenever $au \in \ell_\infty$. Also, it is obvious that (4.2.5) is satisfied and so it is superfluous, where $\lim_{n\to\infty} a_{nk} = \Delta(\hat{a}_{k+1})$ and so $a_k = \Delta(\hat{a}_{k+1})$ for every $k \geq 1$. Further, for each $n > 1$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} |a_{nk} - a_k| &= |a_n u_n + \hat{a}_{n-1} + \Delta(\hat{a}_n)| + |a_n u_n - \Delta(\hat{a}_{n+1})| + \sum_{k=n+1}^{\infty} |\Delta(\hat{a}_{k+1})| \\ &= |a_n u_n + \hat{a}_n| + |a_n u_n + \hat{a}_n - \hat{a}_{n+1}| + \sum_{k=n+1}^{\infty} |\Delta(\hat{a}_{k+1})| \,.\end{aligned}$$

Besides, it can easily be seen that

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} |\Delta(\hat{a}_{k+1})| = 0 \Longleftrightarrow \lim_{n\to\infty} \sum_{k=1}^{n} |\Delta(\hat{a}_{k+1})| \text{ exists } \Longleftrightarrow \hat{a} \in bv_1$$

and

$$(a_n u_n + \hat{a}_n) \in c_0\,,\ (a_n u_n + \hat{a}_n - \hat{a}_{n+1}) \in c_0 \Longleftrightarrow au \in c_0\,,\ \hat{a} \in c_0 \Longleftrightarrow au \in c_0,$$

where $\hat{a} \in c_0$ is implied by $au \in c_0$. To see that, we have $u_k \geq 1$ and so $|a_k| \leq |a_k u_k|$ for all k. Thus $au \in c_0$ implies that $a \in c_0$ as well as $\Delta(au) \in c_0$. Therefore, it follows that $\Delta(au) - a \in c_0$ and so $\hat{a} \in c_0$ by (4.2.1), that is $au \in c_0$ implies both $a \in c_0$ and $\hat{a} \in c_0$. Hence, condition (4.2.6) can equivalently be written as follows:

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk} - a_k| = 0 \Longleftrightarrow au \in c_0 \text{ and } \hat{a} \in bv_1.$$

Consequently, we conclude that

$$a \in \{bs^\lambda\}^\beta \Longleftrightarrow au \in c_0 \text{ and } \hat{a} \in bv_1$$

which proves the given formula of $\{bs^\lambda\}^\beta$ (note that: for our triangle A, both conditions (4.2.4) and (4.2.5) are implied by the strong condition (4.2.6)). Next, in the second and third cases, we have $a \in \{cs^\lambda\}^\beta \Longleftrightarrow A \in (c, c)$, and $a \in \{cs_0^\lambda\}^\beta \Longleftrightarrow A \in (c_0, c)$. Thus, it follows by (2) and (3) of Lemma 1.3.10 that

$$a \in \{cs^\lambda\}^\beta \Longleftrightarrow \text{(4.2.4) and (4.2.5) hold, and } \lim_{n\to\infty} \sum\nolimits_{k=1}^{\infty} a_{nk} \text{ exists,}$$

$$a \in \{cs_0^\lambda\}^\beta \Longleftrightarrow (4.2.4) \text{ and } (4.2.5) \text{ hold.}$$

But, we have already show that (4.2.5) is satisfied and so it is superfluous. Similarly, it can easily be shown that the other condition of existence of the limit $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}$ is also satisfied and so it is redundant, where $\sum_{k=1}^{\infty} a_{nk} = -\hat{a}_1$ for all $n > 1$ and hence $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk} = -\hat{a}_1$ exists. We therefore deduce that

$$a \in \{cs^\lambda\}^\beta \Longleftrightarrow a \in \{cs_0^\lambda\}^\beta \Longleftrightarrow (4.2.4) \text{ holds} \Longleftrightarrow au \in \ell_\infty \text{ and } \hat{a} \in bv_1$$

which proves the given formula of $\{\bar{\eta}^\lambda\}^\beta$ and this ends the proof. □

Moreover, the spaces $\bar{\mu}^\lambda$ have the same γ-dual for the same sequence λ, as follows:

Theorem 4.2.2 *The γ-duals of the spaces $\bar{\mu}^\lambda$ are given by*

$$\{\bar{\mu}^\lambda\}^\gamma = \Big\{a \in w : \ au = (a_k u_k) \in \ell_\infty \text{ and } \hat{a} = (\hat{a}_k) \in bv_1\Big\},$$

where the sequences u and $\hat{a}$ are given by (4.1.1) *and* (4.2.1)*, respectively.*

Proof. Let $x \in \bar{\mu}^\lambda$ be given and let $y \in \mu$ be the sequence connected by $y = \hat{\Lambda}(x)$. Also, for every $a = (a_k) \in w$, let A be the triangle defined as in the proof of Theorem 4.2.1, above. Then, we can similarly show, by using (4.2.3) with Lemma 1.3.9, that

$$a \in \{\bar{\mu}^\lambda\}^\gamma \Longleftrightarrow A \in (\mu, \ell_\infty) \Longleftrightarrow (4.2.4) \text{ holds} \Longleftrightarrow au \in \ell_\infty \text{ and } \hat{a} \in bv_1$$

which proves the present result. □

4.3 Additional Results

We my begin with the following remark concerning with our main results in the previous sections.

Remark 4.3.1 We may note the following:

(1) It must be noticed that $\{cs_0^\lambda\}^\theta = \{cs^\lambda\}^\theta$ for $\theta = \alpha, \beta$ and γ, while $\{bs^\lambda\}^\theta = \{\bar{\eta}^\lambda\}^\theta$ for only $\theta = \alpha$ and γ. Also, we may observe that $\{\bar{\eta}^\lambda\}^\beta = \{\bar{\eta}^\lambda\}^\gamma$.

(2) As we have shown in the proof of Theorem 4.2.1, we have $u_k \geq 1$ and so $|a_k| \leq |a_k u_k|$ for all k. Thus, if au belongs to any of the spaces ℓ_1, c_0 or ℓ_∞; then a must belong to the same space. It follows that $\{\bar{\mu}^\lambda\}^\theta \subset \ell_\infty$, specially $\{\bar{\mu}^\lambda\}^\alpha \subset \ell_1$ and $\{bs^\lambda\}^\beta \subset c_0$. Besides, if $u \notin \ell_\infty$; then we also have $\{\bar{\mu}^\lambda\}^\theta \subset c_0$ for $\theta = \beta$ and γ.

(3) In particular, if $a \in \{bs^\lambda\}^\beta$; then $a \in c_0$ and so $\hat{a} \in c_0$ which implies that $\hat{a} \in bv_0$. Thus $a \in \{bs^\lambda\}^\beta \Longleftrightarrow au \in c_0$ and $\hat{a} \in bv_1 \Longleftrightarrow au \in c_0$ and $\hat{a} \in bv_0$, and it follows that

$$\left\{bs^\lambda\right\}^\beta = \left\{a \in w : \ au \in c_0 \ \text{ and } \ \hat{a} \in bv_0\right\}.$$

Corollary 4.3.2 *We have the following alternative formulae of β- and γ-duals of $\bar{\mu}^\lambda$*

$$\left\{bs^\lambda\right\}^\beta = \left\{a \in w : \ au \in c_0 \ \text{ and } \ \hat{a} \in bs^\beta\right\},$$

$$\left\{\bar{\eta}^\lambda\right\}^\beta = \left\{a \in w : \ au \in \ell_\infty \ \text{ and } \ \hat{a} \in \bar{\eta}^\beta\right\},$$

$$\left\{\bar{\mu}^\lambda\right\}^\gamma = \left\{a \in w : \ au \in \ell_\infty \ \text{ and } \ \hat{a} \in \bar{\mu}^\gamma\right\},$$

where $\bar{\mu}$ is any of the spaces cs_0, cs or bs, while $\bar{\eta}$ is either cs_0 or cs.

Proof. It is immediate by Lemma 1.3.5 and the results of this chapter with (3) of Remark 4.3.1. □

Remark 4.3.3 We have the following:

(1) For any $a \in \{bs^\lambda\}^\beta$ and every $y \in \ell_\infty$, we have $(a_n u_n \Delta(y_n)) \in c_0$ and $(\hat{a}_n y_n) \in c_0$ (since $au \in c_0$ and $\hat{a} \in c_0$).

(2) For any $a \in \{cs^\lambda\}^\beta$ and every $y \in c$, we have $(a_n u_n \Delta(y_n)) \in c_0$ and $(\hat{a}_n y_n) \in c$ (because of $\Delta(y) \in c_0$, $au \in \ell_\infty$ and $\hat{a} \in bv_1 \subset c$).

(3) For any $a \in \{cs_0^\lambda\}^\beta$ and every $y \in c_0$, we have $(a_n u_n \Delta(y_n)) \in c_0$ and $(\hat{a}_n y_n) \in c_0$ (as $y \in c_0$, $au \in \ell_\infty$ and $\hat{a} \in c$).

Corollary 4.3.4 *We have the following facts:*

(1) *If $a \in \{bs^\lambda\}^\beta$; then $\sum_{k=1}^\infty a_k x_k = \sum_{k=1}^\infty \Delta(\hat{a}_{k+1}) y_k$ for every $x \in bs^\lambda$ with $y = \hat{\Lambda}(x)$.*

(2) *If $a \in \{cs^\lambda\}^\beta$; then $\sum_{k=1}^\infty a_k x_k = -L\hat{a}_0 + \sum_{k=1}^\infty \Delta(\hat{a}_{k+1}) y_k$ for every $x \in cs^\lambda$ with $y = \hat{\Lambda}(x)$, where $L = \lim_{n\to\infty} y_n$ and $\hat{a}_0 = \lim_{n\to\infty} \hat{a}_n$.*

(3) *If $a \in \{cs_0^\lambda\}^\beta$; then $\sum_{k=1}^\infty a_k x_k = \sum_{k=1}^\infty \Delta(\hat{a}_{k+1}) y_k$ for every $x \in cs_0^\lambda$ with $y = \hat{\Lambda}(x)$.*

Proof. For any $x \in \bar{\mu}^\lambda$, let $y = \hat{\Lambda}(x) \in \mu$. Then, for every $a \in \{\bar{\mu}^\lambda\}^\beta$, we have $ax \in cs$ which means that $(\sum_{k=1}^n a_k x_k) \in c$ and so $\lim_{n\to\infty} \sum_{k=1}^n a_k x_k$ exists. Thus, by going to the limits in both sides of (4.2.2) as $n \to \infty$ and using Remark 4.3.3, we deduce the present result (the details are left to the reader). $\square$

Corollary 4.3.5 *We have the following facts:*

(1) *The inclusions $\{\bar{\mu}^\lambda\}^\alpha \subset \{\bar{\mu}\}^\alpha$ are always satisfied. Further, if $u \in \ell_\infty$; then the equalities $\{\bar{\mu}^\lambda\}^\alpha = \{\bar{\mu}\}^\alpha$ are satisfied.*

(2) *If $1/\lambda \in \ell_1$ and $\Delta(u) \in bv_1$; then the inclusions $\{\bar{\mu}^\lambda\}^\beta \subset \{\bar{\mu}\}^\beta$ and $\{\bar{\mu}^\lambda\}^\gamma \subset \{\bar{\mu}\}^\gamma$ are satisfied.*

(3) *If $u \in \ell_\infty$ and $\Delta(u) \in bv_0$; then the identities $\{\bar{\mu}^\lambda\}^\beta = \{\bar{\mu}\}^\beta$ and $\{\bar{\mu}^\lambda\}^\gamma = \{\bar{\mu}\}^\gamma$ are satisfied.*

Proof. For (1), it is obvious by Lemma 1.3.5 and Theorem 4.1.2 that $\{\bar{\mu}^\lambda\}^\alpha \subset \ell_1 = \{\bar{\mu}\}^\alpha$. Also, if $u \in \ell_\infty$; then $au \in \ell_1$ whenever $a \in \ell_1$ which implies that $\{\bar{\mu}\}^\alpha \subset \{\bar{\mu}^\lambda\}^\alpha$. For (2), if $1/\lambda \in \ell_1$ and $\Delta(u) \in bv_1$; then $cs \subset cs^\lambda$ and $bs \subset bs^\lambda$ (see Theorem

3.3.1) which implies that $\{cs^\lambda\}^\theta \subset \{cs\}^\theta$ and $\{bs^\lambda\}^\theta \subset \{bs\}^\theta$. Also $\{cs_0^\lambda\}^\theta = \{cs^\lambda\}^\theta \subset \{cs\}^\theta = \{cs_0\}^\theta$ (see Lemma 1.3.5 and Remark 4.3.1). For (3), if $u \in \ell_\infty$ and $\Delta(u) \in bv_0$; then the equalities $cs^\lambda = cs$ and $bs^\lambda = bs$ hold (see Theorem 3.3.1) which implies that $\{cs^\lambda\}^\theta = \{cs\}^\theta$ and $\{bs^\lambda\}^\theta = \{bs\}^\theta$. Also $\{cs_0^\lambda\}^\theta = \{cs^\lambda\}^\theta = \{cs\}^\theta = \{cs_0\}^\theta$. □

Remark 4.3.6 Suppose $u = e = (1, 1, 1, \cdots)$ in (4.1.2); we get $x = \Delta(y)$ and so $y = \sigma(x)$. Thus $x \in \bar{\mu} \Longleftrightarrow y \in \mu$. Also, from (4.2.1), we get $\hat{a}_k = -a_k$ for all k. Besides, relation (4.2.2) will be reduced to the form $\sum_{k=1}^{n} a_k x_k = a_n y_n - \sum_{k=1}^{n-1} \Delta(a_{k+1}) y_k$, where $y = \sigma(x)$. Therefore, by taking $u = (1, 1, 1, \cdots)$ in Theorems 4.1.2, 4.2.1 and 4.2.2 obtaining the θ-duals of the spaces $\bar{\mu}^\lambda$; these results will be reduced to obtain the θ-duals of the spaces $\bar{\mu}$ as given by Lemma 1.3.5. That is, θ-duals of $\bar{\mu}$, as given in Lemma 1.3.5, can be obtained from θ-duals of $\bar{\mu}^\lambda$ in Theorems 4.1.2, 4.2.1 and 4.2.2, by assuming that $u_k = 1$ for all k. Similarly, Remark 4.3.3 and Corollary 4.3.4 will be valid for β-duals of $\bar{\mu}$ with $u_k = 1$, $\hat{a}_k = -a_k$ $(k \geq 1)$, $\hat{a}_0 = -a_0$ and $y = \sigma(x)$ instead of $y = \hat{\Lambda}(x)$, where $x \in \bar{\mu}$ and $a \in \bar{\mu}^\beta$.

Chapter 5

MATRIX OPERATORS

5 MATRIX OPERATORS

In this last chapter, we characterized some matrix classes and their matrix operators related to the λ-sequence spaces $\bar{\mu}^{\lambda}$ of bounded, convergent and null series. More precisely, we essentially deduced the necessary and sufficient conditions for an infinite matrix A to act on, into and between the spaces $\bar{\mu}^{\lambda}$, where $\bar{\mu}$ stands for any of the spaces cs_0, cs or bs. This chapter is divided into three sections, the first is devoted to characterize matrix operators on the spaces $\bar{\mu}^{\lambda}$, the second is for matrix operators into the spaces $\bar{\mu}^{\lambda}$ and the last is for matrix operators between the spaces $\bar{\mu}^{\lambda}$ with some particular cases. The materials of this chapter are part of our research paper [49] which has been published in the Global Sci. J. on 2022.

5.1 Matrix Operators on $\bar{\mu}^{\lambda}$

In this section, we conclude the necessary and sufficient conditions for an infinite matrix A to act on the λ-sequence spaces $\bar{\mu}^{\lambda}$.

Every infinite matrix $A = [a_{nk}]$ will be associated with an infinite matrix $\hat{A}$ called as the associated matrix of A, and we define this associated matrix $\hat{A} = [\hat{a}_{nk}]$ by

$$\hat{a}_{nk} = \frac{\lambda_{k+1}}{\Delta(\lambda_{k+1})} a_{n,k+1} - \frac{\lambda_k}{\Delta(\lambda_k)} a_{nk} - a_{n,k+1} = \lambda_k \left(\frac{a_{n,k+1}}{\Delta(\lambda_{k+1})} - \frac{a_{nk}}{\Delta(\lambda_k)} \right) \qquad (n, k \geq 1)$$

which can be redefined in terms of our notation in (4.1.1) as follows:

$$\hat{a}_{nk} = u_{k+1} a_{n,k+1} - u_k a_{nk} - a_{n,k+1} = (u_{k+1} - 1) a_{n,k+1} - u_k a_{nk} \tag{5.1.1}$$

for all $n, k \geq 1$, where $u_k = \lambda_k / \Delta(\lambda_k)$ $(k \geq 1)$. Also, for simplicity in notations, we may define another associated matrix $\bar{A} = [\bar{a}_{nk}]$ via the entries of $\hat{A}$ by

$$\bar{a}_{nk} = \hat{a}_{n,k+1} - \hat{a}_{nk} \qquad (n, k \geq 1). \tag{5.1.2}$$

Besides, for each $n \geq 1$, we have $A_n = (a_{nk})_{k=1}^{\infty}$, $\hat{A}_n = (\hat{a}_{nk})_{k=1}^{\infty}$ and $\bar{A}_n = (\bar{a}_{nk})_{k=1}^{\infty}$ which are the n-th row sequences in the matrices A, $\hat{A}$ and $\bar{A}$, respectively. That is $\hat{A}_n = (u_{k+1}a_{n,k+1} - u_k a_{nk} - a_{n,k+1})_{k=1}^{\infty}$ and $\bar{A}_n = (\hat{a}_{n,k+1} - \hat{a}_{nk})_{k=1}^{\infty}$ for every $n \geq 1$. Further, we assume the sequences $x, y \in w$ are connected by the relation $y = \hat{\Lambda}(x)$ and so (4.1.2) is satisfied. Then $x \in \bar{\mu}^{\lambda} \iff y \in \mu$. In fact, the sequences x and y are uniquely determined in the spaces $\bar{\mu}^{\lambda}$ and μ, respectively (since $\bar{\mu}^{\lambda}$ and μ are linearly isomorphic to each others by Theorem 2.2.5 and Remark 4.1.1). Also, by using (4.1.2) with the same technique by which (4.2.2) has been derived, we obtain that

$$\sum_{k=1}^{m} a_{nk}x_k = a_{nm}u_m\Delta(y_m) - \hat{a}_{n,m-1}y_{m-1} + \sum_{k=1}^{m-2} \bar{a}_{nk}y_k \qquad (n, m \geq 1), \tag{5.1.3}$$

where $y_0 = 0$ and the sum on the right-hand side is zero when $m = 1, 2$. Moreover, let $\bar{\rho}$ be any of the spaces cs_0 or bs, and so ρ is the respective one of the spaces c_0 or ℓ_∞. Then, if $A_n \in \{\bar{\rho}^{\lambda}\}^{\beta}$ for every $n \geq 1$; then we find by Theorem 4.2.1 that $\hat{A}_n \in bv_1$ and so $\bar{A}_n \in \ell_1$ for every $n \geq 1$, which implies that series on right-hand side of (5.1.3) is absolutely convergent when $m \to \infty$, where $x \in \bar{\rho}^{\lambda}$ and $y \in \rho$. Further, it follows, by (1) and (3) of Remark 4.3.3, that $\lim_{m\to\infty} a_{nm}u_m\Delta(y_m) = 0$ and $\lim_{m\to\infty} \hat{a}_{n,m-1}y_{m-1} = 0$ for every $n \geq 1$. Therefore, by passing to the limits in both sides of (5.1.3) when $m \to \infty$ and using (1) and (3) of Corollary 4.3.4, we deduce that

$$\sum\nolimits_{k=1}^{\infty} a_{nk}x_k = \sum\nolimits_{k=1}^{\infty} \bar{a}_{nk}y_k = \sum\nolimits_{k=1}^{\infty} (\hat{a}_{n,k+1} - \hat{a}_{nk})y_k \qquad (n \geq 1) \tag{5.1.4}$$

which means that $A_n(x) = \bar{A}_n(y)$ for all n, and hence $A(x) = \bar{A}(y)$ for every pair of sequences $x \in \bar{\rho}^{\lambda}$ and $y \in \rho$ which are connected by $y = \hat{\Lambda}(x)$. Also, this means that $A(x) \in X$ for every $x \in \bar{\rho}^{\lambda}$ if and only if $\bar{A}(y) \in X$ for every $y \in \rho$, where X is any sequence space. On other side, if $A_n \in \{cs^{\lambda}\}^{\beta}$ for every $n \geq 1$; we similarly find, by Theorem 4.2.1, that $\hat{A}_n \in bv_1$ and $\bar{A}_n \in \ell_1$ for every $n \geq 1$. Thus, series on right-hand side of (5.1.3) is also absolutely convergent when $m \to \infty$, where $x \in cs^{\lambda}$ and

$y \in c$. Further, it follows, by (2) of Remark 4.3.3, that $\lim_{m\to\infty} a_{nm}u_m\Delta(y_m) = 0$ and $\lim_{m\to\infty} \hat{a}_{n,m}y_m = L\hat{a}_n$ for every $n \geq 1$, where $L = \lim_{m\to\infty} y_m$ and $\hat{a}_n = \lim_{m\to\infty} \hat{a}_{nm}$ for all n which implies that $\sum_{k=1}^{\infty}\bar{a}_{nk} = \hat{a}_n - \hat{a}_{n1}$ $(n \geq 1)$. Therefore, by using (2) of Corollary 4.3.4 and going to the limits in both sides of (5.1.3) as $m \to \infty$, we get

$$\sum_{k=1}^{\infty} a_{nk}x_k = \sum_{k=1}^{\infty} \bar{a}_{nk}y_k - L\hat{a}_n = \sum_{k=1}^{\infty} \bar{a}_{nk}(y_k - L) - L\hat{a}_{n1} \qquad (n \geq 1) \tag{5.1.5}$$

which means that $A_n(x) = \bar{A}_n(y) - L\hat{a}_n$ for all n, and hence $A(x) = \bar{A}(y) - L\hat{a}$ for every pair of sequences $x \in cs^\lambda$ and $y \in c$ which are connected by $y = \hat{\Lambda}(x)$. This also means that $A(x) \in X$ for every $x \in cs^\lambda$ if and only if $\bar{A}(y) - L\hat{a} \in X$ for every $y \in c$, where $\hat{a} = (\hat{a}_n)$ and X is any sequence space. Thus, we have proved the following:

Lemma 5.1.1 *For any infinite matrix A, let $\hat{A}$ and $\bar{A}$ be its associated matrices defined by* (5.1.1) *and* (5.1.2)*, respectively. Then, for each $n \geq 1$, the following conditions are equivalent to each others:*

(1) $A_n \in \{\bar{\mu}^\lambda\}^\beta$.

(2) $uA_n \in \langle\bar{\mu}^\lambda\rangle$ *and* $\hat{A}_n \in bv_1$.

(3) $uA_n \in \langle\bar{\mu}^\lambda\rangle$ *and* $\bar{A}_n \in \mu^\beta$,

where $\mu^\beta = \ell_1$, $\langle bs^\lambda\rangle = c_0$ *and* $\langle cs^\lambda\rangle = \langle cs_0^\lambda\rangle = \ell_\infty$.

Proof. It is immediate by Theorem 4.2.1 and Corollary 4.3.2. □

Lemma 5.1.2 *Let $\bar{\rho}$ be any of the spaces cs_0 or bs, and ρ the respective one of the spaces c_0 or ℓ_∞. Then, for any sequence space X and every infinite matrix A, we have the following facts:*

(1) *If $A_n \in \{\bar{\rho}^\lambda\}^\beta$ for every $n \geq 1$; then $A(x) = \bar{A}(y)$ for every pair of sequences $x \in \bar{\rho}^\lambda$ and $y \in \rho$ which are connected by the relation $y = \hat{\Lambda}(x)$. Also $A(x) \in X$ for all $x \in \bar{\rho}^\lambda$ if and only if $\bar{A}(y) \in X$ for all $y \in \rho$.*

(2) *If $A_n \in \{cs^\lambda\}^\beta$ for every $n \geq 1$; then $A_n(x) = \bar{A}_n(z) - L(\hat{a}_{n1})$ $(n \geq 1)$ for every pair of sequences $x \in cs^\lambda$ and $z \in c_0$ which are connected by $z = \hat{\Lambda}(x) - Le$, where $e = (1, 1, 1, \cdots)$, $L = \lim_{k\to\infty} \hat{\Lambda}_k(x)$ and $(\hat{a}_{n1})$ is the 1st column sequence in $\hat{A}$. Also, we have $A(x) \in X$ for all $x \in cs^\lambda$ if and only if $(\hat{a}_{nk})_{n=1}^\infty \in X$ for every $k \geq 1$ and $\bar{A}(z) \in X$ for all $z \in c_0$.*

Proof. we have already proved part (1) and for part (2) let $A_n \in \{cs^\lambda\}^\beta$ for every $n \geq 1$. Then, we obtain by (5.1.5) that $A_n(x) = \bar{A}_n(z) - L(\hat{a}_{n1})$ $(n \geq 1)$ for every pair of sequences $x \in cs^\lambda$ and $z \in c_0$ which are connected by $z = \hat{\Lambda}(x) - Le$, where $e = (1, 1, 1, \cdots)$ and $L = \lim_{k\to\infty} \hat{\Lambda}_k(x)$. Also, assume that $A(x) \in X$ for all $x \in cs^\lambda$, and for each $k \geq 1$, define a sequence $e^{(k)} = (e_n^{(k)})_{n=1}^\infty$ by

$$e_k^{(k)} = \begin{cases} u_k\,; & (n = k), \\ -(u_{k+1} - 1)\,; & (n = k + 1), \\ 0\,; & (\text{otherwise}). \end{cases}$$

Then, it can easily be seen that $\Lambda(e^{(k)}) = e_k \in cs$ for every $k \geq 1$, where $e_k = (\delta_{nk})_{n=1}^\infty$ for all k. Thus $e^{(k)} \in cs^\lambda$ such that $A(e^{(k)}) = (-\hat{a}_{nk})_{n=1}^\infty$ for all k which implies that $(\hat{a}_{nk})_{n=1}^\infty \in X$ for every $k \geq 1$ (by assumption). Also, for any $y \in c_0$, let $x = (x_k)$ be given by (4.1.2). Then $x \in cs_0^\lambda$ such that $y = \hat{\Lambda}(x)$ and so $\bar{A}(y) \in X$ by part (1). Thus $\bar{A}(y) \in X$ for all $y \in c_0$. Conversely, suppose that $(\hat{a}_{nk})_{n=1}^\infty \in X$ for every $k \geq 1$ and $\bar{A}(z) \in X$ for all $z \in c_0$. Then $(\hat{a}_{n1}) \in X$ and for every $x \in cs^\lambda$ we have $z \in c_0$, where $z = \hat{\Lambda}(x) - Le$ and $L = \lim_{k\to\infty} \hat{\Lambda}_k(x)$. Thus $\bar{A}(z) \in X$ and hence $A(x) = \bar{A}(z) - L(\hat{a}_{n1}) \in X$. That is $A(x) \in X$ for all $x \in cs^\lambda$, and we have done. $\square$

Now, we have the following main results characterizing matrix operators acting from cs_0^λ or bs^λ into an arbitrary sequence space X and from cs^λ into X.

Theorem 5.1.3 *Let $\bar{\rho}$ be any of the spaces cs_0 or bs, and ρ the respective one of the spaces c_0 or ℓ_∞. Then, for any sequence space X and every infinite matrix A, the following statements are equivalent to each others:*

(1) $A \in (\bar{\rho}^\lambda, X)$.

(2) $A_n \in \{\bar{\rho}^\lambda\}^\beta$ *for every* $n \geq 1$ *and* $\bar{A}(y) \in X$ *for all* $y \in \rho$.

(3) $uA_n \in \left\langle \bar{\rho}^\lambda \right\rangle$ *for every* $n \geq 1$ *and* $\bar{A} \in (\rho, X)$,

where $\left\langle bs^\lambda \right\rangle = c_0$ *and* $\left\langle cs_0^\lambda \right\rangle = \ell_\infty$.

Proof. Suppose that (1) is satisfied, that is $A \in (\bar{\rho}^\lambda, X)$. Then $A_n \in \{\bar{\rho}^\lambda\}^\beta$ for every $n \geq 1$ and $A(x) \in X$ for all $x \in \bar{\rho}^\lambda$ by Lemma 1.3.6. Thus, for every $y \in \rho$, let $x = (x_k)$ be given by (4.1.2). Then $x \in \bar{\rho}^\lambda$ such that $y = \hat{\Lambda}(x)$ and so $\bar{A}(y) \in X$ (as $A(x) = \bar{A}(y)$ by (1) of Lemma 5.1.2) and since $y \in \rho$ was arbitrary; we find that $\bar{A}(y) \in X$ for all $y \in \rho$. Hence, we have $A_n \in \{\bar{\rho}^\lambda\}^\beta$ for every $n \geq 1$ and $\bar{A}(y) \in X$ for all $y \in \rho$ which is (2), that is (1) $\Longrightarrow$ (2). Further, assume that (2) is satisfied, that is $A_n \in \{\bar{\rho}^\lambda\}^\beta$ for every $n \geq 1$ and $\bar{A}(y) \in X$ for all $y \in \rho$. This, together with Lemma 5.1.1, implies that $uA_n \in \left\langle \bar{\rho}^\lambda \right\rangle$ and $\bar{A}_n \in \rho^\beta$ for every $n \geq 1$, as well as $\bar{A}(y) \in X$ for all $y \in \rho$. Hence, we deduce that $uA_n \in \left\langle \bar{\rho}^\lambda \right\rangle$ for every $n \geq 1$ and $\bar{A} \in (\rho, X)$ which is (3), that is (2) $\Longrightarrow$ (3). Finally, suppose that (3) is satisfied, that is $uA_n \in \left\langle \bar{\rho}^\lambda \right\rangle$ for every $n \geq 1$ and $\bar{A} \in (\rho, X)$. This implies that $uA_n \in \left\langle \bar{\rho}^\lambda \right\rangle$ and $\bar{A}_n \in \rho^\beta$ for every $n \geq 1$ as well as $\bar{A}(y) \in X$ for all $y \in \rho$. Hence, it follows by Lemma 5.1.1 that $A_n \in \{\bar{\rho}^\lambda\}^\beta$ for every $n \geq 1$. Besides, for every $x \in \bar{\rho}^\lambda$, let $y = \hat{\Lambda}(x)$. Then $y \in \rho$ and $A(x) = \bar{A}(y)$ by (5.1.4) which implies that $A(x) \in X$ for all $x \in \bar{\rho}^\lambda$. Therefore, we have $A_n \in \{\bar{\rho}^\lambda\}^\beta$ for every

$n \geq 1$ and $A(x) \in X$ for all $x \in \bar{\rho}^\lambda$. This means that $A \in (\bar{\rho}^\lambda, X)$ which is (1), that is (3) $\Longrightarrow$ (1) and this completes the proof. $\square$

Theorem 5.1.4 *For any sequence space X and every infinite matrix A, the following statements are equivalent to each others:*

(1) $A \in (cs^\lambda, X)$.

(2) $A_n \in \{cs^\lambda\}^\beta$ *for every* $n \geq 1$, $\bar{A}(z) \in X$ *for all* $z \in c_0$ *and* $(\hat{a}_{nk})_{n=1}^\infty \in X$ *for every* $k \geq 1$.

(3) $uA_n \in \ell_\infty$ *for every* $n \geq 1$, $\bar{A} \in (c_0, X)$ *and* $(\hat{a}_{nk})_{n=1}^\infty \in X$ *for every* $k \geq 1$.

(4) $A \in (cs_0^\lambda, X)$ *and* $(\hat{a}_{nk})_{n=1}^\infty \in X$ *for every* $k \geq 1$.

Proof. The proof of this result is based on (2) of Lemma 5.1.2 with help of (5.1.5) and Lemma 5.1.4. Also, its proof is exactly same as that of Theorem 5.1.3, above. Thus, we may omit the details of proof. $\square$

Now, with help of (4.1.1) and (5.1.1), let's consider the following conditions:

$$\big(u_k a_{nk}\big)_{k=1}^\infty \in c_0 \text{ for every } n \geq 1 \tag{5.1.6}$$

$$\big(u_k a_{nk}\big)_{k=1}^\infty \in \ell_\infty \text{ for every } n \geq 1 \tag{5.1.7}$$

$$\sum_{k=1}^\infty |\hat{a}_{n,k+1} - \hat{a}_{nk}| \text{ converges for every } n \geq 1 \tag{5.1.8}$$

$$\sup_n \sum_{k=1}^\infty |\hat{a}_{n,k+1} - \hat{a}_{nk}| < \infty \tag{5.1.9}$$

$$\sup_n \left| \lim_{k\to\infty} \hat{a}_{nk} \right| < \infty \tag{5.1.10}$$

$$\lim_{n\to\infty} (\hat{a}_{n,k+1} - \hat{a}_{nk}) = \bar{a}_k \text{ exists for every } k \geq 1 \tag{5.1.11}$$

$$\lim_{n\to\infty} \sum_{k=1}^\infty |\hat{a}_{n,k+1} - \hat{a}_{nk} - \bar{a}_k| = 0 \tag{5.1.12}$$

$$\lim_{n\to\infty} \hat{a}_{nk} \text{ exists for every } k \geq 1 \tag{5.1.13}$$

$$\lim_{n\to\infty} \sum\nolimits_{k=1}^{\infty} |\hat{a}_{n,k+1} - \hat{a}_{nk}| = 0 \tag{5.1.14}$$

$$\lim_{n\to\infty} (\hat{a}_{n,k+1} - \hat{a}_{nk}) = 0 \text{ for every } k \geq 1 \tag{5.1.15}$$

$$\lim_{n\to\infty} \hat{a}_{nk} = 0 \text{ for every } k \geq 1 \tag{5.1.16}$$

$$\sum\nolimits_{n=1}^{\infty} |\hat{a}_{nk}|^p \text{ converges for every } k \geq 1 \ (p \geq 1) \tag{5.1.17}$$

$$\sup_{K\in\mathcal{K}} \sum_{n=1}^{\infty} \Big| \sum_{k\in K} (\hat{a}_{n,k+1} - \hat{a}_{nk}) \Big|^p < \infty \text{ for } p \geq 1, \tag{5.1.18}$$

where $\mathcal{K}$ stands for the collection of all non-empty finite subsets of positive integers. Then, by using Lemma 5.1.1, we find that

$$A_n \in \{bs^\lambda\}^\beta \text{ for every } n \geq 1 \Longleftrightarrow \text{(5.1.6) and (5.1.8) are satisfied,}$$

$$A_n \in \{\bar{\eta}^\lambda\}^\beta \text{ for every } n \geq 1 \Longleftrightarrow \text{(5.1.7) and (5.1.8) are satisfied,}$$

where $\bar{\eta}$ stands for any of the spaces cs or cs_0. Therefore, by using Theorems 5.1.3 and 5.1.4 with help of Lemmas 1.3.8, 1.3.9, 1.3.10 and 1.3.11 which characterize the matrix operators on the spaces ℓ_∞, c and c_0 into the classical sequence spaces, we can immediately deduce the following consequences characterizing matrix operators on the spaces $\bar{\mu}^\lambda$:

Corollary 5.1.5 *For an infinite matrix A, we have the following:*

(1) $A \in (bs^\lambda, \ell_\infty)$ *if and only if* (5.1.6) *and* (5.1.9) *are satisfied.*

(2) $A \in (cs_0^\lambda, \ell_\infty)$ *if and only if* (5.1.7) *and* (5.1.9) *are satisfied.*

(3) $A \in (cs^\lambda, \ell_\infty)$ *if and only if* (5.1.7), (5.1.9) *and* (5.1.10) *are satisfied.*

Proof. It follows from Theorems 5.1.3 and 5.1.4, since $\bar{A} \in (\mu, \ell_\infty) \Longleftrightarrow$ (5.1.9) held (note that (5.1.9) implies that $(\sum_{j=k}^{\infty} \bar{a}_{nj})_{n=1}^{\infty} \in \ell_\infty$. Thus (5.1.10) $\Longleftrightarrow$ $(\hat{a}_{nk})_{n=1}^{\infty} \in \ell_\infty$ for every $k \geq 1$, since $\sum_{j=k}^{\infty} \bar{a}_{nj} = \hat{a}_n - \hat{a}_{nk}$ for all n and k). $\square$

Corollary 5.1.6 *For an infinite matrix A, we have the following:*

(1) *$A \in (bs^\lambda, c)$ if and only if (5.1.6), (5.1.9), (5.1.11) and (5.1.12) are satisfied. Further, if $A \in (bs^\lambda, c)$; then $\lim_{n\to\infty} A_n(x) = \sum_{k=1}^{\infty} \bar{a}_k y_k$ for every $x \in bs^\lambda$, where $y = \hat{\Lambda}(x)$ and $\bar{a}_k = \lim_{n\to\infty}(\hat{a}_{n,k+1} - \hat{a}_{nk})$ for all k.*

(2) *$A \in (cs_0^\lambda, c)$ if and only if (5.1.7), (5.1.9) and (5.1.11) are satisfied. Moreover, if $A \in (cs_0^\lambda, c)$; then $\lim_{n\to\infty} A_n(x) = \sum_{k=1}^{\infty} \bar{a}_k y_k$ for every $x \in cs_0^\lambda$, where $y = \hat{\Lambda}(x)$ and $\bar{a}_k = \lim_{n\to\infty}(\hat{a}_{n,k+1} - \hat{a}_{nk})$ for all k.*

(3) *$A \in (cs^\lambda, c)$ if and only if (5.1.7), (5.1.9) and (5.1.13) are satisfied. Further, if $A \in (cs^\lambda, c)$; then $\lim_{n\to\infty} A_n(x) = \sum_{k=1}^{\infty} \bar{a}_k y_k - L(\hat{a}_0 + \sum_{k=1}^{\infty} \bar{a}_k)$ for every $x \in cs^\lambda$, where $y = \hat{\Lambda}(x)$, $L = \lim_{k\to\infty} y_k$, $\hat{a}_0 = \lim_{n\to\infty} \hat{a}_{n1}$ and $\bar{a}_k = \lim_{n\to\infty}(\hat{a}_{n,k+1} - \hat{a}_{nk})$ for all k.*

Proof. It is immediate by noting that: (1) $\bar{A} \in (\ell_\infty, c) \iff$ (5.1.9), (5.1.11) and (5.1.12) are satisfied. (2) $\bar{A} \in (c_0, c) \iff$ (5.1.9) and (5.1.11) are satisfied. □

Corollary 5.1.7 *For an infinite matrix A, we have the following:*

(1) *$A \in (bs^\lambda, c_0)$ if and only if (5.1.6) and (5.1.14) are satisfied.*

(2) *$A \in (cs_0^\lambda, c_0)$ if and only if (5.1.7), (5.1.9) and (5.1.15) are satisfied.*

(3) *$A \in (cs^\lambda, c_0)$ if and only if (5.1.7), (5.1.9) and (5.1.16) are satisfied.*

Proof. It is obtained by observing that: (1) $\bar{A} \in (\ell_\infty, c_0) \iff$ (5.1.14) is satisfied. (2) $\bar{A} \in (c_0, c_0) \iff$ (5.1.9) and (5.1.15) held. □

Corollary 5.1.8 *Let A be an infinite matrix. Then, for every real $p \geq 1$, we have:*

(1) *$A \in (bs^\lambda, \ell_p)$ if and only if (5.1.6), (5.1.8) and (5.1.18) are satisfied.*

(2) $A \in (cs_0^\lambda, \ell_p)$ *if and only if* (5.1.7), (5.1.8) *and* (5.1.18) *are satisfied.*

(3) $A \in (cs^\lambda, \ell_p)$ *if and only if* (5.1.7), (5.1.8), (5.1.17) *and* (5.1.18) *are satisfied.*

Proof. It is immediate by means of the fact: $\bar{A} \in (\mu, \ell_p) \Longleftrightarrow$ (5.1.18) holds. $\square$

It is worth mentioning that (5.1.6) implies $\lim_{k\to\infty} \hat{a}_{nk} = 0$ for all n (see Remark 4.3.1) and so implies all of (5.1.7) and (5.1.10). Further, in light of Remark 4.3.6, it must be noted that Corollaries 5.1.5, 5.1.6, 5.1.7 and 5.1.8 can be reduced, with assumption $u = e$ (i.e. $u_k = 1$ for all $k \geq 1$), to characterize matrix operators on the sequence spaces $\bar{\mu} = bs$, cs and cs_0 as follows:

Remark 5.1.9 The necessary and sufficient conditions for an infinite matrix A in order to belong to any of the classes $(\bar{\mu}, \ell_\infty)$, $(\bar{\mu}, c)$, $(\bar{\mu}, c_0)$ or $(\bar{\mu}, \ell_p)$ are those conditions given respectively in Corollaries 5.1.5, 5.1.6, 5.1.7 or 5.1.8 by removing condition (5.1.7) and taking $u_k = 1$ and $\hat{a}_{nk} = -a_{nk}$ for all $n, k \geq 1$, where $p \geq 1$. For example, let $u_k = 1$ in (3) of Corollaries 5.1.5, 5.1.6 and 5.1.7, we respectively obtain that

$$A \in (cs, \ell_\infty) \Longleftrightarrow \sup_n \left| \lim_{k\to\infty} a_{nk} \right| < \infty \text{ and } \sup_n \sum\nolimits_{k=1}^{\infty} |a_{nk} - a_{n,k+1}| < \infty,$$

$$A \in (cs, c) \Longleftrightarrow \sup_n \sum\nolimits_{k=1}^{\infty} |a_{nk} - a_{n,k+1}| < \infty \text{ and } \lim_{n\to\infty} a_{nk} \text{ exists for every } k \geq 1,$$

$$A \in (cs, c_0) \Longleftrightarrow \sup_n \sum\nolimits_{k=1}^{\infty} |a_{nk} - a_{n,k+1}| < \infty \text{ and } \lim_{n\to\infty} a_{nk} = 0 \text{ for every } k \geq 1$$

which coincide with the familiar results in [58] (see also Lemma 1.3.14).

5.2 Matrix Operators into $\bar{\mu}^\lambda$

In this section, we conclude the necessary and sufficient conditions for an infinite matrix A to act from any sequence space into the λ-sequence spaces $\bar{\mu}^\lambda$.

For this, we will apply the useful result in part (3) of Lemma 1.3.7 to our new spaces $\bar{\mu}^\lambda$. This leads us to the following theorem:

Theorem 5.2.1 *Let X be a sequence space, A an infinite matrix and define the matrix $B = [b_{nk}]$ by*

$$b_{nk} = \sum\nolimits_{j=1}^{n} \left(\sum\nolimits_{i=j}^{n} \frac{1}{\lambda_i} \right) \Delta(\lambda_j)\, a_{jk} \qquad (n, k \geq 1).$$

$A \in (X, \bar{\mu}^{\lambda})$ if and only if $B \in (X, \mu)$, where $\bar{\mu}$ stands for any of the spaces cs_0, cs or bs, and μ stands for the respective one of the spaces c_0, c or ℓ_∞.

Proof. This result is immediate by (3) of Lemma 1.3.7, where $B = \hat{\Lambda}\, A$. □

In particular, if X, in above theorem, is any of the classical sequence spaces; then we obtain the following corollary:

Corollary 5.2.2 *Let A be an infinite matrix and define the matrix $B = [b_{nk}]$ by*

$$b_{nk} = \sum\nolimits_{j=1}^{n} \left(\sum\nolimits_{i=j}^{n} \frac{1}{\lambda_i} \right) \Delta(\lambda_j)\, a_{jk} \qquad (n, k \geq 1).$$

Then A belongs to any one of the classes $(c_0, \bar{\mu}^{\lambda})$, $(c, \bar{\mu}^{\lambda})$, $(\ell_\infty, \bar{\mu}^{\lambda})$ or $(\ell_p, \bar{\mu}^{\lambda})$ if and only if B belongs to the respective one of the classes (c_0, μ), (c, μ), (ℓ_∞, μ) or (ℓ_p, μ), where $p \geq 1$ and μ stands for any of the spaces c_0, c or ℓ_∞.

More precisely, by using Lemmas 1.3.9, 1.3.10, 1.3.11 and 1.3.12 characterizing matrix classes (c_0, μ), (c, μ), (ℓ_∞, μ) or (ℓ_p, μ), where $1 \leq p < \infty$, we conclude the conditions:

$$\sup_n \sum\nolimits_{k=1}^{\infty} |b_{nk}| < \infty \tag{5.2.1}$$

$$\lim_{n\to\infty} b_{nk} = b_k \text{ exists for every } k \geq 1 \tag{5.2.2}$$

$$\lim_{n\to\infty} \sum\nolimits_{k=1}^{\infty} |b_{nk} - b_k| = 0 \tag{5.2.3}$$

$$\lim_{n\to\infty} \sum\nolimits_{k=1}^{\infty} b_{nk} \text{ exists} \tag{5.2.4}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |b_{nk}| = 0 \tag{5.2.5}$$

$$\lim_{n\to\infty} b_{nk} = 0 \text{ for every } k \geq 1 \tag{5.2.6}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{nk} = 0 \tag{5.2.7}$$

$$\sup_{n,\,k} |b_{nk}| < \infty \tag{5.2.8}$$

$$\sup_{n} \sum_{k=1}^{\infty} |b_{nk}|^q < \infty \qquad (\, q = p/(p-1)\,). \tag{5.2.9}$$

Now, with help of Lemmas 1.3.9, 1.3.10, 1.3.11 and 1.3.12, we immediately deduce the following corollaries in which $B = [b_{nk}]$ is as given in Corollary 5.2.2.

Corollary 5.2.3 *We have* $(c_0, bs^\lambda) = (c, bs^\lambda) = (\ell_\infty, bs^\lambda)$, *and* $A \in (\mu, bs^\lambda)$ *if and only if* (5.2.1) *holds.*

Corollary 5.2.4 *We have the following:*

(1) $A \in (\ell_\infty, cs^\lambda)$ *if and only if* (5.2.1), (5.2.2) *and* (5.2.3) *hold.*

(2) $A \in (c, cs^\lambda)$ *if and only if* (5.2.1), (5.2.2) *and* (5.2.4) *hold.*

(3) $A \in (c_0, cs^\lambda)$ *if and only if* (5.2.1) *and* (5.2.2) *hold.*

Corollary 5.2.5 *We have the following:*

(1) $A \in (\ell_\infty, cs_0^\lambda)$ *if and only if* (5.2.5) *holds.*

(2) $A \in (c, cs_0^\lambda)$ *if and only if* (5.2.1), (5.2.6) *and* (5.2.7) *hold.*

(3) $A \in (c_0, cs_0^\lambda)$ *if and only if* (5.2.1) *and* (5.2.6) *hold.*

Corollary 5.2.6 *We have the following:*

(1) $A \in (\ell_1, bs^\lambda)$ *if and only if* (5.2.8) *holds.*

(2) $A \in (\ell_1, cs^\lambda)$ *if and only if* (5.2.2) *and* (5.2.8) *hold.*

(3) $A \in (\ell_1, cs_0^\lambda)$ *if and only if* (5.2.6) *and* (5.2.8) *hold.*

Corollary 5.2.7 *Let $1 < p < \infty$ and $q = p/(p-1)$. Then, we have the following:*

(1) $A \in (\ell_p, bs^\lambda)$ *if and only if* (5.2.9) *holds.*

(2) $A \in (\ell_p, cs^\lambda)$ *if and only if* (5.2.2) *and* (5.2.9) *hold.*

(3) $A \in (\ell_p, cs_0^\lambda)$ *if and only if* (5.2.6) *and* (5.2.9) *hold.*

5.3 Particular Cases

In this final section, we apply our results to some particular cases. Also, we conclude the necessary and sufficient conditions for an infinite matrix A to act between our new spaces.

For this, we will apply (3) of Lemma 1.3.7 to our results in previous section in order to characterize the matrix operators acting from $\bar{\mu}^\lambda$ into the matrix domains of triangles. For instance, we have $cs_0 = (c_0)_\sigma$, $cs = (c)_\sigma$, $bs = (\ell_\infty)_\sigma$, $c_0(\Delta) = (c_0)_\Delta$, $c(\Delta) = (c)_\Delta$, $\ell_\infty(\Delta) = (\ell_\infty)_\Delta$ and $bv_p = (\ell_p)_\Delta$ for $p \geq 1$. Therefore, we conclude the following consequences:

Corollary 5.3.1 *Let A be an infinite matrix and define the matrices $[b_{nk}]$ and $[\hat{b}_{nk}]$ by*

$$b_{nk} = a_{nk} - a_{n-1,k} \quad \textit{and} \quad \hat{b}_{nk} = u_{k+1} b_{n,k+1} - u_k b_{nk} - b_{n,k+1} \qquad (n, k \geq 1).$$

Then, the necessary and sufficient conditions in order that A belongs to any one of the classes $(\bar{\mu}^\lambda, \ell_\infty(\Delta))$, $(\bar{\mu}^\lambda, c(\Delta))$, $(\bar{\mu}^\lambda, c_0(\Delta))$ or $(\bar{\mu}^\lambda, bv_p)$ are those conditions given respectively in Corollaries 5.1.5, 5.1.6, 5.1.7 *or* 5.1.8 *provided that the entries a_{nk} and $\hat{a}_{nk}$ are respectively replaced by b_{nk} and $\hat{b}_{nk}$ for all $n, k \geq 1$, where $p \geq 1$.*

Corollary 5.3.2 *Let A be an infinite matrix and define the matrices $[b_{nk}]$ and $[\hat{b}_{nk}]$ by*

$$b_{nk} = \sum\nolimits_{j=1}^{n} a_{jk} \quad \textit{and} \quad \hat{b}_{nk} = u_{k+1} b_{n,k+1} - u_k b_{nk} - b_{n,k+1} \qquad (n, k \geq 1).$$

Then, the necessary and sufficient conditions in order that A belongs to any one of the classes $(\bar{\mu}^{\lambda}, bs)$, $(\bar{\mu}^{\lambda}, cs)$ *or* $(\bar{\mu}^{\lambda}, cs_0)$ *are those conditions given respectively in Corollaries* 5.1.5, 5.1.6 *or* 5.1.7 *provided that the entries* a_{nk} *and* $\hat{a}_{nk}$ *are respectively replaced by* b_{nk} *and* $\hat{b}_{nk}$ *for all* $n, k \geq 1$.

Finally, we end our work with the following corollaries characterizing matrix operators between our spaces of λ-type. For this, let $\lambda' = (\lambda'_k)$ be a strictly increasing sequence of positive reals (λ and λ' need not be equal). Then $\bar{\mu}^{\lambda'} = (\mu)_{\Lambda'}$, where Λ' is the triangle defined by (2.2.4) with λ' instead of λ. Then, we deduce the following:

Corollary 5.3.3 *Let A be an infinite matrix and define the matrices* $[b_{nk}]$ *and* $[\hat{b}_{nk}]$ *by*

$$b_{nk} = \sum\nolimits_{j=1}^{n} \left(\sum\nolimits_{i=j}^{n} \frac{1}{\lambda'_i} \right) \Delta(\lambda'_j)\, a_{jk} \qquad (n, k \geq 1),$$

$$\hat{b}_{nk} = u_{k+1} b_{n,k+1} - u_k b_{nk} - b_{n,k+1} \qquad (n, k \geq 1).$$

Then, the necessary and sufficient conditions in order that A belongs to any one of the classes $(\bar{\mu}^{\lambda}, bs^{\lambda'})$, $(\bar{\mu}^{\lambda}, cs^{\lambda'})$ *or* $(\bar{\mu}^{\lambda}, cs_0^{\lambda'})$ *are those conditions given respectively in Corollaries* 5.1.5, 5.1.6 or 5.1.7 *provided that* a_{nk} *and* $\hat{a}_{nk}$ *are respectively replaced by* b_{nk} *and* $\hat{b}_{nk}$ *for all* $n, k \geq 1$.

Corollary 5.3.4 *Let A be an infinite matrix and define the matrices* $[b_{nk}]$ *and* $[\hat{b}_{nk}]$ *by*

$$b_{nk} = \frac{1}{\lambda'_n} \sum\nolimits_{j=1}^{n} \Delta(\lambda'_j)\, a_{jk} \quad \text{and} \quad \hat{b}_{nk} = u_{k+1} b_{n,k+1} - u_k b_{nk} - b_{n,k+1} \qquad (n, k \geq 1).$$

Then, the necessary and sufficient conditions in order that A belongs to any one of the classes $(\bar{\mu}^{\lambda}, \ell_{\infty}^{\lambda'})$, $(\bar{\mu}^{\lambda}, c^{\lambda'})$, $(\bar{\mu}^{\lambda}, c_0^{\lambda'})$ *or* $(\bar{\mu}^{\lambda}, \ell_p^{\lambda'})$ *are those conditions given respectively in Corollaries* 5.1.5, 5.1.6, 5.1.7 or 5.1.8 *provided that* a_{nk} *and* $\hat{a}_{nk}$ *are respectively replaced by* b_{nk} *and* $\hat{b}_{nk}$ *for all* $n, k \geq 1$, *where* $1 \leq p < \infty$.

CONCLUSION

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The new λ-sequence spaces of bounded, convergent and null series have been introduced, their isomorphic, algebraic and topological properties have been studied, their inclusion relations have been established, their Schauder bases and Köthe-Toeplitz dual spaces have been constructed and their matrix operators have been characterized. This gives an open scope and a new area for additional future research studies. For instance, the study of compact operators and some fixed point theorems on our new spaces (see [8, 44, 45] for such studies) and study some spectral theorems (see [8, 47] for such studies) with some applications in differential equations and numerical analysis (see [8, 13] for similar studies).

At the end of this thesis, I suggest the researchers to continue in study of our new sequence spaces and their matrix transformations to solve many open problems still left and in need to study.

LIST OF SYMBOLS

$\mathbb{K}$	the scalar field $\mathbb{R}$ or $\mathbb{C}$
n, k	positive integers
x, y	sequences
x_k	k-term of x
$\Delta(x)$	difference sequence of x
$\sigma(x)$	sum sequence of x
w	the space of all sequences
X, Y	sequence spaces
$\\|\cdot\\|$	norm
X^θ	Köthe-Toeplitz duals of X
X^α	α-dual of X
X^β	β-dual of X
X^γ	γ-dual of X
A, B	matrices
a_{nk}	entries of A
$A(x)$	A-transform of x
Δ	band matrix of difference
σ	sum matrix
(X, Y)	matrix class
X_A	matrix domain of A in X

λ	λ-sequence
Λ	λ-matrix
$\hat{\Lambda}$	$\hat{\lambda}$-matrix
X^λ	λ-sequence space
ℓ_∞	space of bounded sequences
c	space of convergent sequences
c_0	space of null sequences
ℓ_p	space of sequences associated with p-absolutely convergent series
bv_p	space of sequences with p-bounded variation
$\ell_\infty(\Delta)$	space of bounded difference sequences
$c(\Delta)$	space of convergent difference sequences
$c_0(\Delta)$	space of null difference sequences
bs	space of sequences associated with bounded series
cs	space of sequences associated with convergent series
cs_0	space of sequences associated with null series
bs^λ	λ-sequence space of bounded series
cs^λ	λ-sequence space of convergent series
cs_0^λ	λ-sequence space of null series
μ	the space c_0, c or ℓ_∞
$\bar{\mu}$	the space cs_0, cs or bs
$\bar{\mu}^\lambda$	the space cs_0^λ, cs^λ or bs^λ
ρ	the space c_0 or ℓ_∞
$\bar{\rho}$	the space cs_0 or bs
$\bar{\rho}^\lambda$	the space cs_0^λ or bs^λ

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ملخص الرسالة

في هذه الرسالة، قدمنا تعريفاً لبعض الفضاءات الجديدة للمتتاليات ذات النمط λ وذلك بواسطة الفضاءات الكلاسيكية للمتتاليات ذات المتسلسلات المحدودة والمتقاربة. أيضاً، قمنا بدراسة الخواص الجبرية والتوبولوجية لتلك الفضاءات الجديدة مع علاقاتها الآيزومورفية وقواعد شاودر لها. كذلك، استنتجنا بعض علاقات الاحتواء المتعلقة بتلك الفضاءات الجديدة وأوجدنا فضاءاتها الثانوية. بالإضافة إلى ذلك، أثبتنا عدد من النتائج الجديدة لتوصيف المؤثرات المصفوفية التي تؤثر على فضاءاتنا وكذلك المؤثرات المصفوفية التي تؤثر بينها. علاوةً على ذلك، قمنا بمناقشة العديد من الحالات الخاصة لبعض النتائج الأساسية وحصلنا على عدد من الاستنتاجات الهامة.

الجمهـــــوريـــــــة اليمنيــــــة
وزارة التعليم العالي والبحث العلمي
جـــــامعـــــــة البيضــــاء
نيابة الدراسات العليا والبحث العلمي

كلية التربية والعلوم
قسم الرياضيـــــات
تخصص: تحليل دالي

بعـــض المجالات المصفوفية الجديدة في فضاءات المتتاليات ومؤثراتها المصفوفية

رسالـــة مقدمـة إلى قسم الرياضيـات بكليـة التربيـة والعلـوم – رداع جامعـة البيضـاء
لنيل درجـة الماجستيـر في الريـاضيـات تخصص: التحليل الدالي

إعـــداد الباحث:
عصام صالح سعد اليعري

إشـــراف:
أ.د / عبدالله قائد نعمان الشراعي

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