A STUDY ON SOME LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

A THESIS IN PARTIAL SATISFACTION OF THE REQUIREMENT FOR THE DEGREE MASTER IN MATHEMATICS

BY

MONA ABDULNASSER ALI HUNAIBER

UNDER THE GUIDANCE OF

Dr. ALI HASAN AL-AATI

DEPARTMENT OF MATHEMATICS

FACULTY OF EDUCATION AND SCIENCE - RADA'A

ALBAYDHA UNIVERSITY

ALBAYDHA -YEMEN

2023 - 1444

DEDICATION

This thesis is wholeheartedly dedicated to my beloved parents, who have been my source of inspiration and gave me strength when I thought of giving up, who continually provide their moral, spiritual, emotional, and financial support.

To my sisters and my brothers especially my brother Ali, who supported me and made a lot of effort and time. Without him, I would not have been able to continue until the end.

Special thanks to my virtuous teachers, who were the source of my inspiration during my research. lastly, I dedicated this thesis to my relatives, mentor and friends who shared their words of advice and encouragement to finish this study.

CERTIFICATE

This is certify that the work presented in this entitled **((A study on Some Linear Partial Differential Equations with Boundary Conditions))** is an authentic and original research work carried out by **Ms. Mona Abdulnasser Ali Hunaiber**

under my supervision and submitted to the Department of Mathematics, Faculty of Education and Sciences-Rada'a, Albaydha University, as a partial fulfillment of the requirements for the Master Degree in Mathematics.

To the best of my knowledge and belief, the present work has fulfilled the prescribed conditions given in the academic ordinances and regulations of Albaydha University and it has not been submitted before to another university for the award of any degree.

Signature of the Research Supervisor and Date

(Dr. Ali Hasan Al-Aati)

DECLARATION

I **Mona Abdulnasser Ali Hunaiber** declare that this thesis is the result of research work done by me under the supervision of Dr. **Ali Hasan Al-Aati** at Department of Mathematics, Albaydha University, and hereby certify that unless stated, all work contained within this thesis is my own independent research.

I am submitting this thesis entitled, " A study on Some Linear Partial Differential Equations with Boundary Conditions " for possible award of Master degree in Mathematics of the Albaydha University.

I further declare that this thesis or any part of it has not been submitted for the award of any other degree/diploma of this or another university, except where due acknowledgment is made in the text.

Signature of the master Candidate with Date:

Signature of the Research Supervisor with Date:

Counter Signed by

Signature of Chairperson with Date:

Acknowledgments

All praises to Allah for the completion of this thesis. I thank God for all the opportunities, trials and strength that have been given to me to finish writing the thesis. I experienced so much during this process, not only from the academic aspect but also from the aspect of personality.

First, I would like to sincerely thank my supervisor Dr. Ali Al-Aati, for his guidance, understanding, patience and most importantly, he has provided positive encouragement to finish this thesis. It has been a great pleasure and honour to have him as my supervisor.

Second, I thank the president of Albaydha University, Prof. Ahmed A. Saleh Al-arami for his support and encouragement.

Furthermore, thanks to everyone in the Department of Mathematics at Faculty of Education and Sciences - Rada'a, Albaydha University.

Finally, I would like to thank my family, my parents, brothers and sisters who encouraged and supported me throughout the time of my research. This thesis is heartily dedicated to my father who without his support I would not have been able to graduate.

MONA ABDULNASSER HUNAIBER

ABSTRACT

Many phenomena that occur in engineering, chemical, physical, biological and social sciences can be modeled mathematically in the form of either ordinary or partial differential equations. We will focus our efforts in this thesis on the equations that are classified as second order linear partial differential equations. More precisely, boundary value problems involving Laplace's and Poisson's equations with Dirichlet, Neumann and Robin problems. There are many studies that dealt with Laplace's equation and Poisson's equation in terms of dimensions applications and methods of solving them. We study these equations in rectangular coordinates and spherical coordinates. We will focus on some particular methods of solution, ones based on the theory of Fourier series. It is a special case of a more general method called *separation of variables*. The idea of this general method is finding solutions that are products of functions of one variable, such that other solutions are obtained from incoming, in general, infinite sums of such product functions.

The other methods are *the double Laplace-Shehu transform*, *double Laplace-Aboodh transform* and *triple Laplace-Aboodh-Sumudu transform.*

This page

PREFACE

The study of partial differential equations is at the crossroads of scientific computing, mathematical analysis, topology, measure theory, differential geometry and many other branches of mathematics. Partial differential equations can be used to describe an extensive diversity of phenomena such as electrostatics, diffusion, sound, quantum mechanics, gravitation, elasticity, fluid dynamics and electrodynamics, etc. Almost all cases in many engineering and scientific fields are designed using these equations. In this thesis, our contribution is to introduce a new double integral transforms to solve linear partial differential equations and study their properties.

This thesis is divided into four chapters, the main results in the last chapter have been published in three research papers which have been presented in the Albaydha University Journal (2021), Global Scientific Journals (2022) and Journal of Applied Mathematics and Computation (2022).

The content of all chapters are concisely summarized below:

Chapter 1: The first chapter begins with the introduction of partial differential equations, some basic concepts and notations. Also, we have given types of linear partial differential equations and some methods to solve them.

Chapter 2: In this chapter, we have introduced the second order linear partial differential equations and dealt with the Laplace equation in rectangular coordinates in two, three and four dimensions with boundary value problems. Also, we deal with Poisson equation with boundary values problems. In addition, we present the fundamental solution of Laplace and Poisson equations and some concepts and theorems.

Chapter 3: The Laplace equation and Poisson equation in spherical coordinates with boundary conditions are considered in the introduce chapter.

Chapter 4: This chapter present new methods to solve some linear partial differential equations such Laplace and Poisson equations. This methods are the double Laplace-Shehu transform, the double Laplace-Aboodh transform and Triple Laplace-Aboodh-Sumudu Transform. We introduced some properties, some elementary functions of these transforms and solving Laplace, Poisson, Heat and Wave equations by these integral transforms.

NOTATION AND SYMBOLS

- $\mathbb{R}^n$ Euclidian space of dimension n.
- O Open subset of $\mathbb{R}^n$.
- $\mathcal{D}^\alpha$ Operator $\frac{\partial|\alpha|}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}, \alpha \in \mathbb{N}^n, |\alpha| = \sum_{i=1}^n \alpha_i$.
- ω_n Area of unit sphere of $\mathbb{R}^n, \omega_n = 2\pi^{n/2}/\Gamma(n/2)$.
- $\Gamma(x)$ The Gamma function: $\Gamma(x) = \int_0^\infty e^{-\rho}\rho^{x-1}d\rho \quad x > 0$.
- ∇^m The collection of all partial derivatives of order m.
- $P_n(x)$ Legendre polynomial.
- $P_m^n(x)$ Associated Legendre functions.
- ∇ Gradient: $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n})$.
- $\frac{\partial u}{\partial \nu} = (\nabla u, \nu), \nu \in \mathbb{R}^n, |\nu| = 1$.
- Δ Laplace's operator (Laplacian): $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$.
- $\mathcal{C}^m(\Omega)$ Space of m times continuously differentiable functions on the open set $\Omega \subset \mathbb{R}^n$.
- $\mathcal{C}^m(\bar{\Omega})$ Space of functions $u \in \mathcal{C}^m(\Omega)$ whose partial derivatives until order m extend continuously on $\bar{\Omega}$.
- $\bar{\Omega} = \Omega \cup \partial\Omega$.
- $B_r(x)$ Open ball of $\mathbb{R}^n$ of center x and radius r.
- τ_f The Newtonian potential of the function f.

Table of Contents

CHAPTER 1

Introduction

1.1 Introduction

Partial differential equations (PDEs) are the principal means of providing mathematical models in engineering, science, physics and other fields. Partial differential equations are important tools for the study of all kinds of natural phenomena and they are widely used to explain various physical laws. In physics for example, the heat flow and the wave propagation phenomena are well described by partial differential equations. They are often too convoluted to be solved analytically. Though one obtain the exact solution of some problems it includes difficult to interpret the solutions or it is much tedious calculations. The effort to overcome this difficulty led to the invention of different analytical methods to solve them.

1.2 Basic Concepts and Notations

Definition 1.2.1.[34] A partial differential equation (PDE) is an equation involving an unknown function of two or more variables and certain of its partial derivatives. Partial differential equations are either linear or nonlinear. In the linear partial differential equations the dependent variable and all its derivatives appear in a linear form. They model, for instance, heat transfer, fluid flows, wave propagations, vibrations, and elastic or plastic deformations or displacements. The most known partial differential equations are Laplace, Poisson, wave and heat equations.

A nonlinear partial differential equation can be described as a partial differential equation involving nonlinear terms. A partial differential equation is an identity that relates the dependent variable u, the independent variables, and the partial derivatives of u [37]. The general partial differential equation in two independent variables of first order can be written as

$$F(x, y, u, u_x, u_y) = F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0, \tag{1.2.1}$$

and the order of an equation is the highest derivative that seems. The general second order partial differential equation in two independent variables is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{1.2.2}$$

For a positive integer number m, the general formula of an m^{th}-order partial differential equation in a domain $\Omega \subset \mathbb{R}^n$ is given by

$$F(\nabla^m u(x), \nabla^{m-1} u(x), ..., \nabla u(x), u(x), x) = 0, \qquad for \;\; x \in \Omega \tag{1.2.3}$$

where $F : \mathbb{R}^{n^m} \times \mathbb{R}^{n^{m-1}} \times \mathbb{R}^{n^{m-2}} \times ... \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ is continuous function and u is a C^m -function in Ω. A C^m-solution u satisfying the above equation in the pointwise sense in Ω is called a classical solution [37].

A linear partial differential equation is an equation of the form

$$Lu = f, \tag{1.2.4}$$

where L is a linear differential operator, such that

$$L : \mathcal{C}^m(\Omega) \to \mathcal{C}(\Omega), \qquad Lu = \sum_{|\alpha| \le m} a_\alpha(x) \mathcal{D}^\alpha u. \tag{1.2.5}$$

Here $a_\alpha, f \in \mathcal{C}(\Omega)$ are given functions, where the functions $a_\alpha(x)(|\alpha| \le m)$ are called the coefficients of the differential operator. If $f = 0$, one says the equation is homogeneous [62]. If $f \neq 0$, then the equation is called nonhomogeneous [27].

1.3 Classification of Partial Differential Equations

The classification of partial differential equation is motivated by the classification of the quadratic equation of the form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{1.3.1}$$

Consider a second-order partial differential equation with two independent variables which has the form:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{1.3.2}$$

where the coefficients A, B, C, D, E and F are real functions of independent variables x and y. Define a discriminant $\Delta(x, y)$ by

$$\Delta(x_0, y_0) = B^2(x_0, y_0) - 4A(x_0, y_0)C(x_0, y_0). \tag{1.3.3}$$

There are three types of linear partial differential equations elliptic, hyperbolic and parabolic, which are invariant under changes of variables. The types are determined by the sign of the discriminant $\Delta(x, y)$.

Definition.1.3.1.[4] An equation (1.3.2) at the point $p(x_0, y_0)$ is called elliptic if $\Delta(x_0, y_0) < 0$. It is hyperbolic if $\Delta(x_0, y_0) > 0$ and parabolic if $\Delta(x_0, y_0) = 0$.

Elliptic equations describe steady-state phenomena, hyperbolic equations describe wave motion and vibrating and parabolic equations describe heat flow and diffusion processes. Laplace, wave and heat equations are three major examples of second-order partial differential equations of elliptic, hyperbolic and parabolic types [4].

The topic of partial differential equations is very important subject, yet there is no general method to solve all the partial differential equations. The behavior of the solutions very much depend fundamentally on the classification of partial differential equations, hence the problem of classification for partial differential equations is well known and very natural since the classification governs the sufficient number and the type of the conditions in order to determine whether the problem is well-posed and has a unique solution [1].

Boundary value problems for Laplace equation are well-posed with respect to class of boundary problem if [34, 37]:

(i) A solution of the problem exists.

(ii) The solution is unique.

(iii) Small variations of the boundary problem yield small variations on the corresponding solutions.

We have described the boundary problems on the separability of a partial differential equation. We will be concerned with boundary value problems. There are numerous types of boundary value problems for partial differential equations. The ones that appear most frequently in problems of applied mathematics and mathematical physics comprise [37]:

(i) Dirichlet problem: u is prescribed on a boundary.

(ii) Neumann problem: $\frac{\partial u}{\partial n}$ is prescribed on a boundary.

(iii) Mixed problem (Robin problem) : $\frac{\partial u}{\partial n} + gu$ is prescribed on a boundary, where $\frac{\partial u}{\partial n}$ is the directional derivative of u along the outward normal to the boundary, and g is a given continuous function on the boundary.

Remark 1.3.2. The Cauchy problem for Laplace equation is ill-posed.

1.4 Some Methods for Solving Partial Differential Equations

Definition 1.4.1. A solution of partial differential equation means a sufficiently smooth function u of the independent variables that satisfies the partial differential equation at every point of its domain of definition.

We do not necessarily require that the solution be defined for all possible values of the independent variables. Actually, usually the differential equation is imposed on some domain Ω contained in the space of independent variables, and we look for a solution defined only on Ω. In general, the domain Ω will be an open subset, usually connected and often bounded, with a logical nice boundary, denoted by $\partial\Omega$ [59].

Modeling industrial mathematics, computer field and other applied sciences can be described as ordinary or partial differential equations. Because of the amplitude of this applied field that it occupies mathematicians have paid attention to differential equations and finding their solutions through several methods, some are analytical and others are numerical to find the approximate or existing solutions. There are many methods to solve partial differential equations.

Some useful techniques to solve partial differential equations are:

1. *Separation of Variables.* This method was introduced by d'Alembert (1747) and Euler (1748) for the wave equation and used by Laplace (1782) and Legendre (1782) for the Laplace's equation and by Fourier (1811-1824) for the heat equation. This procedure reduces a partial differential equation in n variables to n ordinary differential equations [24].

2. *Integral Transform.* The source of the integral transforms can be traced back to the work of Laplace in 1780s and Fourier in 1822. Laplace transform is highly competent for solving some class of ordinary and partial differential equations. This technique reduces a partial differential equation in n independent variables to one in $n-1$ variables.

In the literature, there are many different types of integral transforms such as Fourier transform, Laplace transform, Sumudu transform and so on. These kinds of integral transforms have many applications in various fields of mathematical sciences and engineering such as physics, mechanics, chemistry, acoustic.

3. *Numerical Methods.* The idea of using a variational formulation of a boundary value problem for its numerical solution goes back to Lord Rayleigh (1894,1896) and Ritz (1908), These methods change a partial differential equation to a system of difference equations that can by solved by iterative techniques on a computer [46].

4. *Change of coordinates.* This technique changes a partial differential equation to else of ones or changes the original partial differential equation to an ordinary differential equation by changing the coordinates of the problem.

5. *Transformation of the Dependent Variables.* This method transforms the unknown of a partial differential equation into a new unknown that easier to find.

6. *Eigenfunction Expansion.* This method attempts to find the solution of a partial differential equation as an infinite sum of eigenfunctions. It has been applied to solve differential and integral equations of linear and non-linear problems in mathematics chemistry, physics, biology and up to now a large number of research papers have been published to show the feasibility of the decomposition method.

7. *The Adomian decomposition method.* The Adomian decomposition method was firstly introduced by George Adomian in 1981. This method has been applied to solve

differential and integral equations of linear and non-linear problems in mathematics, biology, chemistry and physics.

Generally, many researchers have turned their attention to solve partial differential equations and to develop new methods for solving such equations. Due to the rapid development in the physical science and engineering models [58].

CHAPTER 2

The Solution of Second-Order Linear Partial Differential Equations in Rectangular Coordinates Using the Separation of Variables Method

2.1 Introduction

Partial differential equations help to find solutions of many complicated theories and contribute to the development of the theoretical side of the equation as well as to the application side. There are many different types of partial differential equations (PDEs) that differ according to order (first order, second order and so on). For second-order linear partial differential equations we might suppose that pertinent boundary would include specifying u, or some of its first derivatives, or both, along a suitable set of boundaries bordering or enclosing the region over which a solution is sought.

Definition 2.1.1. Let Ω be a domain in $\mathbb{R}^n$ and a_{ij}, b_i and $c \in C(\Omega)$ for $i, j = 1, 2, ..., n$. The second order equation in n independent variables, with one dependent function u on a domain Ω is of the form

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = F, \tag{2.1.1}$$

where a_{ij}, b_i and c are called coefficients of $u_{x_i x_j}, u_{x_i}, u$, respectively, and F are functions of $x_1, x_2, ..., x_n$. Here a_{ij} is a symmetric matrix in Ω as $a_{ij} = a_{ji}$ [37].

Three common types of boundary conditions occur and are associated with the names of Dirichlet, Neumann and Robin. It can be proved that the type of boundary problems needed is very closely related to the nature (elliptic, hyperbolic or parabolic) of the partial differential equations, but that complications can arise in some cases. The general considerations involved in determining exactly which boundary problems are appropriate for a particular problem are complex. The great mathematician and philosopher descartes once said, "All problems can be translated into mathematical problems, and all mathematical problems can be transformed into algebraic problems". All algebraic problems can be transformed into equations. It is to convert a physical problem into a mathematical problem (mainly the second order linear partial differential equation) [82].

In this chapter we will deal with Laplace and Poisson equations in rectangular coordinates. For the Laplace and Poisson equations, both variables represent space coordinates, x and y, and the associated boundary value problems model the equilibrium configuration of a planar body, e.g., the deformations of a membrane.

Separation of variables method seeks special solutions that can be written as the product of functions of the individual variables, thereby reducing the partial differential equation to a pair of ordinary differential equations. More general solutions can then be expressed as infinite series in the appropriate separable solutions.

Theorem 2.1.2.(Principle of Superposition)[27] *Let* $u_1, u_2, \cdots, u_m$ *be solutions of* (1.2.3) *and* F *be linear, then* $U = c_1u_1 + c_2u_2 + \cdots + c_mu_m$ *is a solution of* (1.2.3) *for any constants* $c_1, c_2, ..., c_m$.

proof. The proof follows simply from the fact that the differentiation operator is linear.

2.2 Laplace Equation

One of the important equation in partial differential equations is the Laplace equation, which is used in many applications such as applications of geometry, static electricity, potential theory and fluid flow. It serves as a useful model problem for the study of general elliptic partial differential equations. In recent decades, it used various methods for solving Laplace equation in some geometries and boundary conditions. The Laplace equation has the form

$$\Delta u = 0, \tag{2.2.1}$$

where the Laplacian Δ is defined by

$$\Delta = \frac{\partial^2}{\partial {x^2}_1} + \frac{\partial^2}{\partial {x^2}_2} + ... + \frac{\partial^2}{\partial {x^2}_n},$$

where $x \in \Omega$ and the unknown is $u : \bar{\Omega} \to \mathbb{R},\ u = u(x)$, $\Omega \subset \mathbb{R}^n$ is a given open set. Solutions of Laplace equation are very important in many branches of Physics and Engineering [34].

Definition 2.2.1.[37] Let Ω be an open set, then a function $u \in \mathcal{C}^2(\Omega)$ is called harmonic in Ω if

$$\Delta u = 0, \quad in \quad \Omega. \tag{2.2.2}$$

That means, a function is harmonic if it satisfies Laplace equation. The space of harmonic functions can thus be identified as the kernel of the second order linear partial differential operator Δ [43]. Harmonic functions happen as the potential functions for two dimensional gravitational electrostatic. Here, two dimensional means not that the fields lie in the xy-plane, but rather than as fields in three-space, the vectors all lie in horizontal planes, and the field looks the same no matter what horizontal plane it is viewed in.

Theorem 2.2.2.[37] *Let Ω be a domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$ be a harmonic function in Ω. Then u is smooth in Ω.*

2.2.1 Fundamental Solution

The Laplace equation is preserved by rotations about some point in $\mathbb{R}^n$, say the origin. Hence, it is plausible that there exist special solutions that are invariant under rotations. Since the Laplacian operator Δ is symmetric, we are seek a "radial" solution. Due to the symmetry of Laplace equation, radial solutions are natural to look for since the given partial differential equation can be reduced to a ordinary differential equation which is easier to handle. In this way, we can reduce the higher dimensional problems to one dimensional.

Let u be the harmonic functions in $\mathbb{R}^n$ which are radial. That means, functions depending only on $r = |x|$. Set

$$v(r) = u(x)$$

where $r = |x| = (x_1^2 + ... + x_n^2)^{\frac{1}{2}}$ and v is to be selected, so that $\Delta u = 0$ hold.

For $\quad i = 1, 2, ..., n \quad$ and $x \neq 0$, that

$$r_{x_i} = \frac{1}{2}(x_1^2 + ... + x_n^2)^{-\frac{1}{2}} \; 2x_i = \frac{x_i}{r} \quad (x \neq 0),$$

hence

$$u_{x_i} = v'(r)\frac{x_i}{r},$$

and

$$u_{x_i x_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)\Big(\frac{1}{r} - \frac{x_i^2}{r^3}\Big).$$

Thus

$$\Delta u = v'' + \frac{n-1}{r}v' = 0.$$

Hence $\quad \Delta u = 0 \quad$ if and only if

$$v'' + \frac{n-1}{r}v' = 0.$$

If $v' \neq 0$, we reduce

$$(logv')' + \frac{n-1}{r} = 0.$$

A simple integration then yields, for $n = 2$,

$$v(r) = a \; log \; r + c \qquad for \; any \;\; r > 0,$$

and for $n \geq 3$,

$$v(r) = br^{2-n} + c \qquad for\ any \quad r > 0,$$

where a, b and c are constants [34].

Definition 2.2.1.1.[37, 62] The function

$$\Psi(x) = \begin{cases} \frac{-1}{2\pi} log|x| & for\ n = 2, \\ \frac{1}{n(n-2)\omega_n}|x|^{2-n} & for\ n \geq 3, \end{cases} \tag{2.2.3}$$

defined for $x \in \mathbb{R}^n \backslash \{0\}$, is called *the fundamental solution* of Laplace equation. Here ω_n is the area of n-dimensional unit sphere which is given by $\omega_n = (2\pi^{\frac{\pi}{2}}) \setminus (\Gamma(\frac{n}{2}))$, such that

$$\Gamma(\xi) = \int_0^\infty e^{-\rho}\rho^{\xi-1}d\rho, \quad \xi > 0,$$

is the Gamma function.

Theorem 2.2.1.2.[37] *Let Ω be a bounded C^1-domain in $\mathbb{R}^n$ and that $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. Then for any $x \in \Omega$, we have*

$$u(x) = \int_\Omega \Psi(x-y)\Delta_y u(y)dy - \int_{\partial\Omega} \Big(\Psi(x-y)\frac{\partial u}{\partial \upsilon_y}(y) - u(y)\frac{\partial \Psi}{\partial \upsilon_y}(x-y)\Big)dS_y, \tag{2.2.4}$$

where Ψ is the fundamental solution of Laplace equation.

Remark 2.2.1.3. The fundamental solution Ψ is harmonic in $\mathbb{R}^n \backslash \{0\}$, i.e., $\Delta\Psi = 0 \quad in\ \mathbb{R}^n \backslash \{0\}$, and

$$\int_{\partial B_r} \frac{\partial \Psi}{\partial \nu} dS = 1 \qquad for\ any\ r > 0.$$

2.2.2 Mean-Value Property

If u is a harmonic function in Ω, then for all $x \in \Omega$, the value of u at x is the integral average over any sphere centered or ball at x and contained in Ω. The precise statement of the mean-value property is given in the following. There are two versions of the mean-value property, mean values over spheres and mean values over balls.

Definition 2.2.2.1.[37, 38] Let Ω be a connected domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$ be a harmonic. Then

(i) u satisfies the mean-value property over spheres if for any $B_r(x) \subset \Omega$,

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y) dS_y; \tag{2.2.5}$$

(ii) u satisfies the mean-value property over balls if for any $B_r(x) \subset \Omega$,

$$u(x) = \frac{n}{\omega_n r^n} \int_{B(x)} u(y) dy, \tag{2.2.6}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

Not only do harmonic functions have the mean-value property in definition (2.2.2.1) but they are the only continuous functions that have this property.

Theorem 2.2.2.2.(Mean-Value Formulas for Laplace Equation)[34]

Let $u \in C^2(\Omega)$ be a harmonic function in Ω. Then

$$u(x) = \int_{\partial B(x,r)} u \; dS = \int_{B(x,r)} u \; dy \tag{2.2.7}$$

for all each ball $B(x,r) \subset \Omega$.

Theorem 2.2.2.3.(Converse to Mean-Value Property)[34]

If a function $u \in C^2(\Omega)$ satisfies

$$u(x) = \int_{\partial B(x,r)} u \; dS, \tag{2.2.8}$$

for all each ball $B(x,r) \subset \Omega$, then u is harmonic.

2.3 Boundary Value Problems for Laplace Equation

There are three main types of boundary value problems that arise in most applications. Specifying the value of the solution along the boundary of the domain is called a *Dirichlet problem*, to respect the 19^{th} century analyst Johann Peter Gustav Lejeune Dirichlet (1805-1859). If the values of the normal derivative are prescribed on the boundary, the problem is called a *Neumann problem*, named after his contemporary Carl Gottfried Neumann (1832-1925). Prescribing the function along part of the boundary and the normal derivative along the remainder results in a *mixed problem* or *Robin problem.* A boundary value problem is finding a function which satisfies a given partial differential equation and particular boundary problems. Just as initial value problems are associated with hyperbolic partial differential equations, boundary value problems are associated with partial differential equations of elliptic type.

We have described the boundary problems on the separability of a partial differential equation. Also, we focus on the solution of the Laplace equation over the finite two, three and four dimensional by method of separation of variables. Recall that the method reduces the partial differential equation into a system of ordinary differential equations.

2.3.1 The Dirichlet Problem for the Laplace Equation

Let Ω be a bounded open set in $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$, u be a continuous function defined on $\bar{\Omega}$, harmonic in Ω.

Consider the Dirichlet problem for Laplace equation,

$$\Delta u = 0 \qquad in \;\; \Omega, \tag{2.3.1}$$

$$u = g \qquad on \;\; \partial\Omega. \tag{2.3.2}$$

where $g \in C(\partial\Omega)$ and Δ is the Laplacian operator [62, 75].

The Dirichlet problem for Laplace equation in two dimensions x and y over an $a \times b$ rectangle is

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad 0 < x < a, \; 0 < y < b, \tag{2.3.3}$$

$$u(x,0) = f_1(x), \quad u(x,b) = f_2(x), \qquad 0 < x < a, \tag{2.3.4}$$

$$u(0,y) = f_3(y), \quad u(a,y) = f_4(y), \qquad 0 < y < b. \tag{2.3.5}$$

Laplace equation has an extensive variety of solutions [28]. We can solve (2.3.3) when u is stated along the boundary of the rectangle.

Using separation of variables method to solve this equation. This method consists in building the set of basic functions which is used in developing solutions in the form of an infinite series expansion over the basic functions. Hence, we get the solution of Eq. (2.3.3) with boundary conditions as

$$\begin{aligned} u(x,y) = &\sum_{n=1}^{\infty} A_n \ sin\frac{n\pi}{a}x \ sinh\frac{n\pi}{a}(b-y) + \sum_{n=1}^{\infty} B_n \ sin\frac{n\pi}{a}x \ sinh\frac{n\pi}{a}y \\ &+ \sum_{n=1}^{\infty} C_n \ sin\frac{n\pi}{b}y \ sinh\frac{n\pi}{b}(a-x) + \sum_{n=1}^{\infty} D_n \ sin\frac{n\pi}{b}y \ sinh\frac{n\pi}{b}x, \end{aligned} \tag{2.3.6}$$

where A_n, B_n, C_n and D_n are coefficients and for $n = 1, 2, 3, ...,$ given by

$$A_n = \frac{2}{a \ sinh\frac{nb\pi}{a}} \int_0^a f_1(x) \ sin\frac{n\pi}{a}x \ dx, \tag{2.3.7}$$

$$B_n = \frac{2}{a \ sinh\frac{nb\pi}{a}} \int_0^a f_2(x) \ sin\frac{n\pi}{a}x \ dx, \tag{2.3.8}$$

$$C_n = \frac{2}{b \ sinh\frac{na\pi}{b}} \int_0^b f_3(y) \ sin\frac{n\pi}{b}y \ dy, \tag{2.3.9}$$

and

$$D_n = \frac{2}{b \ sinh\frac{na\pi}{b}} \int_0^b f_4(y) \ sin\frac{n\pi}{b}y \ dy. \tag{2.3.10}$$

When we determining the values of functions f_1, f_2, f_3, and f_4, we can find the values of the coefficients A_n, B_n, C_n and D_n using the Fourier transform [20].

Also, consider Laplace equation in three dimensional with Dirichlet problem,

$$u_{xx} + u_{yy} + u_{zz} = 0, \quad 0 < x < a, \quad 0 < y < b, \quad 0 < z < c, \tag{2.3.11}$$

$$u(x, y, 0) = 0, \quad u(x, y, c) = f(x, y), \quad 0 < x < a, \quad 0 < y < b, \tag{2.3.12}$$

$$u(x, 0, z) = 0, \qquad u(x, b, z) = 0, \qquad 0 < x < a, \quad 0 < z < c, \tag{2.3.13}$$

$$u(0, y, z) = 0, \qquad u(a, y, z) = 0, \qquad 0 < y < b, \quad 0 < z < c. \tag{2.3.14}$$

Using the method of separation of variables to solve this equation. Then, the solution of three dimensional Dirichlet problem is

$$u(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\ sin\frac{n\pi}{a}x\ sin\frac{m\pi}{b}y\ sinh(\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\ )\ z, \tag{2.3.15}$$

where

$$A_{nm} = \frac{4}{a\ b\ sinh(\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\ )\ c}\int_0^b\int_0^a f(x, y)\ sin\frac{n\pi}{a}x\ sin\frac{m\pi}{b}y\ dx\ dy. \tag{2.3.16}$$

Since Laplace equation is a homogeneous linear system, then any linear combination of solutions is also a solution.

Now with some details, we consider the four dimensional Dirichlet problem for Laplace equation

$$u_{xx}(x, y, z, w) + u_{yy}(x, y, z, w) + u_{zz}(x, y, z, w) + u_{ww}(x, y, z, w) = 0, \tag{2.3.17}$$

$$u(0, y, z, w) = 0, \quad u(a, y, z, w) = 0, \qquad 0 < x < a, \tag{2.3.18}$$

$$u(x, 0, z, w) = 0, \qquad u(x, b, z, w) = 0, \qquad 0 < y < b, \tag{2.3.19}$$

$$u(x,y,0,w)=0, \qquad u(x,y,c,w)=0, \qquad 0<z<c, \tag{2.3.20}$$

$$u(x,y,z,0)=0, \qquad u(x,y,z,d)=f(x,y,z), \qquad 0<w<d. \tag{2.3.21}$$

Using the separation of variables method for solving Eq. (2.3.17) with boundary conditions.

We first look for product solutions of the form

$$u(x,y,z,w)=X(x)Y(y)Z(z)W(w). \tag{2.3.22}$$

Differentiating (2.3.22) twice with respect to x, y, z and w, gives

$$u_{xx}=X''(x)Y(y)Z(z)W(w), \tag{2.3.23}$$

$$u_{yy}=X(x)Y''(y)Z(z)W(w), \tag{2.3.24}$$

$$u_{zz}=X(x)Y(y)Z''(z)W(w), \tag{2.3.25}$$

$$u_{ww}=X(x)Y(y)Z(z)W''(w). \tag{2.3.26}$$

Substituting into (2.3.17), gives

$$X''YZW+XY''ZW+XYZ''W+XYZW''=0. \tag{2.3.27}$$

Dividing (2.3.27) by $X(x)Y(y)Z(z)W(w)$, we obtain

$$\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}+\frac{Z''(z)}{Z(z)}+\frac{W''(w)}{W(w)}=0, \tag{2.3.28}$$

$$\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}-\frac{Z''(z)}{Z(z)}-\frac{W''(w)}{W(w)}. \tag{2.3.29}$$

Since the left side depends on x only and the right side depends on y, z and w, the expressions on the right and the left sides must be equal to constant.

Consider negative separation constants. Thus

$$\frac{X''(x)}{X(x)} = -\kappa^2, \qquad (\kappa > 0) \tag{2.3.30}$$

where κ is the separation constant. From this equation we get ordinary differential equation

$$X''(x) + \kappa^2 X(x) = 0. \tag{2.3.31}$$

From Eq. (2.3.29) and Eq. (2.3.30), we have

$$-\Big(\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} + \frac{W''(w)}{W(w)}\Big) = -\kappa^2. \tag{2.3.32}$$

Accordingly, setting

$$\frac{Y''(y)}{Y(y)} = -\Big(\frac{Z''(z)}{Z(z)} + \frac{W''(w)}{W(w)} - \kappa^2\Big) = -\mu^2, \qquad (\mu > 0). \tag{2.3.33}$$

Hence

$$Y''(y) + \mu^2 Y = 0. \tag{2.3.34}$$

Also, we have

$$\frac{Z''(z)}{Z(z)} = -\frac{W''(w)}{W(w)} + v^2, \tag{2.3.35}$$

where $v^2 = \kappa^2 + \mu^2$.

Because in the previous equation the right side depends only on w and the left side only on z, we infer that

$$\frac{Z''(z)}{Z(z)} = -\lambda^2 \tag{2.3.36}$$

and

$$-\frac{W''(w)}{W(w)} + v^2 = -\lambda^2 \qquad or \qquad \frac{W''(w)}{W(w)} = v^2 + \lambda^2, \tag{2.3.37}$$

hence, we obtain

$$Z''(z) + \lambda^2 Z(z) = 0, \qquad and \qquad W''(w) - \alpha^2 W(w) = 0, \tag{2.3.38}$$

where $\alpha^2 = v^2 + \lambda^2$.

We arrive to the four second order ordinary differential equations

$$X''(x) + \kappa^2 X(x) = 0, \tag{2.3.39}$$

$$Y''(y) + \mu^2 Y(y) = 0, \tag{2.3.40}$$

$$Z''(z) + \lambda^2 Z(z) = 0, \tag{2.3.41}$$

and

$$W''(w) - \alpha^2 W(w) = 0. \tag{2.3.42}$$

Here κ, μ, λ and α are separation constants.

Solving the second order differential equations (2.3.39)-(2.3.42), gives

$$X(x) = A_1 \; \cos \kappa x + A_2 \; \sin \kappa x, \tag{2.3.43}$$

$$Y(y) = B_1 \; \cos \mu y + B_2 \; \sin \mu y, \tag{2.3.44}$$

$$Z(z) = C_1 \ \cos\lambda z + C_2 \ \sin\lambda z, \tag{2.3.45}$$

and

$$W(w) = D_1 \ \cosh\alpha w + D_2 \ \sinh\alpha w, \tag{2.3.46}$$

where $\alpha = \sqrt{\kappa^2 + \mu^2 + \lambda^2}$ and $A_1, A_2, B_1, B_2, C_1, C_2, D_1$ and D_2 are constants.

Using the homogeneous boundary conditions, gives

$$A_1 = 0, \qquad \kappa_n = \frac{n\pi}{a}, \tag{2.3.47}$$

$$B_1 = 0, \qquad \mu_m = \frac{m\pi}{b}, \tag{2.3.48}$$

$$C_1 = 0, \qquad \lambda_r = \frac{r\pi}{c}, \tag{2.3.49}$$

$$\begin{aligned} D_1 = 0, \qquad \alpha_{nmr} &= \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{r\pi}{c}\right)^2} \\ &= \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{r^2}{c^2}} \ . \end{aligned} \tag{2.3.50}$$

So, that

$$X_n(x) = A_n \ \sin\frac{n\pi}{a}x, \qquad n = 1, 2, ..., \tag{2.3.51}$$

$$Y_m(y) = B_m \ \sin\frac{m\pi}{b}y, \qquad m = 1, 2, ..., \tag{2.3.52}$$

$$Z_r(z) = C_r \ \sin\frac{r\pi}{c}z, \qquad r = 1, 2, ..., \tag{2.3.53}$$

and

$$W_{nmr}(w) = D_{nmr} \ \sinh\alpha_{nmr}w. \tag{2.3.54}$$

Therefore, the product solutions satisfying (2.3.17):

$$u_{nmr}(x,y,z,w) = \gamma_{nmr}\ \sin\frac{n\pi}{a}x\ \sin\frac{m\pi}{b}y\ \sin\frac{r\pi}{c}z\ \sinh\Big(\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{r^2}{c^2}}\Big)w, \tag{2.3.55}$$

where $\gamma_{nmr} = A_n B_m C_r D_{nmr}$.

Now, we find the solution of boundary value problem (2.3.17) though linear combination of $u_{nmr}(x,y,z,w)$.

Let $u(x,y,z,w) = \sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nmr}(x,y,z,w)$. i.e.,

$$u = \sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\gamma_{nmr}\ \sin\frac{n\pi}{a}x\ \sin\frac{m\pi}{b}y\ \sin\frac{r\pi}{c}z\ \sinh\Big(\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{r^2}{c^2}}\Big)w \tag{2.3.56}$$

Using the nonhomogeneous boundary condition $u(x,y,z,d) = f(x,y,z)$ to find

$$\sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\gamma_{nmr}\ \sin\frac{n\pi}{a}x\ \sin\frac{m\pi}{b}y\ \sin\frac{r\pi}{c}z\ \sinh\alpha_{nmr}d = f(x,y,z). \tag{2.3.57}$$

We have $\gamma_{nmr}\ \sin\frac{n\pi}{a}x\ \sin\frac{m\pi}{b}y\ \sin\frac{r\pi}{c}z\ \sinh\alpha_{nmr}w$ are "orthogonal" over the rectangle $0 \le x \le a, \quad 0 \le y \le b, \quad 0 \le z \le c, \quad 0 \le w \le d.$

$$\int_0^c\int_0^b\int_0^a \sin\frac{n\pi}{a}x\ \sin\frac{m\pi}{b}y\ \sin\frac{r\pi}{c}z\ \sin\frac{\acute{n}\pi}{a}x\ \sin\frac{\acute{m}\pi}{b}y\ \sin\frac{\acute{r}\pi}{c}z\ dxdydz = 0, \tag{2.3.58}$$

if $(n,m,r) \neq (\acute{n},\acute{m},\acute{r})$.

Also, if $(n,m,r) = (\acute{n},\acute{m},\acute{r})$, then we obtain

$$\int_0^c\int_0^b\int_0^a \sin^2\frac{n\pi}{a}x\ \sin^2\frac{m\pi}{b}y\ \sin^2\frac{r\pi}{c}z\ dxdydz = \frac{abc}{8}. \tag{2.3.59}$$

Using the orthogonality properties, we obtain

$$\gamma_{nmr}\ \sinh \alpha_{nmr} d = \frac{8}{abc} \int_0^c \int_0^b \int_0^a f(x,y,z)\ \sin\frac{n\pi}{a}x\ \sin\frac{m\pi}{b}y\ \sin\frac{r\pi}{c}z\ dxdydz. \tag{2.3.60}$$

Thus, the solution of the four dimensional Dirichlet problem for Laplace equation is

$$u(x,y,z,w) = \sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \gamma_{nmr}\ \sin\frac{n\pi}{a}x\ \sin\frac{m\pi}{b}y\ \sin\frac{r\pi}{c}z\ \sinh \alpha_{nmr} w, \tag{2.3.61}$$

where

$$\alpha_{nmr} = \pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2 + \left(\frac{r}{c}\right)^2}, \tag{2.3.62}$$

and

$$\gamma_{nmr} = \frac{8}{abc\ \sinh(\alpha_{nmr})d} \int_0^c \int_0^b \int_0^a \sin\frac{n\pi}{a}x\ \sin\frac{m\pi}{b}y\ \sin\frac{r\pi}{c}z\ dxdydz. \tag{2.3.63}$$

2.3.2 The Neumann Problem for the Laplace Equation

Let Ω be an open set in $\mathbb{R}^n$ with the boundary $\partial\Omega$. The Neumann boundary for Laplace equation which consists of searching for a continuous function u defined on Ω, satisfying

$$\Delta u = 0 \quad in \quad \Omega, \tag{2.3.64}$$

$$\frac{\partial u}{\partial \nu} = g \quad on \quad \partial\Omega, \tag{2.3.65}$$

where $\frac{\partial u}{\partial \nu} = \nabla u.\nu$ is the normal derivative of the field variable u, where ν is the unit outward-pointing normal to Ω [75].

We can establish a necessary condition for the existence of the solution of the Neumann problem. Setting $\sigma = grad\ f$ in Gauss' theorem

$$\int_S \sigma_\nu dS = \int_\Omega div \sigma d\tau.$$

Hence

$$\int_S \frac{\partial f}{\partial \nu} dS = \int_\Omega \Delta f\ d\tau.$$

We have $\frac{\partial f}{\partial \nu} = g$. Thus

$$\int_S g dS = \int_\Omega \Delta f\ d\tau.$$

Therefore, if $\Delta f = 0$, we have

$$\int_S g\ dS = 0,$$

where $S = \partial\Omega$ is the boundary of Ω.

For simplification, we discuss the Neumann problem in two dimensional x and y [56].

$$u_{xx}(x, y) + u_{yy}(x, y) = 0, \qquad 0 < x < a, \quad 0 < y < b. \tag{2.3.66}$$

$$u_y(x, 0) = f_1(x), \quad u_y(x, b) = f_2(x), \qquad 0 < x < a, \tag{2.3.67}$$

$$u_x(0, y) = f_3(y), \quad u_x(a, y) = f_4(y), \qquad 0 < y < b. \tag{2.3.68}$$

The compatibility condition that must be fulfilled in this case is

$$\int_0^a \Big(f_1(x) - f_2(x)\Big) dx + \int_0^b \Big(f_3(y) - f_4(y)\Big) dy = 0. \tag{2.3.69}$$

Setting a solution in the form

$$u(x, y) = u_1(x, y) + u_2(x, y), \tag{2.3.70}$$

where $u_1(x,y)$ is a solution of the following problem

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial y} = 0, \qquad 0 < x < a, 0 < y < b, \tag{2.3.71}$$

$$\frac{\partial u_1}{\partial y}(x,0) = 0, \qquad \frac{\partial u_1}{\partial y}(x,b) = 0, \qquad 0 < x < a, \tag{2.3.72}$$

$$\frac{\partial u_1}{\partial x}(0,y) = f_3(y), \qquad \frac{\partial u_1}{\partial x}(a,y) = f_4(y), \qquad 0 < y < b, \tag{2.3.73}$$

We look for product solutions

$$u_1(x,y) = X(x)Y(y) \tag{2.3.74}$$

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\mu \tag{2.3.75}$$

Hence

$$X'' - \mu X = 0, \tag{2.3.76}$$

and

$$Y'' + \mu Y = 0, \tag{2.3.77}$$

where μ is the separation constant. Solving (2.3.77) with the boundary conditions

$$Y'(0) = 0, \qquad Y'(b) = 0. \tag{2.3.78}$$

Thus, we have

$$\mu_0 = 0, \qquad \mu_n = \left(\frac{n\pi}{b}\right)^2, \qquad n = 1, 2, ..., \tag{2.3.79}$$

and hence

$$Y_0(y) = 1, \qquad Y_n(y) = \cos\frac{n\pi}{b}y, \qquad n = 1, 2, ... \tag{2.3.80}$$

Solving (2.3.76) with the boundary conditions, we get

$$\mu_0 = 0, \qquad X_0(x) = A_0 + B_0\ x, \tag{2.3.81}$$

$$\mu_n = \left(\frac{n\pi}{b}\right)^2, \qquad X_n(x) = A_n \cosh\frac{n\pi}{b}x + B_n \sinh\frac{n\pi}{b}x, \qquad n = 1, 2, \dots\ . \tag{2.3.82}$$

We get the general form of the solution

$$u_1(x, y) = A_0 + B_0\ x + \sum_{n=1}^{\infty} \cos\frac{n\pi}{b}y\Big(A_n \cosh\frac{n\pi}{b}x + B_n \sinh\frac{n\pi}{b}x\Big). \tag{2.3.83}$$

Determining the constants by boundary conditions, we obtain

$$f_3(y) = B_0 + \sum_{n=1}^{\infty} \frac{n\pi}{b}\ B_n\ \cos\frac{n\pi}{b}y, \tag{2.3.84}$$

hence

$$B_0 = \frac{1}{b}\int_0^b f_3(y)\ dy, \tag{2.3.85}$$

and

$$B_n = \frac{2}{n\pi}\int_0^b f_3(y) \cos\frac{n\pi}{b}y\ dy, \qquad n = 1, 2, \dots\ . \tag{2.3.86}$$

Also, we have

$$f_4(y) = B_0 + \sum_{n=1}^{\infty} \frac{n\pi}{b}\ \cos\frac{n\pi}{b}y\Big(A_n \sinh\frac{n\pi a}{b} + B_n \cosh\frac{n\pi a}{b}\Big). \tag{2.3.87}$$

Thus

$$B_0 = \frac{1}{b}\int_0^b f_4(y)\ dy, \tag{2.3.88}$$

and

$$A_n = \frac{2}{n\pi \sinh\frac{n\pi a}{b}}\int_0^b f_4(y)\ \cos\frac{n\pi}{b}y\ dy - \frac{n\pi}{b}B_n\ \cosh\frac{n\pi a}{b}, \qquad n = 1, 2, \dots\ . \tag{2.3.89}$$

Now, we can find the solution $u_2(x, y)$ of the following problem

$$\frac{\partial u_2}{\partial x} + \frac{\partial u_2}{\partial y} = 0, \qquad 0 < x < a, 0 < y < b, \tag{2.3.90}$$

$$\frac{\partial u_2}{\partial y}(x, 0) = f_1(x), \quad \frac{\partial u_2}{\partial y}(x, b) = f_2(x), \qquad 0 < x < a, \tag{2.3.91}$$

$$\frac{\partial u_2}{\partial x}(0, y) = 0, \qquad \frac{\partial u_2}{\partial x}(a, y) = 0, \qquad 0 < y < b, \tag{2.3.92}$$

where f_1 and f_2 satisfy the compatibility condition [30, 56],

$$\int_0^a f_1(x)dx = \int_0^a f_2(x)dx. \tag{2.3.93}$$

Remarks 2.3.2.1.[30]

(i) There are infinity many solutions of Neumann problem for Laplace equation, so this Neumann problem is not well-posed.

(ii) The functions f_3 and f_4 are satisfying the compatibility condition

$$\int_0^b f_3(y)dy = \int_0^b f_4(y)dy.$$

Theorem 2.3.2.2. Let u be a harmonic function in Ω and $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$, then u is a constant in $\bar{\Omega}$ [65].

Theorem 2.3.2.3. If the Neumann problem for Laplace equation in a bounded open set $\Omega \subset \mathbb{R}^n$ has a solution, then, it is either unique or it differs from one another by a constant only [65] .

Proof. Let u_1 and u_2 be two solutions of the Neumann problem. Then we have boundary condition in two coordinates given by

$$\Delta u_1 = 0, \quad in \quad \Omega,$$

$$\frac{\partial u_1}{\partial \nu} = g \qquad on \;\; \partial\Omega.$$

and

$$\Delta u_2 = 0, \quad in \qquad \Omega,$$

$$\frac{\partial u_2}{\partial \nu} = g \qquad on \;\; \partial\Omega.$$

Let $v = u_1 - u_2$. Then

$$\Delta v = \Delta u_1 - \Delta u_2 = 0 \qquad in \;\; \Omega,$$

$$\frac{\partial v}{\partial \nu} = \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} = 0 \qquad on \;\; \partial\Omega.$$

Using theorem (2.3.2.2), u is a constant on $\bar{\Omega}$.

Consequently, the solution of the Neumann problem for Laplace equation is not unique. Hence, the solution of a definite Neumann problem can differ from one another by a constant only. □

2.3.3 The Robin Problem for the Laplace Equation

The Robin problem for the Laplace equation given as

$$\Delta u = 0, \qquad in \;\; \Omega, \tag{2.3.94}$$

$$\frac{\partial u}{\partial \nu} + \rho u = g \qquad on \;\; \partial\Omega, \tag{2.3.95}$$

where ρ is a continuous function on $\partial\Omega$. Dirichlet and Neumann problems are special cases of Robin problem [27].

Consider the Robin problem for Laplace equation in two dimensional x and y [57, 74].

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a, \quad 0 < y < b, \tag{2.3.96}$$

$$u_x(0, y) = 0, \qquad u_x(a, y) = 0, \qquad 0 \le y \le b, \tag{2.3.97}$$

$$u(x, 0) = f(x), \qquad u(x, b) = 0, \qquad 0 \le x \le a. \tag{2.3.98}$$

Separating variables, we will look for the product solutions

$$u(x, y) = X(x)Y(y) \tag{2.3.99}$$

Substituting into Eq. (2.3.96), we get

$$X''(x)Y(y) + X(x)Y''(y) = 0, \tag{2.3.100}$$

which separates into

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\kappa, \tag{2.3.101}$$

where κ is the separation constant. Hence, we get two ordinary differential equations

$$X'' + \kappa X = 0, \tag{2.3.102}$$

$$Y'' - \kappa Y = 0. \tag{2.3.103}$$

Using the boundary condition (2.3.97), we observe that

$$X'(0) = X'(a) = 0. \tag{2.3.104}$$

Hence, we obtain

$$\kappa_n = \left(\frac{n\pi}{a}\right)^2, \quad n = 0, 1, ..., \tag{2.3.105}$$

corresponding solutions

$$X_n(x) = A_n \cos\Big(\frac{n\pi}{a}\Big)x, \tag{2.3.106}$$

where A_n's are arbitrary constants.

putting $\kappa = \kappa_n$ in Eq. (2.3.103), and solving for Y yields

$$Y_0 = B_0 + C_0\ y, \tag{2.3.107}$$

$$Y_n = B_n \cosh\Big(\frac{n\pi}{a}\Big)y + C_n \sinh\Big(\frac{n\pi}{a}\Big)y, \quad n = 1, 2, \ldots\ . \tag{2.3.108}$$

The boundary condition $u(x, b) = 0$ will be satisfied if $Y(b) = 0$. putting $y = b$ in (2.3.107), gives

$$B_0 = -bC_0, \qquad B_n = -C_n \tanh\Big(\frac{n\pi b}{a}\Big)y, \tag{2.3.109}$$

or equivalently,

$$Y_n(y) = D_n \sinh\Big[\frac{n\pi}{a}(y - b)\Big], \quad n = 1, 2, \ldots\ . \tag{2.3.110}$$

Thus, we obtain

$$u_0(x, y) = A_0C_0(y - b) = E_0(y - b), \tag{2.3.111}$$

$$u_n(x, y) = E_n \cos\Big(\frac{n\pi}{a}\Big)x\ \sinh\Big[\frac{n\pi}{a}(y - b)\Big], \qquad n = 1, 2, \ldots, \tag{2.3.112}$$

where E_n's are constants. Then, we get the general form of the solution

$$u(x, y) = E_0(y - b) + \sum_{n=1}^{\infty} E_n \cos\Big(\frac{n\pi}{a}\Big)x\ \sinh\Big[\frac{n\pi}{a}(y - b)\Big]. \tag{2.3.113}$$

Applying the boundary condition $u(x, 0) = f(x)$, we get

$$f(x) = -E_0 b + \sum_{n=1}^{\infty} E_n \cos\Big(\frac{n\pi}{a}\Big)x\ \sinh\Big(\frac{-n\pi b}{a}\Big). \tag{2.3.114}$$

Determining the constants by boundary conditions, we get

$$E_0 = \frac{-1}{bA} \int_0^a f(x)dx, \tag{2.3.115}$$

$$E_n = \frac{2}{A \ \sinh\left(\frac{-n\pi b}{a}\right)} \int_0^a f(x) \cos\left(\frac{-n\pi}{a} x\right) \, dx, \quad n = 1, 2, \dots \, . \tag{2.3.116}$$

In this section, we used the separation of variables method to solve Laplace equation with boundary condition. But, the method of separation of variables results in a solution that is given in terms of a Fourier series which naturally incorporates the boundary conditions. This is a result of the fact that the boundary problems were used in structuring the series. This approach gives a compact way of expressing the solution, however, since an entirely new series must be constructed every time the boundary problems change, it is not an effective way to study boundary value problems. Also, the domain must be nice enough to find a coordinate system such that the partial differential equation is separable and such that the boundary conditions can be used in the solution. Therefore, for more complex domains it is difficult to use this method.

2.4 Poisson Equation

Poisson equation is an elliptic partial differential equation with wide applications in many fields, such as fluid dynamics, mechanical engineering, magnetism, electrostatics. In 1813, French mathematician Siméon-Denis Poisson (1781-1840) pointed out Laplace error and showed that the partial differential equation must be nonhomogeneous. The Laplace equation is the special case of the Poisson equation when $f = 0$.

The Poisson equation has the form

$$\Delta u = f, \tag{2.4.1}$$

where Δ is the Laplacian operator and $x \in \Omega, \quad \Omega \subset \mathbb{R}^n$ is a given open set and the unknown is $u : \bar{\Omega} \to \mathbb{R}$ and the function f is known as a source function.

2.4.1 Fundamental Solution of Poisson Equation

By construction the function $x \to \Psi(x)$ is harmonic for $x \neq 0$. Shifting the origin to a new point y, the Eq.(2.2.1) is unchanged and so $x \to \Psi(x - y)$ is harmonic as a function of x where $x \neq y$. The mapping $x \to \Psi(x-y)f(y)$ is harmonic for each point $y \in \mathbb{R}^n$, and so is the sum of finitely many such expression constructed for different points y. Consider the convolution

$$u(x) = \int_{\mathbb{R}^n} \Psi(x - y)f(y)dy. \tag{2.4.2}$$

From equations (2.2.3) and (2.4.2), we have

$$u(x) = \begin{cases} \frac{-1}{2\pi} \int_{\mathbb{R}^n} log(|x - y|)f(y)dy & for\ n = 2, \\ \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy & for\ n \geq 3. \end{cases} . \tag{2.4.3}$$

For simplicity, we assume that the function f used in Poisson equation is twice continuously differentiable [66].

Definition 2.4.1.1.[62] Let Ω be a domain in $\mathbb{R}^n$ and f be a continuous function in Ω. The function

$$\tau_f(x) = \int_\Omega \Psi(x-y)f(y)dy, \tag{2.4.4}$$

is called the Newtonian potential of function f in Ω, where Ψ is the fundamental solution of the Laplace operator as in (2.2.3).

Lemma 2.4.1.2. Let Ω be a bounded domain, f be a bounded function in Ω and τ_f be defined as (2.4.4). If f is smooth in Ω, then τ_f is smooth in Ω. If the function $f \in C^{m-1}(\Omega)$ for some integer $m \geq 2$, then $\tau_f \in C^m(\Omega)$ and $\tau_f = f$ in Ω.

2.5 Boundary Value Problems for the Poisson Equation

The fundamental boundary value problems for the Poisson equation are the Dirichlet problem which consist in finding a solution of u on the domain Ω such that on the boundary of Ω is equal to some given functions and the Neumann problem specify not the function itself on the boundary of Ω, but its normal derivative. A linear combination of these two boundary value problems is the Robin problem.

2.5.1 Dirichlet Problem for Poisson Equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, we denote by $\partial\Omega$ its boundary by $\bar{\Omega}$ and $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ be given functions. Finding $u \in C^2(\Omega) \cap C(\bar{\Omega})$ which satisfying the

Poisson equation with Dirichlet problem

$$\Delta u = f \quad in \quad \Omega,$$

$$u = g \quad on \quad \partial\Omega. \tag{2.5.1}$$

The Dirihlet problem for Poisson equation can be solved by the method of the separation of variables [5]. We treat Poisson equation with zero boundary problem and take the solution of the Dirichlet problem for Laplace equation, which is a special case of problem when $f = 0$.

Consider the function

$$\psi_{nm}(x,y) = sin(\frac{n\pi}{a})x\ sin(\frac{m\pi}{b})y.$$

We have

$$\Delta\psi_{nm}(x,y) = -\lambda_{nm}\ sin(\frac{n\pi}{a})x\ sin(\frac{m\pi}{b})y,$$

where $\lambda_{nm} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2},\ and\ \ n,m = 1,2,3,...$. The constant λ_{nm} is called an eigenvalue of the Laplacian and the function $\psi_{nm}(x,y)$ the corresponding eigenfunction. The Laplacian of $\psi_{nm}(x,y)$ is a constant multiple of $\psi_{nm}(x,y)$.

Consider the solution

$$u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{nm}\ sin(\frac{n\pi}{a})x\ sin(\frac{m\pi}{b})y, \tag{2.5.2}$$

where T_{nm} are constants to be determined. Assume that can interchange the sums and the derivatives.

Differentiating twice in (2.5.2) and plugging into (2.5.1), we obtain

$$f(x,y) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{nm}\ \lambda_{nm}\ sin(\frac{n\pi}{a})x\ sin(\frac{m\pi}{b})y. \tag{2.5.3}$$

Concluding that

$$T_{nm} = \frac{-4}{ab\lambda_{nm}}\int_0^b\int_0^a f(x,y)\ sin(\frac{n\pi}{a})x\ sin(\frac{m\pi}{b})y. \tag{2.5.4}$$

The general solution of Dirichlet problem for Poisson equation is

$$\begin{aligned} u(x,y) = &\sum_{n=1}^{\infty} A_n\ sin\frac{n\pi}{a}x\ sinh\frac{n\pi}{a}(b-y) + \sum_{n=1}^{\infty} B_n\ sinh\frac{nb\pi}{a}\ sin\frac{n\pi}{a}x \\ &+\sum_{n=1}^{\infty} C_n\ sin\frac{n\pi}{b}y\ sinh\frac{n\pi}{b}(a-x) + \sum_{n=1}^{\infty} D_n\ sin\frac{n\pi}{b}y\ sinh\frac{n\pi}{b}x \\ &+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{nm}\ sin(\frac{n\pi}{a})x\ sin(\frac{m\pi}{b})y, \end{aligned} \tag{2.5.5}$$

where $A_n, B_n, C_n,\ D_n$ and T_{nm} are coefficients and given by (2.3.7), (2.3.8), (2.3.9), (2.3.10), (2.5.4) [20].

Corollary 2.5.1.2.[37] Let Ω be a bounded domain in $\mathbb{R}^n$. Then for any function $f \in C(\Omega)$ and $g \in C(\partial\Omega)$, there exists at most one solution $u \in C(\Omega) \cap C(\bar{\Omega})$ of the problem

$$\Delta u = f \quad in \quad \Omega, \tag{2.5.6}$$

$$u = g \quad on \quad \partial\Omega. \tag{2.5.7}$$

2.5.2 Neumann and Robin Problems for the Poisson Equation

Other problems besides Dirichlet problem, which achieve the Poisson equation in rectangular coordinates are Neumann and Robin value problems which have great importance and wide applications in the sciences of mathematics, physics and others. We will briefly discuss these boundary problems as follows

Definition 2.5.2.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $f \in C(\Omega)$ and $g \in \partial\Omega$ be given functions. Seek $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that

$$\Delta u = f \quad in \quad \Omega, \tag{2.5.8}$$

$$\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu = g \quad on \quad \partial\Omega, \tag{2.5.9}$$

where $\frac{\partial u}{\partial \nu}$ is the normal derivative.

Definition 2.5.2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ be given functions. The Robin problem for Poisson equation is given by

$$\Delta u = f \quad in \quad \Omega, \tag{2.5.10}$$

$$\frac{\partial u}{\partial \nu} + \alpha u = g \quad on \quad \partial\Omega, \tag{2.5.11}$$

where α is a continuous function on $\partial\Omega$ and $\frac{\partial u}{\partial \nu}$ is the normal derivative.

Note that by setting α to zero, the Robin problem is reduced to Neumann problem. For the case of Dirichlet problem or mixed problem, the solution to Poisson equation always exists and is unique when it is smooth, compact, and with a piecewise smooth boundary. For the case of the Neumann problem, a solution may or may not exist (depending on whether a certain condition is satisfied).

Finally, the Laplace equation in its homogeneous and heterogeneous form (Poisson) is one of the most important equations that had applications in various fields, which attracted for researchers and those interested to develop and find many methods of solution that removed difficulties and facilitated understanding and solving complex problems in different branches of science.

CHAPTER 3

The Solution of Second-Order Linear Partial Differential Equations in Spherical Coordinates Using the Separation of Variables Method

3.1 Introduction

Lagrange and Euler both had written Laplace equation in spherical coordinates. It was then Legendre, while studying gravitational attraction, who solved the spherical version in the 1780s, with some help from Laplace. Poisson was accountable for proving that the gravitational potential must satisfy the nonhomogeneous Laplace equation, that is, Poisson equation in regions where mass is present. Also, it was while studying these problems that he provided his neat closed form solution to Laplace equation in polar coordinates. Moreover, around 1813, Poisson was the first to apply Laplace and Poisson equations to the study of electricity.

In this chapter, we deal with Laplace and Poisson equations in spherical coordinates with boundary conditions.

3.2 Laplace Equation in Spherical Coordinates

The coordinate systems which encounter most frequently are cartesian, spherical and cylindrical. We examined Laplace equation in cartesian coordinates in class and just began investigating its solution in spherical coordinates.

We define rectangular coordinate system for Laplace equation in three dimensions x, y and z by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{3.2.1}$$

3.2.1 The Dirichlet Problem for the Laplace Equation

We will change the rectangular coordinates system to spherical coordinates system and induced in the x, y, z space are the spherical coordinates r, θ, ϕ related to rectangular ones by

$$x = rsin\theta cos\phi, \quad y = rsin\theta sin\phi, \quad z = rcos\theta.$$

Hence, we have

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}}, \quad \theta = arctan\Big(\frac{\sqrt{x^2 + y^2}}{z}\Big) \quad and \quad \phi = arctan\Big(\frac{y}{x}\Big)$$

Under the spherical coordinate system radial distance r, polar angle θ and azimuthal angle ϕ.

The Laplace equation is written in spherical coordinates as [54],

$$r^2 \frac{\partial^2 u}{\partial r^2} + 2r \frac{\partial u}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(sin\theta \frac{\partial u}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0. \tag{3.2.2}$$

where $0 < r < a, \quad 0 < \theta < \pi, \quad and \quad 0 < \phi < 2\pi$, with boundary problems

$$u(a, \theta, \phi) = f(\theta, \phi), \qquad 0 < \theta < \pi, \quad 0 < \phi < 2\pi. \tag{3.2.3}$$

Also, we can written (3.2.2) as [39, 66]

$$\Delta u(r, \theta, \phi) = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\csc^2\theta}{r^2}\frac{\partial^2 u}{\partial\phi^2} = 0. \tag{3.2.4}$$

Using separation of variables method, we obtain the product solutions as

$$u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi). \tag{3.2.5}$$

Eq. (3.2.2) becomes

$$\left[\frac{r^2}{R}\ R'' + \frac{2r}{R}\ R' + \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\Big(\sin\theta\Theta'\Big)\right] = -\frac{1}{\Phi}\ \Phi'' = m^2. \tag{3.2.6}$$

Hence, we get

$$\Phi'' + m^2\Phi = 0, \quad m = 0, 1, 2, \dots\ . \tag{3.2.7}$$

The solutions of Eq. (3.2.7) are the $2\pi-$periodic, which we combined in complex form as

$$\Phi(\phi) = e^{im\phi}. \tag{3.2.8}$$

Also, we have

$$\frac{r^2}{R}\ R'' + \frac{2r}{R}\ R' = -\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\Big(\sin\theta\Theta'\Big) + \frac{m^2}{\sin^2\theta} = n(n+1). \tag{3.2.9}$$

Thus, we obtain

$$r^2R'' + 2rR' - n(n+1)R = 0, \quad 0 < r < a, \tag{3.2.10}$$

$$\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\Big(\sin\theta\Theta'\Big) + \left[n(n+1) - \frac{m^2}{sin^2\theta}\right] = 0. \tag{3.2.11}$$

The equation (3.2.10) is the Euler equation. Hence, It is being homogeneous.

Putting $r = e^z$, reduces to

$$(D-n)(D+(n+1))R = 0, \tag{3.2.12}$$

where $D \equiv \frac{d}{dz}$. Then, the roots are $D = n$ and $D = -(n+1)$.

Therefore, the solutions of Euler equation (3.2.10) are

$$R(r) = A\ r^n, \quad R^*(r) = B\ r^{-(1+n)}, \quad n = 0, 1, 2, \dots\ . \tag{3.2.13}$$

We choose only the solution $R(r) = A\ r^n$.

To solve (3.2.11), we make the change of variables $\gamma = \cos\theta$; $\frac{d\gamma}{d\theta} = -\sin\theta$. Hence, by the chain rule,

$$\Theta' = \frac{d\Theta}{d\theta} = \frac{d\Theta}{d\gamma}\frac{d\gamma}{d\theta} = -sin\theta\frac{d\Theta}{d\theta},$$

and

$$\Theta'' = sin^2\theta\ \frac{d^2\Theta}{d\gamma^2} - cos\theta\ \frac{d\Theta}{d\gamma} = (1-\gamma^2)\frac{d^2\Theta}{d\gamma^2} - \gamma\ \frac{d\Theta}{d\gamma}.$$

Plugging into (3.2.11) and simplifying, we arrive at the associated Legendre differential equation

$$(1-\gamma^2)\Theta'' - 2\gamma\Theta' + \Big[n(n+1) - \frac{m^2}{1-\gamma^2}\Big]\Theta = 0. \tag{3.2.14}$$

This equation has bounded solutions in the interval $[-1, 1]$. Then, the corresponding bounded solutions of Eq. (3.2.14) are denoted by $P_n^m(\gamma)$ and are called the associated Legendre functions, we substitute $\gamma = cos\theta$. Hence, the bounded solutions of (3.2.7) are given by

$$\Theta(\theta) = P_n^m(\cos\theta). \tag{3.2.15}$$

Therefore, we arrive at the product solutions of Eq. (3.2.4):

$$u(r,\theta,\phi) = r^n e^{im\phi} P_n^m(\cos\theta). \tag{3.2.16}$$

Remark 3.2.1.1. In the solutions of the Euler equation (3.2.13), we see that for problems inside the ball with $0 < r < a$, we choose the bounded solutions $R(r) = Ar^n$, and reject $R^*(r) = B\ r^{-(1+n)}$ and for problems outside the ball with $r > a$, we choose $R(r) = B\ r^{-(1+n)}$ and discard $R(r) = Ar^n$, which is unbounded as $r \to \infty$ [20].

3.2.2 The Spherical Harmonics

Spherical harmonics arise in a large variety of physical problems is solved by separation of variables in spherical coordinates. The spherical harmonic $Y_n^m(\theta, \phi),\ \ -n \leq m \leq n$, is a function of the two coordinates θ, ϕ on the surface of a sphere. The spherical harmonics are orthogonal for different n and m, and they are normalized so that their integrated square over the sphere is unity:

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d(cos\theta)\bar{Y}_{n'm'}(\theta, \phi)\ Y_{nm}(\theta, \phi) = \delta_{nn'}\ \delta_{mm'}, \tag{3.2.17}$$

where $\bar{Y}_{n',m'}(\theta, \phi)$ is a complex conjugation of $Y_{n'm'}$ [20, 28].

The spherical harmonics can be used to expand functions defined on the sphere, much as we used Fourier series and other orthogonal series expansions.

Definition 3.2.2.1. The spherical harmonics $Y_{n,m}(\theta, \phi)$ is define by

$$Y_{n,m}(\theta, \phi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}\ P_n^m(\cos\theta)e^{im\phi}, \tag{3.2.18}$$

where $n = 0, 1, ...,$ and $m = -n, -n+1, ..., n-1, n$, and $P_n^m(\cos\theta)$ is the associated Legendre function of the mth derivative of the Legendre polynomial of degree n.

The coefficient in (3.2.18) is chosen. Therefore, the spherical harmonics become an orthonormal set of functions on the surface of the sphere when the element of surface area $\sin\theta \ d\theta d\phi$.

Theorem 3.2.2.2.[20] Let $f(\theta, \phi)$ be a function defined for all $0 < \phi < 2\pi, \ 0 < \theta < \pi$, and suppose that f is 2π-periodic in ϕ. Then, we have the spherical harmonics series expansion

$$f(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{n,m} \ Y_{n,m}(\theta, \phi), \tag{3.2.19}$$

where the spherical harmonics coefficients are given by

$$A_{n,m} = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \ \bar{Y}_{n,m} \ \sin\theta \ d\theta d\phi. \tag{3.2.20}$$

Theorem 3.2.2.3.[20] Let $f(r, \theta, \phi)$ be a square integrable function. Then

$$f(r, \theta, \phi) = \sum_{\xi=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{\xi nm} \ \xi_n(\gamma_{n\xi} \ r) Y_{n,m}(\theta, \phi), \tag{3.2.21}$$

where ξ_n is the spherical Bessel function, which defined by

$$\xi_n(r) = \sqrt{\frac{\pi}{2r}} \xi_{\frac{2n+1}{2}}(r). \tag{3.2.22}$$

$$Here \ A_{\xi nm} = \frac{2a^{-3}}{\xi_{n+1}^2(\alpha_{n+\frac{1}{2},\xi})} \int_0^a \int_0^{2\pi} \int_0^{\pi} f(r, \theta, \phi) \ \xi_n(\gamma_{n,\xi} \ r) \bar{Y}_{n,m} r^2 \sin\theta \ d\theta d\phi dr. \tag{3.2.23}$$

3.2.3 The Dirichlet problem for the Laplace equation in a ball

Since Laplace equation is homogenous, we choose any scalar multiple of these equation. Using (3.2.18), we can replace $P_n^m(cos\theta) \ e^{im\phi}$ in (3.2.16) by spherical harmonics $Y_{n,m}(cos\theta, \phi)$ and take the product solutions

$$\frac{r^n}{a^n} \ Y_{n,m}(\theta, \phi). \tag{3.2.24}$$

Superposing scalar multiplies of these solutions, we obtain

$$u(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}A_{nm}\,\frac{r^n}{a^n}\,Y_{n,m}(\theta,\phi), \tag{3.2.25}$$

where A_{nm} is given by (3.2.20).

Theorem 3.2.3.1.[20] The solution of (3.2.2) subject to the boundary condition of (3.2.3) in a ball is given by (3.2.25).

The solution of Laplace equation in a ball has nice expression in terms of the Fourier coefficients of the boundary function.

3.3 Poisson Equation in Spherical Coordinates

One of the most commonly used three-dimensional coordinate systems is spherical coordinates. When we need to solve a partial differential equation, we try to propose a solution function comprised of the multiplication of single-variable functions. Then, the partial differential equation can be decomposed into a set of ordinary differential equations, which are more accessible to solve and whose solutions establish the solution of the original partial differential equation.

In this section, we will briefly discuss solving the Poisson equation in spherical coordinates.

3.3.1 Poisson Equation with Dirichlet Condition

In the previous chapter, we have solved Poisson equation in rectangular coordinates, and see the importance of this equation and its applications have been shown. Now the Poisson equation will be solved in spherical coordinates with boundary conditions. Consider Poisson equation with boundary problem in spherical coordinates

$$\Delta u(r, \theta, \phi) = f(r, \theta, \phi), \tag{3.3.1}$$

where $0 < r < a, \quad 0 < \theta < \pi, \quad 0 < \phi < 2\pi$, with boundary condition

$$u(a, \theta, \phi) = g(\theta, \phi). \tag{3.3.2}$$

The previous problem is the Dirichlet problem for Poisson equation [20, 48]. We reduce to Poisson problem with zero boundary condition and a Dirichlet problem with boundary condition given by (3.3.2). Hence, it remains to solve (3.3.1) with homogeneous boundary condition

$$u(a, \theta, \phi) = 0. \tag{3.3.3}$$

We will use the method of eigenfunction expansion to solve (3.3.1) and (3.3.3). Hence, we get

$$u(r, \theta, \phi) = \sum_{\xi=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{\xi nm} \; \xi_n(\gamma_{n,s} \; r) Y_{n,m}(\theta, \phi). \tag{3.3.4}$$

To determine $C_{\xi nm}$, we plug the triple series into(3.3.1), and assume that each term satisfies (3.2.4) with $\kappa = \gamma_{n,s}^2$, we get

$$\sum_{\xi=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \gamma_{n,\xi}^2 C_{\xi nm}\ \xi_n(\gamma_{n,\xi}\ r) Y_{n,m}(\theta,\phi) = f(r,\theta,\phi). \tag{3.3.5}$$

Thinking of this last equation as an eigenfunction expansion of f, we get

$$C_{snm} = \frac{-1}{\gamma_{n,\xi}^2} A_{\xi nm} \tag{3.3.6}$$

where $A_{\xi nm}$ is given by (3.2.23) [20].

Theorem 3.3.1.1.[20] The solution of Poisson problem (3.3.1)-(3.3.2) is

$$u(r,\theta,\phi) = \sum_{\xi=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{-1}{\gamma_{n,\xi}^2} A_{\xi nm}\ \xi_n(\gamma_{n,\xi}\ r) Y_{n,m}(\theta,\phi) + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}(\frac{r}{a})^n\ Y_{n,m}(\theta,\phi) \tag{3.3.7}$$

where the spherical harmonic coefficients A_{nm} is given by

$$A_{nm} = \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)\ \bar{Y}_{n,m}(\theta,\phi)\ \sin\theta\ d\theta d\phi.$$

Here the spherical harmonic series expansion $f(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}\ Y_{n,m}(\theta,\phi)$.

Finally, we can say that as most researches have been focused on the solution in rectangular coordinates with simple boundary conditions, very little have been published for the solution in spherical coordinates. However, solutions in spherical coordinates are still of attention; for example, they are the natural coordinates for many astrophysical collapse problems (e.g. formation of galaxies, proto-star formation, supernovae). In such collapse calculations one has to solve the equations of conservation together with Poisson equation.

CHAPTER 4

Solving Second-Order Linear Partial Differential Equations by Integral Transforms

4.1 Introduction

In previous years, numerous notice has been given to deal with the single, double and triple transform, which have many applications in various field of mathematical sciences and engineering such as acoustics, physics, chemistry, etc.,.

The method of integral transforms is one of the most effective and easy methods for solving problems which are defined by differential equations and integral equations. Also, the integral transforms are very efficient methodology to solve the linear and nonlinear differential and integral equations. Many problems of physical concern are described by ordinary or partial differential equations with proper initial or boundary conditions, these problems are usually formulated as initial value problems or boundary value problems. The solutions of initial and boundary value problems are given by numerous integral transforms methods. Solving such equations using single integral transforms is more difficult than using the double integral transforms. In this chapter, we introduce double Laplace-Aboodh transform, double Laplace-Shehu transform and triple Laplace-Aboodh-Sumudu transform with their properties and applications.

4.2 Double Laplace-Aboodh Transform

Aboodh transform was introduced by Khalid Aboodh to facilitate the process of solving ordinary and partial differential equations in the time domain. The Aboodh transform is a new integral transform similar to the Laplace transform and other integral transforms that are defined in the time domain. We applied new double Laplace-Aboodh transform to solve Laplace, Poisson, wave and heat equations. We present method of double Laplace-Aboodh transform for solving these equations with initial and boundary conditions.

Definition 4.2.1. The Laplace transform of the continuous function $f(x)$ is defined by

$$\mathcal{L}[f(x)] = \int_0^\infty e^{-\rho x} f(x)dx = F(\rho), \tag{4.2.1}$$

where $\mathcal{L}$ is the Laplace operator [6].

Provided that the integral exists. If the integral is convergent for some value of ρ, then the Laplace transformation of $f(x)$ exists, otherwise not.

The inverse Laplace transform is defined by

$$\mathcal{L}^{-1}[F(\rho)] = f(x) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\rho x} F(\rho) d\rho, \tag{4.2.2}$$

where α is a real constant [31].

Definition 4.2.2. Let $f(y)$ be an exponential order function in the set

$$\mathcal{H} = \Big\{ f(y) : \exists \mathcal{B}, \alpha_1, \alpha_2 > 0, |f(y)| < \mathcal{B} e^{|y|\alpha_i}, \quad for \ \ y \in (-1)^i \times [0, \infty), \quad i = 1, 2 \Big\}.$$

where $\mathcal{B}$ is a finite number and α_1, α_2 may finite or infinite. Then the Aboodh transform of the function $f(y)$ is given by

$$\mathcal{A}[f(y)] = F(\lambda) = \frac{1}{\lambda} \int_0^\infty e^{-\lambda y} f(y) dy, \qquad \alpha_1 < y < \alpha_2, \tag{4.2.3}$$

where $\mathcal{A}$ is called the Aboodh transform operator.

The inverse Aboodh transform is given by

$$\mathcal{A}^{-1}[F(\lambda)] = f(y) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \lambda e^{\lambda y} F(\lambda) d\lambda; \beta \geq 0. \tag{4.2.4}$$

where β is a real constant[2, 9].

In the next definition, we introduce the double Laplace-Aboodh transform.

Definition 4.2.3. The double Laplace-Aboodh transform of the function f of two variables $x, y > 0$ is denoted by $\quad \mathcal{L}_x \mathcal{A}_y[f(x, y)] = F(\rho, \lambda) \quad$ and defined by

$$\mathcal{L}_x \mathcal{A}_y[f(x, y)] = F(\rho, \lambda) = \frac{1}{\lambda} \int_0^\infty \int_0^\infty e^{-(\rho x + \lambda y)} f(x, y) dx dy, \tag{4.2.5}$$

provided the integral exists.

Definition 4.2.4. The inverse double Laplace-Aboodh transform of the function $f(x, y)$ is defined by

$$f(x, y) = \mathcal{L}_x^{-1} \mathcal{A}_y^{-1}[F(\rho, \lambda)] = \frac{1}{(2\pi i)^2} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\rho x} \Big(\int_{\beta - i\infty}^{\beta + i\infty} \lambda e^{\lambda y} F(\rho, \lambda) d\lambda \Big) d\rho, \tag{4.2.6}$$

where α and β are real constants.

4.2.1 Existence and Uniqueness of The Double Laplace-Aboodh Transform

Definition 4.2.1.1. A function $f(x,y)$ is said to be of exponential orders $\alpha, \beta > 0$, on $0 \leq x, y < \infty$, if there exists positive constants K, X and Y such that

$$|f(x,y)| \leq Ke^{(\alpha x+\beta y)}, \qquad for\ all \quad x > X, \quad y > Y,$$

and we write

$$f(x,y) = o(e^{\alpha x+\beta y}) \qquad as \quad x, y \to \infty.$$

Or, equivalently,

$$\lim_{x\to\infty, y\to\infty} e^{-(\rho x+\lambda y)}|f(x,y)| \leq K \lim_{x\to\infty, y\to\infty} e^{-(\rho-\alpha)x}e^{-(\lambda-\beta)y} = 0, \qquad \rho > \alpha, \ \ \lambda > \beta.$$

Theorem 4.2.1.2. Let $f(x,y)$ be a continuous function in every finite intervals $(0,X)$ and $(0,Y)$ and of exponential order $e^{(\alpha x+\beta y)}$, then the double Laplace-Aboodh transform of $f(x,y)$ exists for all $\rho > \alpha$ and $\lambda > \beta$.

Proof. Let $f(x,y)$ be of exponential order $e^{(\alpha x+\beta y)}$, such that

$$\Big|f(x,y)\Big| \leq Ke^{(\alpha x+\beta y)}, \qquad for \qquad all \quad x > X, \quad y > Y.$$

Then, we have

$$
\begin{aligned}
\Big|F(\rho,\lambda)\Big| &= \Big|\frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}f(x,y)dxdy\Big| \\
&\le \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}|f(x,y)|dxdy \\
&\le \frac{K}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}e^{(\alpha x+\beta y)}dxdy \\
&= \frac{K}{\lambda}\int_0^\infty e^{-(\rho-\alpha)x}dx\int_0^\infty e^{-(\lambda-\beta)y}dy \\
&= \frac{K}{(\rho-\alpha)(\lambda^2-\beta\lambda)}.
\end{aligned}
$$

Thus, the proof is complete.

Theorem 4.2.1.3. Let $F_1(\rho,\lambda)$ and $F_2(\rho,\lambda)$ be the double Laplace-Aboodh transform of the continuous functions $f_1(x,y)$ and $f_2(x,y)$ defined for $x, y \ge 0$ respectively. If $F_1(\rho,\lambda) = F_2(\rho,\lambda)$, then $f_1(x,y) = f_2(x,y)$.

Proof. Assume that α and β are adequately large, since

$$
f(x,y) = \mathcal{L}_x^{-1}\mathcal{A}_y^{-1}[F(\rho,\lambda)] = \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\rho x}\Big(\int_{\beta-i\infty}^{\beta+i\infty}\lambda e^{\lambda y}F(\rho,\lambda)d\lambda\Big)d\rho,
$$

we deduce that

$$
\begin{aligned}
f_1(x,y) &= \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\rho x}\Big(\int_{\beta-i\infty}^{\beta+i\infty}\lambda e^{\lambda y}F_1(\rho,\lambda)d\lambda\Big)d\rho \\
&= \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\rho x}\Big(\int_{\beta-i\infty}^{\beta+i\infty}\lambda e^{\lambda y}F_2(\rho,\lambda)d\lambda\Big)d\rho \\
&= f_2(x,y).
\end{aligned}
$$

This proves the uniqueness of the double Laplace-Aboodh transform.

4.2.2 Some Useful Properties of Laplace-Aboodh Transform

Linearity property 4.2.2.1. If the double Laplace-Aboodh transform of functions $f_1(x,y)$ and $f_2(x,y)$ are $F_1(\rho,\lambda)$ and $F_2(\rho,\lambda)$ respectively, then double Laplace-Aboodh transform of $\alpha f_1(x,y)+\beta f_2(x,y)$ is given by $\alpha F_1(\rho,\lambda)+\beta F_2(\rho,\lambda)$, where α and β are arbitrary constants.

Proof.

$$\begin{aligned}
\mathcal{L}_x\mathcal{A}_y[\alpha f_1(x,y)+\beta f_2(x,y)] &= \int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}\Big(\alpha f_1(x,y)+\beta f_2(x,y)\Big)dxdy \\
&= \alpha\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}f_1(x,y)dxdy \\
&+ \beta\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}f_2(x,y)dxdy \\
&= \alpha F_1(\rho,\lambda)+\beta F_2(\rho,\lambda). \qquad (4.2.7)
\end{aligned}$$

Change of scale property 4.2.2.2. Let $f(x,y)$ be a function such that

$$\mathcal{L}_x\mathcal{A}_y[f(x,y)] = F(\rho,\lambda).$$

Then for α and β are positive constants, we have

$$\mathcal{L}_x\mathcal{A}_y[f(\alpha x,\beta y)] = \frac{1}{\alpha\beta^2}F\Big(\frac{\rho}{\alpha},\frac{\lambda}{\beta}\Big). \qquad (4.2.8)$$

Proof.

$$\mathcal{L}_x\mathcal{A}_y[f(\alpha x,\beta y)] = \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}f(\alpha x,\beta y)dxdy.$$

Let $u = \alpha x, \quad v = \beta y$, then

$$\begin{aligned}
\mathcal{L}_x\mathcal{A}_y[f(u,v)] &= \frac{1}{\alpha\beta\lambda}\int_0^\infty\int_0^\infty e^{-(\frac{\rho}{\alpha}u+\frac{\lambda}{\beta}v)}f(u,v)dudv \\
&= \frac{1}{\alpha\beta^2\frac{\lambda}{\beta}}\int_0^\infty\int_0^\infty e^{-(\frac{\rho}{\alpha}u+\frac{\lambda}{\beta}v)}f(u,v)dudv \\
&= \frac{1}{\alpha\beta^2}F\Big(\frac{\rho}{\alpha},\frac{\lambda}{\beta}\Big).
\end{aligned}$$

Shifting property 4.2.2.3. If $\mathcal{L}_x\mathcal{A}_y[f(x,y)] = F(\rho,\lambda)$, then for any pair of real constants $\alpha, \beta > 0$,

$$\mathcal{L}_x\mathcal{A}_y[e^{(\alpha x+\beta y)}f(x,y)] = \frac{\lambda-\beta}{\lambda}F(\rho-\alpha,\lambda-\beta). \tag{4.2.9}$$

Proof. Using the definition of double Laplace-Aboodh transform, we get

$$\begin{aligned}
\mathcal{L}_x\mathcal{A}_y[e^{(\alpha x+\beta y)}f(x,y)] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}e^{(\alpha x+\beta y)}f(x,y)dxdy \\
&= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-\big((\rho-\alpha)x+(\lambda-\beta)y\big)}f(x,y)dxdy \\
&= \frac{\lambda-\beta}{\lambda(\lambda-\beta)}\int_0^\infty\int_0^\infty e^{-\big((\rho-\alpha)x+(\lambda-\beta)y\big)}f(x,y)dxdy \\
&= \frac{\lambda-\beta}{\lambda}F(\rho-\alpha,\lambda-\beta).
\end{aligned}$$

Derivatives properties 4.2.2.4. If $\mathcal{L}_x\mathcal{A}_y[f(x,y)] = F(\rho,\lambda)$, then

$$(1). \qquad \mathcal{L}_x\mathcal{A}_y\Big[\frac{\partial f(x,y)}{\partial x}\Big] = \rho F(\rho,\lambda) - \mathcal{A}[f(0,y)]. \tag{4.2.10}$$

Proof.

$$\begin{aligned}
\mathcal{L}_x\mathcal{A}_y\Big[\frac{\partial f(x,y)}{\partial x}\Big] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}\frac{\partial f(x,y)}{\partial x}dxdy \\
&= \frac{1}{\lambda}\int_0^\infty e^{-\lambda y}dy\Big(\int_0^\infty e^{-\rho x}f_x(x,y)dx\Big).
\end{aligned}$$

Using integration by parts, let $u = e^{-\rho x}, \quad dv = f_x(x,y)dx$, then we obtain

$$\begin{aligned}\mathcal{L}_x\mathcal{A}_y\Big[\frac{\partial f(x,y)}{\partial x}\Big] &= \frac{1}{\lambda}\int_0^\infty e^{-\lambda y}dy\Big(-f(0,y)+\rho\int_0^\infty e^{-\rho x}f(x,y)dx\Big)\\ &= \rho F(\rho,\lambda)-\mathcal{A}[f(0,y)].\end{aligned}$$

$$(2). \qquad \mathcal{L}_x\mathcal{A}_y\Big[\frac{\partial f(x,y)}{\partial y}\Big]=\lambda F(\rho,\lambda)-\frac{1}{\lambda}\mathcal{L}[f(x,0)]. \qquad (4.2.11)$$

Proof.

$$\begin{aligned}\mathcal{L}_x\mathcal{A}_y\Big[\frac{\partial f(x,y)}{\partial y}\Big] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}\frac{\partial f(x,y)}{\partial y}dxdy\\ &= \frac{1}{\lambda}\int_0^\infty e^{-\rho x}dx\Big(\int_0^\infty e^{-\lambda y}f_y(x,y)dy\Big).\end{aligned}$$

Using integration by parts, let $u = e^{-\lambda y}, \quad dv = f_y(x,y)dy$, then we obtain

$$\begin{aligned}\mathcal{L}_x\mathcal{A}_y\Big[\frac{\partial f(x,y)}{\partial y}\Big] &= \frac{1}{\lambda}\int_0^\infty e^{-\rho x}dx\Big(-f(x,0)+\lambda\int_0^\infty e^{-\lambda y}f(x,y)dy\Big)\\ &= \lambda F(\rho,\lambda)-\frac{1}{\lambda}\mathcal{L}[f(x,0)].\end{aligned}$$

$$(3). \qquad \mathcal{L}_x\mathcal{A}_y\Big[\frac{\partial^2 f(x,y)}{\partial x^2}\Big]=\rho^2F(\rho,\lambda)-\rho\mathcal{A}[f(0,y)]-\mathcal{A}[f_x(0,y)]. \qquad (4.2.12)$$

Proof.

$$\begin{aligned}\mathcal{L}_x\mathcal{A}_y\Big[\frac{\partial^2 f(x,y)}{\partial x^2}\Big] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}\frac{\partial^2 f(x,y)}{\partial x^2}dxdy\\ &= \frac{1}{\lambda}\int_0^\infty e^{-\lambda y}dy\Big(\int_0^\infty e^{-\rho x}f_{xx}(x,y)dx\Big).\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}\mathcal{L}_x\mathcal{A}_y\Big[\frac{\partial^2 f(x,y)}{\partial x^2}\Big] &= \frac{1}{\lambda}\int_0^\infty e^{-\lambda y}dy\Big(-f_x(0,y)+\rho\Big\{-f(0,y)+\rho\int_0^\infty e^{-\rho x}f(x,y)dx\Big\}\Big)\\ &= \rho^2F(\rho,\lambda)-\rho\mathcal{A}[f(0,y)]-\mathcal{A}[f_x(0,y)].\end{aligned}$$

Similarly, we can prove that:

$$(4).\qquad \mathcal{L}_x\mathcal{A}_y\Big[\frac{\partial^2 f(x,y)}{\partial y^2}\Big]=\lambda^2 F(\rho,\lambda)-\mathcal{L}[f(x,0)]-\frac{1}{\lambda}\mathcal{L}[f_y(x,0)].$$

$$(5).\qquad \mathcal{L}_x\mathcal{A}_y\Big[\frac{\partial^2 f(x,y)}{\partial x\partial y}\Big]=\rho\lambda F(\rho,\lambda)-\frac{\rho}{\lambda}\mathcal{L}[f(x,0)]-\mathcal{A}[f_y(0,y)].$$

4.2.3 The Double Laplace-Aboodh Transform of Some Elementary Functions

(1). If the function $f(x,y)=1$, then

$$\mathcal{L}_x\mathcal{A}_y[f(x,y)]=\frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}dxdy=\frac{1}{\rho\lambda^2}. \tag{4.2.13}$$

(2). If the function $f(x,y)=xy$, then

$$\mathcal{L}_x\mathcal{A}_y[f(x,y)]=\frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}xy\ dxdy=\frac{1}{\rho^2\lambda^3}. \tag{4.2.14}$$

(3). If the function $f(x,y)=x^2y^2$, then

$$\mathcal{L}_x\mathcal{A}_y[f(x,y)]=\frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}x^2y^2\ dxdy=\frac{4}{\rho^3\lambda^4}. \tag{4.2.15}$$

(4). If the function $f(x,y)=x^ny^m,\ \ n,m=0,1,2,\ \ \ldots\ \ ,$ then

$$\mathcal{L}_x\mathcal{A}_y[f(x,y)]\ =\ \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}x^ny^m\ dxdy=\frac{n!m!}{\rho^{n+1}\lambda^{m+2}}. \tag{4.2.16}$$

(5). If the function $f(x,y)=x^\sigma y^\nu,\ \ \sigma\geq-1,\ \ \nu\geq-1$, then

$$\mathcal{L}_x\mathcal{A}_y[f(x,y)]=\frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}x^\sigma y^\nu\ dxdy\ \ =\ \int_0^\infty e^{-\rho x}x^\sigma dx\int_0^\infty\frac{1}{\lambda}e^{-\lambda y}y^\nu dy.$$

Let $\zeta = \rho x \quad$ and $\quad \eta = \lambda y$

$$\begin{aligned}
\mathcal{L}_x\mathcal{A}_y[f(x,y)] &= \frac{1}{\rho^{\sigma+1}}\int_0^\infty e^{-\zeta}\zeta^\sigma d\zeta\Big(\frac{1}{\lambda^{\nu+2}}\int_0^\infty e^{-\eta}\eta^\nu d\eta\Big)\\
&= \Gamma(\sigma+1)\Big(\frac{1}{\rho^{\sigma+1}}\Big)\Gamma(\nu+1)\frac{1}{\lambda^{\nu+2}}. \qquad (4.2.17)
\end{aligned}$$

Where $\Gamma(.)$ is the Euler gamma function.

(6). If the function $f(x,y) = e^{(\alpha x+\beta y)}, \quad \alpha,\beta = 0,1,2,...,$ then

$$\mathcal{L}_x\mathcal{A}_y[f(x,y)] = \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}e^{(\alpha x+\beta y)}dxd\lambda = \frac{1}{(\rho-\alpha)(\lambda^2-\beta\lambda)}. \qquad (4.2.18)$$

Similarly,

$$\begin{aligned}
\mathcal{L}_x\mathcal{A}_y[e^{i(\alpha y+\beta y)}] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}e^{i(\alpha x+\beta y)}dxdy = \frac{1}{\lambda(\rho-i\alpha)}\frac{1}{(\lambda-i\beta)}\\
&= \frac{(\rho\lambda-\alpha\beta)+i(\rho\beta+\alpha\lambda)}{(\rho^2+\alpha^2)(\lambda^3+\beta^2\lambda)}. \qquad (4.2.19)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathcal{L}_x\mathcal{A}_y[\cos(\alpha x+\beta y)] &= \frac{\rho\lambda-\alpha\beta}{(\rho^2+\alpha^2)(\lambda^3+\beta^2\lambda)},\\
\mathcal{L}_x\mathcal{A}_y[\sin(\alpha x+\beta y)] &= \frac{\rho\beta+\alpha\lambda}{(\rho^2+\alpha^2)(\lambda^3+\beta^2\lambda)}.
\end{aligned}$$

(7). If the function $f(x,y) = \cosh(\alpha x+\beta y), \quad \alpha,\beta = 0,1,2,...,$ then

$$\begin{aligned}
\mathcal{L}_x\mathcal{A}_y[f(x,y)] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}\cosh(\alpha x+\beta y)dxdy\\
&= \frac{\rho\lambda+\alpha\beta}{(\rho^2-\alpha^2)(\lambda^3-\beta^2\lambda)}. \qquad (4.2.20)
\end{aligned}$$

(8). If the function $f(x,y) = \sinh(\alpha x+\beta y), \quad \alpha,\beta = 0,1,2, \ \ ... \ ,$ then

$$\begin{aligned}
\mathcal{L}_x\mathcal{A}_y[f(x,y)] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho x+\lambda y)}\sinh(\alpha x+\beta y)dxdy\\
&= \frac{\rho\beta+\alpha\lambda}{(\rho^2-\alpha^2)(\lambda^3-\beta^2\lambda)}. \qquad (4.2.21)
\end{aligned}$$

4.2.4 Applications of the Double Laplace-Aboodh Transform

In this section, to establish the efficiency of the suggestion method we consider second-order linear partial differential equations with initial and boundary problems. Let the second-order nonhomogeneous linear partial differential equation in two independent variables (x, y) be in the form:

$$Af_{xx}(x,y) + Bf_{yy}(x,y) + Cf_x(x,y) + Df_y(x,y) + Ef(x,y) = h(x,y), (x,y) \in \mathbb{R}^2_+, \tag{4.2.22}$$

with the initial conditions:

$$f(x,0) = T_1(x), \qquad f_y(x,0) = T_2(x), \tag{4.2.23}$$

and the boundary conditions:

$$f(0,y) = T_3(y), \qquad f_x(0,y) = T_4(y), \tag{4.2.24}$$

where A, B, C, D and E are constants and $h(x,y)$ is the source term.

Using the property of partial derivative of the double Laplace-Aboodh transform for equation (4.2.22), single Laplace transform for equation (4.2.23) and single Aboodh transform for equation (4.2.24) and simplifying, we obtain that:

$$F(\rho,\lambda) = \frac{(B + \frac{D}{\lambda})T_1(\rho) + \frac{B}{\lambda}T_2(\rho) + (A\rho + C)T_3(\lambda) + AT_4(\lambda) + H(\rho,\lambda)}{(A\rho^2 + B\lambda^2 + C\rho + D\lambda + E)}, \tag{4.2.25}$$

where $H(\rho,\lambda) = \mathcal{L}_x\mathcal{A}_y[h(x,y)]$.

Lastly, solving this algebraic equation in $F(\rho,\lambda)$ and taking the inverse double Laplace-Aboodh transform on both sides of equation (4.2.25), yields

$$f(x,y) = \mathcal{L}_x^{-1}\mathcal{A}_y^{-1}\Big[\frac{(B+\frac{D}{\lambda})T_1(\rho) + \frac{B}{\lambda}T_2(\rho) + (A\rho + C)T_3(\lambda) + AT_4(\lambda) + H(\rho,\lambda)}{(A\rho^2 + B\lambda^2 + C\rho + D\lambda + E)}\Big], \quad (4.2.26)$$

which represent the general formula for the solution of equation (4.2.22) by double Laplace-Aboodh transform method.

Example 4.2.4.1. Consider the following boundary Laplace equation

$$f_{xx}(x,y) + f_{yy}(x,y) = 0, \qquad (x,y) \in \mathbb{R}_+^2, \quad (4.2.27)$$

with the conditions:

$$\begin{cases} f(x,0) = \sinh x = T_1(x), & f_y(x,0) = 0 = T_2(x), \\ f(0,y) = 0 = T_3(y), & f_x(0,y) = \cos y = T_4(y). \end{cases}$$

Solution:

Substituting

$$T_1(\rho) = \frac{1}{\rho^2 - 1}, \qquad T_2(\rho) = 0, \qquad T_3(\lambda) = 0, \qquad T_4(\lambda) = \frac{1}{\lambda^2+1}, \qquad H(\rho,\lambda) = 0,$$

in (4.2.25) and simplifying, we get a solution of (4.2.27)

$$f(x,y) = \mathcal{L}_x^{-1}\mathcal{A}_y^{-1}\Big[\frac{1}{\rho^2+\lambda^2}\Big(\frac{1}{\rho^2-1} + \frac{1}{\lambda^2+1}\Big)\Big] = \sinh x \cos y. \quad (4.2.28)$$

Example 4.2.4.2. Consider the following boundary Poisson equation

$$f_{xx}(x,y) + f_{yy}(x,y) = 2e^{-x+y}, \qquad (x,y) \in \mathbb{R}_+^2, \quad (4.2.29)$$

with the conditions:

$$\begin{cases} f(x,0) = e^{-x} + \cos x = T_1(x), & f_y(x,0) = e^{-x} + \cos x = T_2(x), \\ f(0,y) = 2e^{y} = T_3(y), & f_x(0,y) = -e^{y} = T_4(y). \end{cases}$$

Solution:

Substituting

$$\begin{cases} T_1(\rho) = \frac{1}{\rho+1} + \frac{\rho}{\rho^2+1}, & T_2(\rho) = \frac{1}{\rho+1} + \frac{\rho}{\rho^2+1}, \\ T_3(\lambda) = \frac{2}{\lambda(\lambda-1)}, & T_4(\lambda) = \frac{-1}{\lambda(\lambda-1)}, \\ H(\rho,\lambda) = \frac{2}{\lambda(\rho+1)(\lambda-1)}, \end{cases}$$

in (4.2.25) and simplifying, we get

$$\begin{aligned} F(\rho,\lambda) &= \frac{\left(\frac{2}{\lambda(\rho+1)(\lambda-1)} + \frac{2\rho}{\lambda(\lambda-1)} - \frac{1}{\lambda(\lambda-1)} + \frac{1}{\rho+1} + \frac{\rho}{\rho^2+1} + \frac{1}{\lambda(\rho+1)} + \frac{\rho}{\lambda(\rho^2+1)}\right)}{(\rho^2+\lambda^2)} \\ &= \frac{1}{\lambda(\rho+1)(\lambda-1)} + \frac{\rho}{\lambda(\rho^2+1)(\lambda-1)}. \end{aligned} \tag{4.2.30}$$

Taking the inverse double Laplace-Aboodh transform of equation (4.2.30), we get a solution of (4.2.29)

$$\begin{aligned} f(x,y) &= \mathcal{L}_x^{-1}\mathcal{A}_y^{-1}\Big[\frac{1}{\lambda(\rho+1)(\lambda-1)} + \frac{\rho}{\lambda(\rho^2+1)(\lambda-1)}\Big] \\ &= e^{-x+y} + e^{y}\cos x. \end{aligned} \tag{4.2.31}$$

Example 4.2.4.3. Consider the following nonhomogeneous wave equation

$$f_{tt}(x,t) = f_{xx}(x,t) + 6t + 2x, \qquad (x,t) \in \mathbb{R}_+^2, \tag{4.2.32}$$

with the conditions:

$$\begin{cases} f(x,0) = 0 = T_1(x), & f_t(x,0) = \sin x = T_2(x), \\ f(0,t) = t^3 = T_3(t), & f_x(0,t) = t^2 + \sin t = T_4(t). \end{cases}$$

Solution:

Substituting

$$\begin{cases} T_1(\rho) = 0, & T_2(\rho) = \frac{1}{\rho^2+1}, \\ T_3(\lambda) = \frac{6}{\lambda^5}, & T_4(\lambda) = \frac{2}{\lambda^4} + \frac{1}{\lambda(\lambda^2+1)}, \\ H(\rho,\lambda) = \frac{6}{\rho\lambda^3} + \frac{2}{\rho^2\lambda^2}, \end{cases}$$

in (4.2.25) and simplifying, we get

$$\begin{aligned} F(\rho,\lambda) &= \frac{1}{\rho^2-\lambda^2}\Big(\frac{6\rho}{\lambda^5} + \frac{2}{\lambda^4} + \frac{1}{\lambda(\lambda^2+1)} - \frac{1}{\lambda(\rho^2+1)} - \frac{2}{\rho^2\lambda^2} - \frac{6}{\rho\lambda^3}\Big) \\ &= \frac{6}{\rho\lambda^5} + \frac{2}{\rho^2\lambda^4} + \frac{1}{\lambda(\lambda^2+1)(\rho^2+1)}. \end{aligned} \tag{4.2.33}$$

Taking the inverse double Laplace-Aboodh transform of equation (4.2.33), we get a solution of (4.2.32)

$$\begin{aligned} f(x,t) &= \mathcal{L}_x^{-1}\mathcal{A}_t^{-1}\Big[\frac{6}{\rho\lambda^5} + \frac{2}{\rho^2\lambda^4} + \frac{1}{\lambda(\lambda^2+1)(\rho^2+1)}\Big] \\ &= t^3 + xt^2 + \sin x \sin t. \end{aligned} \tag{4.2.34}$$

Example 4.2.4.4. Consider the following nonhomogeneous heat equation

$$f_t(z,t) = f_{xx}(x,t) - f(x,t) + 1, \qquad (x,t) \in \mathbb{R}^2_+, \tag{4.2.35}$$

with the conditions:

$$\begin{cases} f(x,0) = 1 + \sin x = T_1(x), & f_t(x,0) = -2\sin x = T_2(x), \\ f(0,t) = 1 = T_3(t), & f_x(0,t) = e^{-2t} = T_4(t). \end{cases}$$

Solution:

Substituting

$$\begin{cases} T_1(\rho) = \frac{1}{\rho} + \frac{1}{\rho^2+1}, & T_2(\rho) = \frac{-2}{\rho^2+1}, \\ T_3(\lambda) = \frac{1}{\lambda^2}, & T_4(\lambda) = \frac{1}{\lambda(\lambda+2)}, \\ H(\rho,\lambda) = \frac{1}{\rho\lambda^2}, \end{cases}$$

in (4.2.25) and simplifying, we get a solution of (4.2.35)

$$\begin{aligned} f(x,t) &= \mathcal{L}_x^{-1}\mathcal{A}_t^{-1}\Big[\frac{1}{\rho^2-\lambda-1}\Big(\frac{\rho}{\lambda^2}+\frac{1}{\lambda(\lambda+2)}-\frac{1}{\rho\lambda}-\frac{1}{\lambda(\rho^2+1)}-\frac{1}{\rho\lambda^2}\Big)\Big] \\ &= \mathcal{L}_x^{-1}\mathcal{A}_t^{-1}\Big[\frac{1}{\rho\lambda^2}+\frac{1}{\lambda(\rho^2+1)(\lambda+2)}\Big] \\ &= 1+e^{-2t}\sin x. \end{aligned} \tag{4.2.36}$$

4.3 Double Laplace-Shehu Transform

Eltayeb and Kilicman, applied double Laplace transform to solve wave, Laplace and heat equations with convolution terms, general linear telegraph and partial integro-differential equations. Alfaqeih and Misirli dealt with double Shehu transform to get the solution of initial and boundary value problems in different areas of real life science and engineering. We applied new double Laplace-Shehu transform to solve Laplace, Poisson, wave and heat equations, through the derivation of general formula for solutions of these equations, or by applying the double Laplace-Shehu transform directly to the given equation.

In this section, we introduce method for solving some partial differential equations subject to the initial and boundary conditions called double Laplace-Shehu transform, the definition of double Laplace-Shehu transform and its inverse. We also discuss some theorems, properties, elementary functions about the double Laplace-Shehu transform. Moreover, we implement the double Laplace-Shehu transform method to some examples.

Definition 4.3.1. The Shehu transform of the real function $f(y)$ of exponential order is defined over the set of functions,

$$\mathcal{M} = \Big\{ f(y) : \exists N, \tau_1, \tau_2 > 0, |f(y)| < Ke^{\frac{|y|}{\tau_i}}, \quad for \quad y \in (-1)^i \times [0, \infty), \quad i = 1, 2 \Big\},$$

by the following integral

$$S[f(y)] = \int_0^{\infty} e^{-\frac{\delta}{\mu}y} f(y)dy = F(\delta, \mu), \quad \delta, \mu > 0, \tag{4.3.1}$$

where $e^{-\frac{\delta}{\mu}}$ is the kernel function, and S is the operator of Shehu transform.

The inverse Shehu transform is defined by

$$S^{-1}[F(\delta, \mu)] = f(y) = \frac{1}{2\pi i} \int_{\omega - i\infty}^{\omega + i\infty} \frac{1}{\mu} e^{\frac{\delta}{\mu}y} F(\delta, \mu) d\delta, \tag{4.3.2}$$

where ω is a real constant [49].

In the next definition, we introduce the double Laplace-Shehu transform.

Definition 4.3.2. The double Laplace-Shehu transform of the function $f(x, y)$ of two variables $x > 0$ and $y > 0$ is denoted by $L_x S_y[f(x, y)] = F(\gamma, \delta, \mu)$ and defined as

$$\begin{aligned} L_x S_y[f(x, y)] = F(\gamma, \delta, \mu) &= \int_0^{\infty} \int_0^{\infty} e^{-(\gamma x + \frac{\delta}{\mu} y)} f(x, y) \; dxdy \\ &= \lim_{\alpha \to \infty, \beta \to \infty} \int_0^{\alpha} \int_0^{\beta} e^{-(\gamma x + \frac{\delta}{\mu} y)} f(x, y) \; dxdy. \end{aligned} \tag{4.3.3}$$

It converges if the limit of the integral exists, and diverges if not.

Definition 4.3.3. The inverse double Laplace-Shehu transform of a function $F(\gamma, \delta, \mu)$

is given by

$$L_x^{-1}S_y^{-1}[F(\gamma,\delta,\mu)] = f(x,y).$$

Equivalently,

$$f(x,y) = L_x^{-1}S_y^{-1}[F(\gamma,\delta,\mu)] = \frac{1}{(2\pi i)^2}\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\gamma x}d\gamma \int_{\omega-i\infty}^{\omega+i\infty} \frac{1}{\mu}e^{\frac{\delta}{\mu}y}F(\gamma,\delta,\mu)d\delta, \quad (4.3.4)$$

where κ and ω are real constants.

4.3.1 Existence and Uniqueness of The Double Laplace-Shehu Transform

Definition 4.3.1.1. A function $f(x,y)$ is said to be of exponential orders $\alpha > 0$, $\beta > 0$, on $0 \leq x < \infty$, $0 \leq y < \infty$, if there exists positive constants K, X and Y such that

$$|f(x,y)| \leq Ke^{\alpha x+\beta y}, \qquad for\ all \quad x > X, \ \ y > Y,$$

and we write

$$f(x,y) = o(e^{\alpha x+\beta y}) \qquad as \quad x \to \infty, \quad y \to \infty.$$

Or, equivalently,

$$\lim_{x\to\infty,y\to\infty} e^{-(\alpha x+\beta y)}|f(x,y)| = K \lim_{x\to\infty,y\to\infty} e^{-(\gamma-\alpha)x}e^{-(\frac{\delta}{\mu}-\beta)y} = 0, \qquad \gamma > \alpha, \quad \frac{\delta}{\mu} > \beta.$$

Theorem 4.3.1.2. Let $f(x,y)$ be a continuous function in every finite intervals $(0,X)$ and $(0,Y)$ and of exponential order $e^{(\alpha x+\beta y)}$, then the double Laplace-Shehu transform of $f(x,y)$ exists for all $\gamma > \alpha$ and $\frac{\delta}{\mu} > \beta$.

Proof. Let $f(x,y)$ be of exponential order $exp(\alpha x + \beta y)$ such that

$$\Big|f(x,y)\Big| \leq Ke^{(\alpha x+\beta y)}, \qquad \forall \;\; x > X, \;\; y > Y.$$

Then, we have

$$\begin{aligned}
\Big|F(\gamma,\delta,\mu)\Big| &= \Big|\int_0^\infty \int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)} f(x,y)dxdy\Big| \\
&\leqq \int_0^\infty \int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)} |f(x,y)|dxdy \\
&\leqq K\int_0^\infty \int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)} e^{(\alpha x+\beta y)}dxdy \\
&= K\int_0^\infty e^{-(\gamma-\alpha)x}dx \int_0^\infty e^{-(\frac{\delta}{\mu}-\beta)y}dy \\
&= \frac{K\mu}{(\gamma-\alpha)(\delta-\beta\mu)}.
\end{aligned}$$

Thus, the proof is complete.

Theorem 4.3.1.3. Let $F_1(\gamma,\delta,\mu)$ and $F_2(\gamma,\delta,\mu)$ be the double Laplace-Shehu transform of the continuous functions $f_1(x,y)$ and $f_2(x,y)$ defined for $x,y \geq 0$ respectively. If $F_1(\gamma,\delta,\mu) = F_2(\gamma,\delta,\mu)$, then $f_1(x,y) = f_2(x,y)$.

Proof. Assume that κ and ω are sufficiently large, since

$$f(x,y) = L_x^{-1}S_y^{-1}[F(\gamma,\delta,\mu)] = \Big(\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\gamma x}d\gamma\Big)\Big(\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{1}{\mu}e^{\frac{\delta}{\mu}y}F(\gamma,\delta,\mu)d\delta\Big),$$

we deduce that

$$\begin{aligned}
f_1(x,y) &= \Big(\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\gamma x}d\gamma\Big)\Big(\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{\mu}e^{\frac{\delta}{\mu}y}F_1(\gamma,\delta,\mu)d\delta\Big)\\
&= \Big(\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\gamma x}d\gamma\Big)\Big(\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{\mu}e^{\frac{\delta}{\mu}y}F_2(\gamma,\delta,\mu)d\delta\Big)\\
&= f_2(x,y).
\end{aligned}$$

This ends the proof of the theorem. □

4.3.2 Some Basic Properties of the Double Laplace-Shehu Transform

Linearity property 4.3.2.1. If the double Laplace-Shehu transform of functions $f_1(x,y)$ and $f_2(x,y)$ are $F_1(\gamma,\delta,\mu)$ and $F_2(\gamma,\delta,\mu)$ respectively, then the double Laplace-Shehu transform of $\alpha f_1(x,y)+\beta f_2(x,y)$ is given by $\alpha F_1(\gamma,\delta,\mu)+\beta F_2(\gamma,\delta,\mu)$, where α and β are arbitrary constants.

Proof.

$$\begin{aligned}
L_xS_y[\alpha f_1(x,y)+\beta f_2(x,y)] &= \int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}\Big(\alpha f_1(x,y)+\beta f_2(x,y)\Big)dxdy\\
&= \alpha\int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}f_1(x,y)dxdy\\
&+ \beta\int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}f_2(x,y)dxdy\\
&= \alpha F_1(\gamma,\delta,\mu)+\beta F_2(\gamma,\delta,\mu). \qquad (4.3.5)
\end{aligned}$$

Shifting property 4.3.2.2. If the double Laplace-Shehu transform of $f(x,y)$ is $F(\gamma,\delta,\mu)$, then double Laplace-Shehu transform of the function $e^{(\alpha x+\beta y)}f(x,y)$ is given by $F(\gamma-\alpha,\delta-\beta\mu,\mu)$.

Proof.

$$\begin{aligned} L_xS_y[e^{(\alpha x+\beta y)}f(x,y)] &= \int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}e^{(\alpha x+\beta y)}f(x,y)dxdy \\ &= \int_0^\infty\int_0^\infty e^{-\left((\gamma-\alpha)x+(\frac{\delta-\beta\mu}{\mu})y\right)}f(x,y)dxdy \\ &= F(\gamma-\alpha,\delta-\beta\mu,\mu). \qquad (4.3.6) \end{aligned}$$

Change of scale property 4.3.2.3. If the double Laplace-Shehu transform of the function $f(x,y)$ is $F(\gamma,\delta,\mu)$, then the double Laplace-Shehu transform of $f(\alpha x,\beta y)$ is given by $\frac{1}{\alpha\beta}F(\frac{\gamma}{\alpha},\frac{\delta}{\beta},\mu)$.

Proof.

$$L_xS_y[f(\alpha x,\beta y)] = \int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}f(\alpha x,\beta y)dxdy.$$

Let $\upsilon=\alpha x,\quad \tau=\beta y$, then

$$\begin{aligned} L_xS_y[f(\alpha x,\beta y)] &= \frac{1}{\alpha\beta}\int_0^\infty\int_0^\infty e^{-\frac{\gamma}{\alpha}\upsilon}e^{-\frac{\delta}{\beta\mu}\tau}f(\upsilon,\tau)d\upsilon d\tau \\ &= \frac{1}{\alpha\beta}F(\frac{\gamma}{\alpha},\frac{\delta}{\beta},\mu). \qquad (4.3.7) \end{aligned}$$

Derivatives properties 4.3.2.4. If $L_xS_y[f(x,y)]=F(\gamma,\delta,\mu)$, then:

$$(1). \qquad L_xS_y\Big[\frac{\partial f(x,y)}{\partial x}\Big]=\gamma F(\gamma,\delta,\mu)-S[f(0,y)]. \qquad (4.3.8)$$

Proof.

$$\begin{aligned} L_xS_y\Big[\frac{\partial f(x,y)}{\partial x}\Big] &= \int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}\frac{\partial f(x,y)}{\partial x}dxdy \\ &= \int_0^\infty e^{-\frac{\delta}{\mu}y}dy\Big(\int_0^\infty e^{-\gamma x}f_x(x,y)dx\Big). \end{aligned}$$

Using integration by parts, let $u = e^{-\gamma x}, \quad dv = f_x(x,y)dx$, then we obtain

$$\begin{aligned} L_xS_y\Big[\frac{\partial f(x,y)}{\partial x}\Big] &= \int_0^\infty e^{-\frac{\delta}{\mu}y}dy\Big(-f(0,y)+\gamma\int_0^\infty e^{-\gamma x}f(x,y)dx\Big) \\ &= \gamma F(\gamma,\delta,\mu) - S[f(0,y)]. \end{aligned}$$

$$(2). \qquad L_xS_y\Big[\frac{\partial f(x,y)}{\partial y}\Big] = \frac{\delta}{\mu}F(\gamma,\delta,\mu) - L[f(x,0)]. \tag{4.3.9}$$

Proof.

$$\begin{aligned} L_xS_y\Big[\frac{\partial f(x,y)}{\partial y}\Big] &= \int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}\frac{\partial f(x,y)}{\partial y}dxdy \\ &= \int_0^\infty e^{-\gamma x}dx\Big(\int_0^\infty e^{-\frac{\delta}{\mu}y}f_y(x,y)dy\Big). \end{aligned}$$

Using integration by parts, let $u = e^{-\frac{\delta}{\mu}y}, \quad dv = f_y(x,y)dy$, then we obtain

$$\begin{aligned} L_xS_y\Big[\frac{\partial f(x,y)}{\partial y}\Big] &= \int_0^\infty e^{-\gamma x}dx\Big(-f(x,0)+\frac{\delta}{\mu}\int_0^\infty e^{-\frac{\delta}{\mu}y}f(x,y)dy\Big) \\ &= \frac{\delta}{\mu}F(\gamma,\delta,\mu) - L[f(x,0)]. \end{aligned}$$

We can prove that:

$$(3). \qquad L_xS_y\Big[\frac{\partial^2 f(x,y)}{\partial x^2}\Big] = \gamma^2F(\gamma,\delta,\mu) - \gamma S[f(0,y)] - S[f_x(0,y)]. \tag{4.3.10}$$

$$(4). \qquad L_xS_y\Big[\frac{\partial^2 f(x,y)}{\partial y^2}\Big] = \frac{\delta^2}{\mu^2}F(\gamma,\delta,\mu) - \frac{\delta}{\mu}L[f(x,0)] - L[f_y(x,0)]. \tag{4.3.11}$$

$$(5). \qquad L_xS_y\Big[\frac{\partial^2 f(x,y)}{\partial x\partial y}\Big] = \frac{\gamma\delta}{\mu}F(\gamma,\delta,\mu) - \gamma L[f(x,0)] - S[f_y(0,y)]. \tag{4.3.12}$$

4.3.3 The Double Laplace-Shehu Transform of Some Elementary Functions

(1). If the function $f(x,y)=1$, then

$$L_xS_y[f(x,y)]=\int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}dxdy=\frac{\mu}{\gamma\delta}. \tag{4.3.13}$$

(2). If the function $f(x,y)=xy$, then

$$L_yS_y[f(x,y)]=\int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}xy\ dxdy=\frac{\mu^2}{\gamma^2\delta^2}. \tag{4.3.14}$$

(3). If the function $f(x,y)=x^ny^m,\quad n,m=0,1,2,...$, then

$$L_xS_y[f(x,y)]\ =\ \int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}x^ny^m\ dxdy=\frac{n!m!}{\gamma^{n+1}}\Big(\frac{\mu}{\delta}\Big)^{m+1}. \tag{4.3.15}$$

(4). If the function $f(x,y)=x^\sigma y^\nu,\ \sigma\geq-1,\ \nu\geq-1$, then

$$L_xS_y[f(x,y)]=\int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}x^\sigma y^\nu\ dxdy\ =\ \int_0^\infty e^{-\gamma x}x^\sigma dx\int_0^\infty e^{-\frac{\delta}{\mu}y}y^\nu dy.$$

Let $\xi=\gamma x\quad$ and $\quad\eta=\frac{\delta}{\mu}y$, then we have

$$\begin{aligned}L_xS_y[f(x,y)] &= \frac{1}{\gamma^{\sigma+1}}\int_0^\infty e^{-\xi}\xi^\sigma d\xi\ (\frac{\mu}{\delta})^{\nu+1}\int_0^\infty e^{-\eta}\eta^\nu d\eta\\ &= \Gamma(\sigma+1)\Big(\frac{1}{\gamma^{\sigma+1}}\Big)\Gamma(\nu+1)\Big(\frac{\mu}{\delta}\Big)^{\nu+1}.\end{aligned} \tag{4.3.16}$$

Where $\Gamma(.)$ is the Euler gamma function.

(5). If the function $f(x,y)=e^{nx+my},\quad n,m=0,1,2,...$, then

$$L_xS_y[f(x,y)]=\int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}e^{nx+my}\ dxdy=\frac{\mu}{(\gamma-n)(\delta-m\mu)}. \tag{4.3.17}$$

(6). If the function $f(x,y)=\cos(nx+my),\quad n,m=0,1,2,\quad ...$, then

$$\begin{aligned} L_xS_y[f(x,y)] &= \int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}\cos(nx+my)\ dxdy \\ &= \frac{\mu(\gamma\delta-nm\mu)}{(\gamma^2+n^2)(\delta^2+m^2\mu^2)}. \end{aligned} \tag{4.3.18}$$

(7). If the function $f(x,y)=\sin(nx+my),\quad n,m=0,1,2,\quad ...$, then

$$\begin{aligned} L_xS_y[f(x,y)] &= \int_0^\infty\int_0^\infty e^{-(\gamma x+\frac{\delta}{\mu}y)}\sin(nx+my)\ dxdy \\ &= \frac{\mu(n\delta+m\gamma\mu)}{(\gamma^2+n^2)(\delta^2+m^2\mu^2)}. \end{aligned} \tag{4.3.19}$$

Consequently,

$$L_xS_y[\cosh(nx+my)]=\frac{\mu(\gamma\delta+nm\mu)}{(\gamma^2-n^2)(\delta^2-m^2\mu^2)}, \tag{4.3.20}$$

$$L_xS_y[\sinh(nx+my)]=\frac{\mu(n\delta+m\gamma\mu)}{(\gamma^2-n^2)(\delta^2-m^2\mu^2)}. \tag{4.3.21}$$

4.3.4 Applications of The Double Laplace-Shehu Transform

We apply the double Laplace-Shehu transform method to linear partial differential equations. Let the second-order nonhomogeneous partial differential equation in two independent variables be in the form:

$$A\ f_{xx}+B\ f_{yy}+C\ f_x+D\ f_y+E\ f=h(x,y),\qquad (x,y)\in\mathbb{R}^2_+, \tag{4.3.22}$$

with the initial conditions:

$$f(x,0)=\hbar_1(x),\qquad f_y(x,0)=\hbar_2(x), \tag{4.3.23}$$

and the boundary conditions:

$$f(0,y) = \hbar_3(y), \qquad f_x(0,y) = \hbar_4(y), \tag{4.3.24}$$

where A, B, C, D and E are constants and $h(x,y)$ is the source term.

Using the property of partial derivative of the double Laplace-Shehu transform for equation (4.3.22), single Laplace transform for equation (4.3.23) and single Shehu transform for equation (4.3.24) and simplifying, we obtain that:

$$F(\gamma,\delta,\mu) = \Big[\frac{(B\frac{\delta}{\mu} + D)\hbar_1(\gamma) + B\hbar_2(\gamma) + (A\gamma + C)\hbar_3(\delta,\mu) + A\hbar_4(\delta,\mu) + H(\gamma,\delta,\mu)}{(A\gamma^2 + B\frac{\delta^2}{\mu^2} + C\gamma + D\frac{\delta}{\mu} + E)}\Big] \tag{4.3.25}$$

where $H(\gamma,\delta,\mu) = L_x S_y[h(x,y)]$.

Finally, Solving this algebraic equation in $F(\gamma,\delta,\mu)$ and taking the inverse double Laplace-Shehu transform on both sides of equation (4.3.25), yields:

$$f(x,y) = L_x^{-1} S_y^{-1}\Big[\frac{(B\frac{\delta}{\mu} + D)\hbar_1(\gamma) + B\hbar_2(\gamma) + (A\gamma + C)\hbar_3(\delta,\mu) + A\hbar_4(\delta,\mu) + H(\gamma,\delta,\mu)}{(A\gamma^2 + B\frac{\delta^2}{\mu^2} + C\gamma + D\frac{\delta}{\mu} + E)}\Big] \tag{4.3.26}$$

which represent the general formula for the solution of equation (4.3.22) by the double Laplace-Shehu transform method.

Example 4.3.4.1. Consider the following boundary Laplace equation

$$f_{xx}(x,y) + f_{yy}(x,y) = 0, \tag{4.3.27}$$

with the conditions:

$$f(x,0) = \cos x = \hbar_1(x), \qquad f_y(x,0) = 0 = \hbar_2(x),$$

$$f(0,y) = \cosh y = \hbar_3(y), \qquad f_x(0,y) = 0 = \hbar_4(y).$$

Solution:

Substituting

$$\hbar_1(\gamma) = \frac{\gamma}{\gamma^2+1}, \qquad \hbar_2(\gamma) = 0, \qquad \hbar_3(\delta,\mu) = \frac{\delta\mu}{\delta^2-\mu^2}, \qquad \hbar_4(\delta,\mu) = 0,$$

in (4.3.25) and simplifying, we get a solution of (4.3.27)

$$f(x,y) = L_x^{-1}S_y^{-1}\Big[\frac{\gamma}{\gamma^2+1} \ \frac{\delta\mu}{\delta^2-\mu^2}\Big] = \cos x \cosh y. \tag{4.3.28}$$

Example 4.3.4.2. Consider the following boundary Poisson equation

$$f_{xx}(x,y) + f_{yy}(x,y) = y\sin x, \tag{4.3.29}$$

with the conditions:

$$f(x,0) = 0 = \hbar_1(x), \qquad f_y(x,0) = -\sin x = \hbar_2(x),$$

$$f(0,y) = 0 = \hbar_3(y), \qquad f_x(0,y) = -y = \hbar_4(y).$$

Solution:

Applying the double Laplace-Shehu transform on both sides of equation (4.3.29), we get

$$\begin{aligned}\gamma^2 F(\gamma,\delta,\mu) \quad &- \quad \gamma S[f(0,y)] - S[f_x(0,y)] + \frac{\delta^2}{\mu^2}F(\gamma,\delta,\mu) - \frac{\delta}{\mu}L[f(x,0)] - L[f_y(x,0)] \\ &= \quad \frac{\mu^2}{\delta^2(\gamma^2+1)}. \end{aligned} \tag{4.3.30}$$

Substituting

$$L[\hbar_1(x)] = 0, \qquad L[\hbar_2(x)] = \frac{-1}{\gamma^2+1}, \qquad S[\hbar_3(y)] = 0, \qquad S[\hbar_4(y)] = -\frac{\mu^2}{\delta^2},$$

in (4.3.25) and simplifying, we get

$$F(\gamma,\delta,\mu)=\frac{-\mu^2}{\delta^2(\gamma^2+1)}. \tag{4.3.31}$$

Taking the inverse double Laplace-Shehu transform of previous equation, we get the solution

$$f(x,y)=L_x^{-1}S_y^{-1}\Big[\frac{-\mu^2}{\delta^2(\gamma^2+1)}\Big]=-L_x^{-1}S_y^{-1}\Big[\frac{\mu^2}{\delta^2}\ \frac{1}{\gamma^2+1}\Big]=-y\sin x. \tag{4.3.32}$$

In the following two examples, we will replace the independent variable η with the time variable t. Therefore, $S_y\equiv S_t$ and $S_y^{-1}\equiv S_t^{-1}$ will be used.

Example 4.3.4.3. Consider the following nonhomogeneous wave equation

$$f_{tt}(x,t)=f_{xx}(x,t)-3f(x,t)+3, \qquad t>0, \tag{4.3.33}$$

with the conditions:

$$f(x,0)=1=\hbar_1(x), \qquad f_t(x,0)=2\sin x=\hbar_2(x),$$

$$f(0,t)=1=\hbar_3(t), \qquad f_x(0,t)=\sin 2t=\hbar_4(t).$$

Solution:

Substituting

$$\hbar_1(\gamma)=\frac{1}{\gamma},\quad \hbar_2(\gamma)=\frac{2}{\gamma^2+1},\quad \hbar_3(\delta,\mu)=\frac{\mu}{\delta},\quad \hbar_4(\delta,\mu)=\frac{2\mu^2}{\delta^2+4\mu^2},\quad F(\gamma,\delta,\mu)=\frac{-3\mu}{\gamma\delta},$$

in (4.3.25) and simplifying, we get a solution of (4.3.33)

$$f(x,t)=L_x^{-1}S_t^{-1}\Big[\frac{\mu}{\gamma\delta}+\frac{1}{\gamma^2+1}\ \frac{2\mu^2}{\delta^2+4\mu^2}\Big]=1+\sin x\sin 2t. \tag{4.3.34}$$

Example 4.3.4.4. Consider the following nonhomogeneous heat equation

$$f_t(x,t) = f_{xx}(x,t) - 6x, \qquad t > 0, \tag{4.3.35}$$

with the conditions:

$$f(x,0) = x^3 + \sin x = \hbar_1(x), \qquad f_t(x,0) = -\sin x = \hbar_2(x),$$

$$f(0,t) = 0 = \hbar_3(t), \qquad f_x(0,t) = e^{-t} = \hbar_4(t).$$

Solution:

Substituting

$$\begin{aligned} \hbar_1(\gamma) = \frac{6}{\gamma^4} + \frac{1}{\gamma^2+1}, \quad \hbar_2(\gamma) = \frac{-1}{\gamma^2+1}, \\ \hbar_3(\delta,\mu) = 0, \quad \hbar_4(\delta,\mu) = \frac{\mu}{\delta+\mu}, \quad F(\gamma,\delta,\mu) = \frac{6}{\gamma^2}, \end{aligned} \tag{4.3.36}$$

in (4.3.25) and simplifying, we get a solution of (4.3.35)

$$f(x,t) = L_x^{-1}S_t^{-1}\Big[\frac{6}{\gamma^4} + \frac{\mu}{\delta+\mu}\ \ \frac{1}{\gamma^2+1}\Big] = x^3 + e^{-t}\sin x. \tag{4.3.37}$$

4.4 Triple Laplace-Aboodh-Sumudu Transform

Definition 4.4.1.[7, 22] The Sumudu transform of the continuous function $f(x)$ for all $x \geq 0$ is defined by

$$\mathcal{S}[f(x)] = \frac{1}{p}\int_0^\infty e^{-\frac{x}{p}} f(x)dx = F(p), \qquad \kappa_1 \leq p \leq \kappa_2, \tag{4.4.1}$$

where $\mathcal{S}$ is the Sumudu operator. Provided that the integral exists. If the integral is convergent for some value of x, then the Sumudu transformation of $f(x)$ exists otherwise not. The inverse Sumudu transform of the function $F(p)$ is defined by

$$f(x) = \mathcal{S}^{-1}[F(p)] = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{p} e^{\frac{x}{p}} F(p)dp, \tag{4.4.2}$$

where α is a real constant and S^{-1} is an operator and it is called inverse Sumudu transform operator [79].

Definition 4.4.2. [8] The double Laplace-Sumudu transform of the continuous function $f(x,t)$ and $x, t > 0$ is defined by

$$\begin{aligned} L_x S_t[f(x,t)] = F(p,r) &= \frac{1}{r}\int_0^\infty \int_0^\infty e^{-(px+\frac{t}{r})} f(x,t)dxdt \\ &= \frac{1}{r} \lim_{\alpha\to\infty,\beta\to\infty} \int_0^\alpha \int_0^\beta e^{-(px+\frac{t}{r})} f(x,t)dxdt. \end{aligned} \tag{4.4.3}$$

It converges if the limit of the integral exists, and diverges if not.

The inverse of double Laplace-Sumudu transform is given by

$$f(x,t) = L_x^{-1} S_t^{-1}[F(p,r)] = \frac{1}{(2\pi i)^2}\int_{\omega_1-i\infty}^{\omega_1+i\infty} e^{px}\Big\{\int_{\omega_2-i\infty}^{\omega_2+i\infty} \frac{1}{r} e^{\frac{t}{r}} F(p,r)dr\Big\}dp, \tag{4.4.4}$$

where ω_1 and ω_2 are real constants.

Definition 4.4.3.[9] The double Aboodh-Sumudu transform of the continuous function $h(y,t)$, $y, t > 0$ is denoted by the operator $A_y S_t[h(y,t)] = H(q,r)$ and defined by

$$A_y S_t[h(y,t)] = H(q,r) = \frac{1}{qr}\int_0^\infty \int_0^\infty e^{-(qy+\frac{t}{r})} h(y,t)dydt. \tag{4.4.5}$$

And the inverse double Aboodh-Sumudu transform is defined by

$$h(y,t)=A_y^{-1}S_t^{-1}[H(q,r)]=\frac{1}{(2\pi i)^2}\int_{\bar{\gamma}_1-i\infty}^{\bar{\gamma}_1+i\infty}qe^{qy}\Big\{\int_{\bar{\gamma}_2-i\infty}^{\bar{\gamma}_2+i\infty}\frac{1}{r}e^{\frac{t}{r}}H(q,r)dr\Big\}dq, \quad (4.4.6)$$

where $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are real constants.

In the next definition, we introduce the triple Laplace-Aboodh-Sumudu transform.

Definition 4.4.4. Let f be a continuous function of three variables say $x, y, t > 0$; then, the triple Laplace-Aboodh-Sumudu transform of $f(x,y,t)$ is denoted by the operator $L_xA_yS_t[f(x,y,t)] = F(p,\lambda,r)$ and defined by

$$L_xA_yS_t[f(x,y,t)]=F(p,\lambda,r)=\frac{1}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+\lambda y+\frac{t}{r})}f(x,y,t)dxdydt \quad (4.4.7)$$

Provided the integral exists.

The inverse triple Laplace-Aboodh-Sumudu transform is defined by

$$\begin{aligned} f(x,y,t) &= L_x^{-1}A_y^{-1}S_t^{-1}[F(p,\lambda,r)] \\ &= \frac{1}{(2\pi i)^3}\int_{\kappa_1-i\infty}^{\kappa_1+i\infty}e^{px}\left\{\int_{\kappa_2-i\infty}^{\kappa_2+i\infty}\lambda e^{\lambda y}\left\{\int_{\kappa_3-i\infty}^{\kappa_3+i\infty}\frac{1}{r}e^{\frac{t}{r}}F(p,\lambda,r)dr\right\}d\lambda\right\}dp, \end{aligned}$$

where κ_1, κ_2 and κ_3 are real constants.

4.4.1 Some Properties of Triple Laplace-Aboodh-Sumudu Transform

Linearity property 4.4.1.1.

Let $f(x,y,t)$ and $g(x,y,t)$ be two functions such that

$$L_xA_yS_t[f(x,y,t)] = F(p,\lambda,r),$$

$$L_xA_yS_t[g(x,y,t)] = G(p,\lambda,r).$$

Then for any constants α and β, we have

$$L_xA_yS_t[\alpha f(x,y,t) + \beta g(x,y,t)] = \alpha L_xA_yS_t[f(x,y,t)] + \beta L_xA_yS_t[g(x,y,t)]. \quad (4.4.8)$$

Proof. By using definition of triple Laplace-Aboodh-Sumudu transform, we obtain

$$\begin{aligned} L_xA_yS_t[\alpha f(x,y,t) + \beta g(x,y,t)] &= \frac{1}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+\lambda y+\frac{t}{r})}\Big\{\alpha f(x,y,t) + \beta g(x,y,t)\Big\}\, dxd \\ &= \frac{\alpha}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+\lambda y+\frac{t}{r})} f(x,y,t)dxdydt \\ &+ \frac{\beta}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+\lambda y+\frac{t}{r})} g(x,y,t)dxdydt \\ &= \alpha L_xA_yS_t[f(x,y,t)] + \beta L_xA_yS_t[g(x,y,t)]. \end{aligned}$$

Shifting property 4.4.1.2.

If the triple Laplace-Aboodh-Sumudu transform of $f(x,y,t)$ is $F(p,\lambda,r)$, then for real constants a, b and c, we have

$$L_xA_yS_t[e^{(ax+by+ct)}f(x,y,t)] = \frac{\lambda-b}{\lambda(1-cr)}F(p-a,\lambda-b,\frac{r}{1-cr}). \quad (4.4.9)$$

Proof. Using the definition of triple Laplace-Aboodh-Sumudu transform, we get

$$\begin{aligned}
L_xA_yS_t[e^{(ax+by+ct)}f(x,y,t)] &= \frac{1}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+\lambda y+\frac{t}{r})}e^{(ax+by+ct)}f(x,y,t)dxdydt \\
&= \frac{1}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left((p-a)x+(\lambda-b)y+(\frac{1}{r}-c)t\right)}f(x,y,t)dxdydt \\
&= \frac{\lambda-b}{\lambda(\lambda-b)\frac{(1-cr)r}{(1-cr)}}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left((p-a)x+(\lambda-b)y+(\frac{1}{r}-c)t\right)}f(x,y,t)dx \\
&= \frac{\lambda-b}{\lambda(1-cr)}F\Big(p-a,\lambda-b,\frac{r}{1-cr}\Big).
\end{aligned}$$

Changing of scale property 4.4.1.3.

Let $f(x,y,t)$ be a function such that

$$L_xA_yS_t[f(x,y,t)] = F(p,\lambda,r).$$

Then for a,b and c are positive constants, we have

$$L_xA_yS_t[f(ax,by,ct)] = \frac{1}{ab^2}F\Big(\frac{p}{a},\frac{\lambda}{b},cr\Big). \tag{4.4.10}$$

Proof. Using the definition of triple Laplace-Aboodh-Sumudu transform, we have

$$L_xA_yS_t[f(ax,by,ct)] = \frac{1}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+\lambda y+\frac{t}{r})}f(ax,by,ct)dxdydt,$$

Let $\tau = ax, \quad \upsilon = by, \quad \varphi = ct$, then

$$\begin{aligned}
L_xA_yS_t[f(\tau,\upsilon,\varphi)] &= \frac{1}{abc\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{p}{a}\tau+\frac{\lambda}{b}\upsilon+\frac{\varphi}{cr})}f(\tau,\upsilon,\varphi)d\tau d\upsilon d\varphi \\
&= \frac{1}{ab^2\frac{\lambda}{b}cr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{p}{a}\tau+\frac{\lambda}{b}\upsilon+\frac{\varphi}{cr})}f(\tau,\upsilon,\varphi)d\tau d\upsilon d\varphi \\
&= \frac{1}{ab^2}F(\frac{p}{a},\frac{\lambda}{b},cr).
\end{aligned}$$

Derivatives properties 4.4.1.4.

If $L_xA_yS_t[f(x,y,t)] = F(p,\lambda,r)$, then:

$$(1). \qquad L_xA_yS_(t)\Big[\frac{\partial f(x,y,t)}{\partial x}\Big] = pF(p,\lambda,r) - A_yS_t[f(0,y,t)]. \qquad (4.4.11)$$

Proof.

$$\begin{aligned} L_xA_yS_t\Big[\frac{\partial f(x,y,t)}{\partial x}\Big] &= \frac{1}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+\lambda y+\frac{t}{r})}\frac{\partial f(x,y,t)}{\partial x}dxdydt \\ &= \frac{1}{\lambda r}\int_0^\infty e^{-\lambda y}dy\int_0^\infty e^{-\frac{t}{r}}dt\Bigg\{\int_0^\infty e^{-px}f_x(x,y)dx\Bigg\}. \end{aligned}$$

Using integration by parts, let $u = e^{-px}$, $\qquad dv = \frac{\partial f(x,y,t)}{\partial x}dx$, then we obtain

$$\begin{aligned} L_xA_yS_t\Big[\frac{\partial f(x,y,t)}{\partial x}\Big] &= \frac{1}{\lambda r}\int_0^\infty e^{-\lambda y}dy\int_0^\infty e^{-\frac{t}{r}}dt\Bigg\{-f(0,y,t) + p\int_0^\infty e^{-px}f(x,y,t)dx\Bigg\} \\ &= pF(p,\lambda,r) - A_yS_t[f(0,y,t)] \end{aligned}$$

$$(2). \qquad L_xA_yS_t\Big[\frac{\partial f(x,y,t)}{\partial y}\Big] = \lambda F(p,\lambda,r) - \frac{1}{\lambda}L_xS_t[f(x,0,t)]. \qquad (4.4.12)$$

Proof.

$$\begin{aligned} L_xA_yS_t\Big[\frac{\partial f(x,y,t)}{\partial y}\Big] &= \frac{1}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+\lambda y+\frac{t}{r})}\frac{\partial f(x,y,t)}{\partial y}dxdydt \\ &= \frac{1}{\lambda r}\int_0^\infty e^{-px}dx\int_0^\infty e^{-\frac{t}{r}}dt\Bigg\{\int_0^\infty e^{-\lambda y}f_y(x,y,t)dy\Bigg\}. \end{aligned}$$

By using integration by parts, let $u = e^{-\lambda y}$, $\qquad dv = \frac{\partial f(x,y,t)}{\partial y}dy$, we get

$$\begin{aligned} L_xA_yS_t\Big[\frac{\partial f(x,y,t)}{\partial y}\Big] &= \frac{1}{\lambda r}\int_0^\infty e^{-px}dx\int_0^\infty e^{-\frac{t}{r}}dt\Bigg\{-f(x,0,t) + \lambda\int_0^\infty e^{-\lambda y}f(x,y,t)dy\Bigg\} \\ &= \lambda F(p,\lambda,r) - \frac{1}{\lambda}L_xS_t[f(x,0,t)]. \end{aligned}$$

Similarly, we can prove that:

$$\begin{aligned}
L_xA_yS_t\Big[\frac{\partial f(x,y,t)}{\partial t}\Big] &= \frac{1}{r}F(p,\lambda,r)-\frac{1}{r}L_xA_y[f(x,y,0)],\\
L_xA_yS_t\Big[\frac{\partial^2 f(x,y,t)}{\partial x^2}\Big] &= p^2F(p,\lambda,r)-pA_yS_t[f(0,y,t)]-A_yS_t[f_x(0,y,t)],\\
L_xA_yS_t\Big[\frac{\partial^2 f(x,y,t)}{\partial y^2}\Big] &= \lambda^2F(p,\lambda,r)-L_xS_t[f(x,0,t)]-\frac{1}{\lambda}L_xS_t[f_y(x,0,t)],\\
L_xA_yS_t\Big[\frac{\partial^2 f(x,y,t)}{\partial t^2}\Big] &= \frac{1}{r^2}F(p,\lambda,r)-\frac{1}{r^2}L_xA_y[f(x,y,0)]-\frac{1}{r}L_xA_y[f_t(x,y,0)],\\
L_xA_yS_t\Big[\frac{\partial^2 f(x,y,t)}{\partial x\partial y}\Big] &= p\lambda F(p,\lambda,r)-\frac{p}{\lambda}L_xS_t[f(x,0,t)]-A_yS_t[f_y(0,y,t)],\\
L_xA_yS_t\Big[\frac{\partial^2 f(x,y,t)}{\partial x\partial t}\Big] &= \frac{p}{r}F(p,\lambda,r)-\frac{p}{r}L_xA_y[f(x,y,0)]-A_yS_t[f_t(0,y,t)],\\
L_xA_yS_t\Big[\frac{\partial^2 f(x,y,t)}{\partial y\partial t}\Big] &= \frac{\lambda}{r}F(p,\lambda,r)-\frac{\lambda}{r}L_xA_y[f(x,y,0)]-\frac{1}{\lambda}L_xS_t[f_t(x,0,y)].
\end{aligned}$$

4.4.2 Triple Laplace-Aboodh-Sumudu Transform of Some Elementary Functions

(1). If $f(x,y,t)=1$, then

$$L_xA_yS_t[f(x,y,t)]=\frac{1}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+\lambda y+\frac{t}{r})}\ dxdydt=\frac{1}{p\lambda^2}.$$

(2). If $f(x,y,t)=xyt$, then

$$L_xA_yS_t[f(x,y,t)]=\frac{1}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+\lambda y+\frac{t}{r})}xyt dxdydt=\frac{r}{p^2\lambda^3}.$$

(3). If $f(x,y,t)=x^ny^mt^k$, $\quad n,m,k=0,1,2,\ \ldots$, then

$$\begin{aligned}
L_xA_yS_t[f(x,y,t)] &= \frac{1}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+\lambda y+\frac{t}{r})}x^ny^mt^k\ dxdydt\\
&= \frac{n!}{p^{n+1}}\cdot\frac{m!}{\lambda^{m+2}}\cdot k!r^k.
\end{aligned}$$

(4). If $f(x,y,t) = x^{\sigma}y^{\nu}t^{\rho}$, $\sigma \geq -1$, $\nu \geq -1$ $\rho \geq -1$, then

$$\begin{aligned}
L_xA_yS_t[f(x,y,t)] &= \frac{1}{\lambda r}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(px+\lambda y+\frac{t}{r})}x^{\sigma}y^{\nu}t^{\rho}\ dxdydt \\
&= \int_0^{\infty} e^{-px}x^{\sigma}dx\ \frac{1}{\lambda}\int_0^{\infty} e^{-qy}y^{\nu}dy\ \frac{1}{r}\int_0^{\infty} e^{-\frac{t}{r}}t^{\rho}dt,
\end{aligned}$$

let $\xi = px,\quad \zeta = \lambda y\ \ and\ \ \eta = \frac{t}{r}$

$$\begin{aligned}
L_xA_yS_t[f(x,y,t)] &= \frac{1}{p^{\sigma+1}}\int_0^{\infty} e^{-\xi}\xi^{\sigma}d\xi\ \frac{1}{\lambda^{\nu+2}}\int_0^{\infty} e^{-\zeta}\zeta^{\nu}d\zeta\ \ r^{\rho}\int_0^{\infty} e^{-\eta}\eta^{\rho}d\eta \\
&= \frac{\Gamma(\sigma+1)}{p^{\sigma+1}}\ \frac{\Gamma(\nu+1)}{\lambda^{\nu+2}}\ \Gamma(\rho+1)r^{\rho},
\end{aligned}$$

where $\Gamma(.)$ is the Euler gamma function.

(5). If $f(x,y,t) = e^{(ax+by+ct)}$, then

$$\begin{aligned}
L_xA_yS_t[f(x,y,t)] &= \frac{1}{\lambda r}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(px+\lambda y+\frac{t}{r})}e^{(ax+by+ct)}\ dxdydt \\
&= \frac{1}{\lambda(p-a)(\lambda-b)(1-cr)}.
\end{aligned}$$

(6). If $f(x,y,t) = \sinh(ax+by+ct)$ or $\cosh(ax+by+ct)$.

Recall that

$$\sinh(ax+by+ct) = \frac{e^{(ax+by+ct)} - e^{-(ax+by+ct)}}{2},\ \ \cosh(ax+by+ct) = \frac{e^{(ax+by+ct)} + e^{-(ax+by+ct)}}{2}.$$

Therefore,

$$\begin{aligned}
L_xA_yS_t[\cosh(ax+by+ct)] &= \frac{p\lambda + ab + bcpr + ac\lambda r}{\lambda(p^2-a^2)(\lambda^2-b^2)(1-c^2r^2)}, \\
L_xA_yS_t[\sinh(ax+by+ct)] &= \frac{bp + a\lambda + cp\lambda r + abcr}{\lambda(p^2-a^2)(\lambda^2-b^2)(1-c^2r^2)}.
\end{aligned}$$

(7). If $f(x,y,t) = e^{i(ax+by+ct)}$, then

$$\begin{aligned}
L_xA_yS_t[f(x,y,t)] &= \frac{1}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+\lambda y+\frac{t}{r})}e^{i(ax+by+ct)}\,dxdydt\\
&= \frac{1}{\lambda(p-ia)(\lambda-ib)(1-icr)}\\
&= \frac{(p\lambda - bcpr - ac\lambda r - ab) + i(cp\lambda r + bp + a\lambda - abcr)}{\lambda(p^2+a^2)(\lambda^2+b^2)(1+c^2r^2)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
L_xA_yS_t[\cos(ax+by+ct)] &= \frac{p\lambda - bcpr - ac\lambda r - ab}{\lambda(p^2+a^2)(\lambda^2+b^2)(1+c^2r^2)},\\
L_xA_yS_t[\sin(ax+by+ct)] &= \frac{cp\lambda r + bp + a\lambda - abcr}{\lambda(p^2+a^2)(\lambda^2+b^2)(1+c^2r^2)}.
\end{aligned}$$

(8). If $f(x,y,t) = f_1(x)f_2(y)f_3(t)$, then

$$\begin{aligned}
L_xA_yS_t[f(x,y,t)] &= \frac{1}{\lambda r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+\lambda y+\frac{t}{r})}\Big(f_1(x)f_2(y)f_3(t)\Big)\,dxdydt\\
&= \int_0^\infty e^{-px}f_1(x)\,\left\{\frac{1}{\lambda}\int_0^\infty e^{-\lambda y}f_2(y)\,\left\{\frac{1}{r}\int_0^\infty e^{-\frac{t}{r}}f_3(t)dt\right\}dy\right\}dx\\
&= L_x[f_1(x)]\;A_y[f_2(y)]\;S_t[f_3(t)].
\end{aligned}$$

Therefore,

$$\begin{aligned}
L_xA_yS_t[\cos(ax)\cos(by)\cos(ct)] &= \frac{p}{(p^2+a^2)}\frac{1}{(\lambda^2+b^2)}\frac{1}{(1+c^2r^2)},\\
L_xA_yS_t[\sin(ax)\sin(by)\sin(ct)] &= \frac{a}{(p^2+a^2)}\frac{b}{\lambda(\lambda^2+b^2)}\frac{cr}{(1+c^2r^2)}.
\end{aligned}$$

4.4.3 Applications of Triple Laplace-Aboodh-Sumudu Transform

In this section, we apply the triple Laplace-Aboodh-Sumudu transform operator for solving some kinds of linear partial differential equations.

Example 4.4.3.1. Consider the following boundary Laplace equation

$$U_{xx}(x,y,t)+U_{yy}(x,y,t)+U_{tt}(x,y,t)=0, \qquad (x,y,t)\in\mathbb{R}^3_+, \tag{4.4.13}$$

subject to the conditions

$$\begin{cases} U(0,y,t)=0, & U_x(0,y,t)=\sin y\ \sinh\sqrt{2}t,\\ U(x,0,t)=0, & U_y(x,0,t)=\sin x\ \sinh\sqrt{2}t,\\ U(x,y,0)=0, & U_t(x,y,0)=\sqrt{2}\sin x\ \sin y. \end{cases} \tag{4.4.14}$$

Solution:

Applying triple Laplace-Aboodh-Sumudu transform to the equation gives and by linearity property and partial derivative properties of triple Laplace-Aboodh-Sumudu transform, we get

$$\begin{aligned} 0 &= p^2F(p,\lambda,r)-pA_yS_t[U(0,y,t)]-A_yS_t[U_x(0,y,t)]\\ &+ \lambda^2F(p,\lambda,r)-L_xS_t[U(x,0,t)]-\frac{1}{\lambda}L_xS_t[U_y(x,0,t)]\\ &+ \frac{1}{r^2}F(p,\lambda,r)-\frac{1}{r^2}L_xA_y[U(x,y,0)]-\frac{1}{r}L_xA_y[U_t(x,y,0)]. \end{aligned} \tag{4.4.15}$$

Substituting

$$\begin{aligned} A_yS_t[U_x(0,y,t)] &= \frac{\sqrt{2}r}{\lambda(\lambda^2+1)(1-2r^2)},\ L_xS_t[U_y(x,0,t)]=\frac{\sqrt{2}r}{(p^2+1)(1-2r^2)},\\ L_xA_y[U_t(x,y,0)] &= \frac{\sqrt{2}}{\lambda(p^2+1)(\lambda^2+1)}, \end{aligned}$$

in equation (4.4.15) and simplifying, we get

$$\begin{aligned} F(p,\lambda,r) &= \frac{r^2}{(p^2r^2+\lambda^2r^2+1)}\left\{\frac{\sqrt{2}(p^2r^2+\lambda^2r^2+1)}{\lambda r(p^2+1)(\lambda^2+1)(1-2r^2)}\right\}\\ &= \frac{\sqrt{2}r}{\lambda(p^2+1)(\lambda^2+1)(1-2r^2)}. \end{aligned} \tag{4.4.16}$$

Taking the inverse of triple Laplace-Aboodh-Sumudu transform, we get

$$\begin{aligned} U(x,y,t) &= L_x^{-1}A_y^{-1}S_t^{-1}\Big[\frac{\sqrt{2}r}{\lambda(p^2+1)(\lambda^2+1)(1-2r^2)}\Big] \\ &= \sin x \sin y \sinh \sqrt{2}t. \end{aligned} \tag{4.4.17}$$

Which is the required solution of the considered Laplace equation.

Example 4.4.3.2. Consider the following Poisson partial differential equation

$$U_{xx}(x,y,t)+U_{yy}(x,y,t)+U_{tt}(x,y,t) = 2\sin x\cos y \sinh 2t,\ (x,y,t)\in\mathbb{R}^3_+, \tag{4.4.18}$$

subjected to the conditions

$$\begin{cases} U(0,y,t)=0, & U_x(0,y,t)=\cos y\sinh 2t, \\ U(x,0,t)=\sin x\sinh 2t, & U_y(x,0,t)=0, \\ U(x,y,0)=0, & U_t(x,y,0)=2\sin x\cos y. \end{cases} \tag{4.4.19}$$

Solution:

Applying triple Laplace-Aboodh-Sumudu transform on both sides of equation (4.4.18) and by using properties of triple Laplace-Aboodh-Sumudu transform, then we have

$$\begin{aligned} (p^2+\lambda^2+\frac{1}{r^2})F(p,\lambda,r) &= pA_yS_t[U(0,y,t)]+A_yS_t[U_x(0,y,t)]+L_xS_t[U(x,0,t)] \\ &+ \frac{1}{\lambda}L_xS_t[U_y(x,0,t)]+\frac{1}{r^2}L_xA_y[U(x,y,0)]+\frac{1}{r}L_xA_y[U_t(x,y,0)] \\ &+ \frac{4r}{(p^2+1)(\lambda^2+1)(1-4r^2)}, \end{aligned} \tag{4.4.20}$$

where

$$L_xA_yS_t[2\sin x\cos y\sinh 2t] = \frac{4r}{(p^2+1)(\lambda^2+1)(1-4r^2)}.$$

Substituting

$$\begin{aligned} A_y S_t[U_x(0,y,t)] &= \frac{2r}{(\lambda^2+1)(1-4r^2)}, \qquad L_x S_t[U(x,0,t)] = \frac{2r}{(p^2+1)(1-4r^2)}, \\ L_x A_y[U_t(x,y,0)] &= \frac{2}{(p^2+1)(\lambda^2+1)}, \end{aligned}$$

in equation (4.4.20), we obtain

$$\begin{aligned} (p^2+\lambda^2+\frac{1}{r^2})F(p,\lambda,r) &= \Big\{\frac{4r}{(p^2+1)(\lambda^2+1)(1-4r^2)} + \frac{2r}{(\lambda^2+1)(1-4r^2)} \\ &+ \frac{2r}{(p^2+1)(1-4r^2)} + \frac{2}{r(p^2+1)(\lambda^2+1)}\Big\}. \end{aligned} \tag{4.4.21}$$

After some simple algebraic operations, we get

$$\begin{aligned} F(p,\lambda,r) &= \frac{r^2}{(p^2r^2+\lambda^2r^2+1)}\Big\{\frac{2(p^2r^2+\lambda^2r^2+1)}{r(p^2+1)(\lambda^2+1)(1-4r^2)}\Big\} \\ &= \frac{2r}{(p^2+1)(\lambda^2+1)(1-4r^2)}. \end{aligned} \tag{4.4.22}$$

Taking $L_x^{-1}A_y^{-1}S_t^{-1}$ for equation (4.4.22), we get

$$\begin{aligned} U(x,y,t) &= L_x^{-1}A_y^{-1}S_t^{-1}\Big[\frac{2r}{(p^2+1)(\lambda^2+1)(1-4r^2)}\Big] \\ &= \sin x \cos y \sinh 2t. \end{aligned} \tag{4.4.23}$$

Which is the required solution of Poisson equation (4.4.18).

Example 4.4.3.3. Consider the following nonhomogeneous heat equation

$$U_t(x,y,t) = U_{xx}(x,y,t) + U_{yy}(x,y,t) + 2\cos(x+y), \quad (x,y) \in \mathbb{R}^2_+, \quad t > 0, \tag{4.4.24}$$

subject to the boundary and initial conditions

$$\begin{aligned} U(0,y,t) &= e^{-2t}\sin y+\cos y, \qquad U_x(0,y,t)=e^{-2t}\cos y-\sin y, && (4.4.25)\\ U(x,0,t) &= e^{-2t}\sin x+\cos x, \qquad U_y(x,0,t)=e^{-2t}\cos x-\sin x, && (4.4.26)\\ U(x,y,0) &= \sin(x+y)+\cos(x+y). && (4.4.27)\end{aligned}$$

Solution:

Applying triple Laplace-Aboodh-Sumudu transform on both sides of Eq. (4.4.24), we have

$$L_xA_yS_t[U_t(x,y,t)] = L_xA_yS_t[U_{xx}(x,y,t)+U_{yy}(x,y,t)+2\cos(x+y)]. \quad (4.4.28)$$

By linearity property and partial derivative properties of triple Laplace-Aboodh-Sumudu transform, we get

$$\begin{aligned}\frac{1}{r}F(p,\lambda,r)-\frac{1}{r}L_xA_y[U(x,y,0)] &= p^2F(p,\lambda,r)-pA_yS_t[U(0,y,t)]-A_yS_t[U_x(0,y,t)]\\ &+ \lambda^2F(p,\lambda,r)-L_xS_t[U(x,0,t)]-\frac{1}{\lambda}L_xS_t[U_y(x,0,t)]\\ &+ \frac{2(p\lambda-1)}{\lambda(p^2+1)(\lambda^2+1)}, \qquad (4.4.29)\end{aligned}$$

where

$$L_xA_yS_t[2\cos(x+y)] = \frac{2(p\lambda-1)}{\lambda(p^2+1)(\lambda^2+1)}.$$

Rearranging the terms, we have

$$\begin{aligned}F(p,\lambda,r) &= \frac{r}{(p^2r+\lambda^2r-1)}\Bigg\{pA_yS_t[U(0,y,t)]+A_yS_t[U_x(0,y,t)]+L_xS_t[U(x,0,t)]\\ &+ \frac{1}{\lambda}L_xS_t[U_y(x,0,t)]-\frac{1}{r}L_xA_y[U(x,y,0)]-\frac{2(p\lambda-1)}{\lambda(p^2+1)(\lambda^2+1)}\Bigg\} \quad (4.4.30)\end{aligned}$$

Using double Aboodh-Sumudu transform for equations (4.4.25), double Laplace-Sumudu transform for equations (4.4.26) and Double Laplace-Aboodh transform for equation (4.4.27), we obtain

$$\begin{aligned}
A_y S_t[U(0,y,t)] &= \frac{1}{\lambda(\lambda^2+1)(1+2r)} + \frac{1}{(\lambda^2+1)}, && (4.4.31)\\
A_y S_t[U_x(0,y,t)] &= \frac{1}{(\lambda^2+1)(1+2r)} - \frac{1}{\lambda(\lambda^2+1)}, && (4.4.32)\\
L_x S_t[U(x,0,t)] &= \frac{1}{(p^2+1)(1+2r)} + \frac{p}{(p^2+1)}, && (4.4.33)\\
L_x S_t[U_y(x,0,t)] &= \frac{p}{(p^2+1)(1+2r)} - \frac{1}{(p^2+1)}, && (4.4.34)\\
L_x A_y[U(x,y,0)] &= \frac{p+\lambda}{\lambda(p^2+1)(\lambda^2+1)} + \frac{p\lambda-1}{\lambda(p^2+1)(\lambda^2+1)}. && (4.4.35)
\end{aligned}$$

Substitute equations (4.4.31)-(4.4.35) into equation (4.4.30) and simplify to obtain

$$\begin{aligned}
F(p,\lambda,r) &= \frac{r}{(p^2r+\lambda^2r-1)}\left\{\frac{(p+\lambda)(p^2r+\lambda^2r-1)}{\lambda r(p^2+1)(\lambda^2+1)(1+2r)} + \frac{(p\lambda-1)(p^2r+\lambda^2r-1)}{\lambda r(p^2+1)(\lambda^2+1)}\right\}\\
&= \frac{p+\lambda}{\lambda(p^2+1)(\lambda^2+1)(1+2r)} + \frac{p\lambda-1}{\lambda(p^2+1)(\lambda^2+1)}. \qquad (4.4.36)
\end{aligned}$$

Taking the inverse triple Laplace-Aboodh-Sumudu transform of equation (4.4.36), we get

$$\begin{aligned}
U(x,y,t) &= L_x^{-1}A_y^{-1}S_t^{-1}\Big[\frac{p+\lambda}{\lambda(p^2+1)(\lambda^2+1)(1+2r)} + \frac{p\lambda-1}{\lambda(p^2+1)(\lambda^2+1)}\Big]\\
&= e^{-2t}\sin(x+y) + \cos(x+y). \qquad (4.4.37)
\end{aligned}$$

Which is the desired solution of (4.4.24).

CONCLUSION

We solved linear partial differential equations by the double Laplace-Aboodh transform and the double Laplace-Shehu transform. Also, we showed properties of these integrative transforms and proving theorems and their transforms for some basic functions. Therefore, we presented the triple Laplace-Aboodh-Sumudu transform with some of its basic properties and used it in solving linear partial differential equations.

At the end of this thesis, we suggest that researchers continue to research and develop new methods for solving linear and nonlinear partial differential equations, due to the importance of these equations and their multiple applications in various aspects of science.

Bibliography

[1] A. Abd Aswhad and M. M. Helal, *A general solution of some linear partial differential equations via two integral transforms*, Int. J. Nonlinear Anal. Appl., 13(1)(2022) 2087-2093.

[2] K. S. Aboodh, *The new integral transform "Aboodh transform"*, Global Journal of Pure and Applied Mathematics, 9(1)(2013) 35-43.

[3] K. S. Aboodh, A. Idris and R. I. Nuruddeen, *On the Aboodh transform connections with some famous integral transforms*, International Journal of Engineering and Information Systems, 1(9)(2017) 143-151.

[4] H. J. Abood and A. H. Hussain, *Existence and uniqueness linear partial differential equations depending initial and boundary conditions*, Journal of Asian Scientific Research, 3(8)(2013) 786-799.

[5] F. Accioa, F. Tommasoa and O. NajouaSiara, *Solving Poisson equation with Dirichlet conditions through multinode shepard operators*, elsevier, Computers and Mathematics with Applications, 98(2021) 254-260.

[6] S. Aggarwal, A.Singh, A. Kumar and N. Kumar, *Application of Laplace transform for solving improper integrals whose integrand consisting error function*, Journal of Advanced Research in Applied Mathematics and Statistics, 4(2)(2019) 1-7.

[7] S. Aggarwal and A. R. Gupta, *Sumudu transform for the solution of Abel's integral equation*, JETIR, 6(4)(2019) 423-431.

[8] S. Ahmed, T. Elzaki, M. Elbadri and M. Mohamed, *Solution of partial differential equations by new double integral transform (Laplace-Sumudu transform)*, Ain Shams Engineering Journal, 12(2021) 4045-4049.

[9] A. Al-Aati and Y. Ouideen, *Solving the heat and wave equations using the double Aboodh-Sumudu transform*, Journal of Albaydha University, 4(1)(2022) 173-183.

[10] A. Albukhuttar, B. Jubear and M. Neamah, *Solve the Laplace, Poisson and Telegraph equations using the Shehu transform*, Turkish Journal of Computer and Mathematics Education, 10(12)(2021) 1759-1768.

[11] A. Albukhuttar, Z. Ridha and H. Kadhim, *Applications of a Shehu transform to the heat and transport equations*, International Journal of Psychosocial Rehabilitation, 24(5)(2020) 4254-4263.

[12] S. Alfaqeih and E. Misirli, *On double Shehu transform and its properties with applications*, International Journal of Analysis and Applications, 3(2020) 381-395.

[13] S. Alfaqeih and T. Ozis, *First Aboodh transform of fractional order and its properties*, International Journal of Progressive Sciences and Technologies, 13(2)(2019) 252-256.

[14] S. Alfaqeih and T. Ozis, *Note on double Aboodh transform of fractional order and its properties*, International Journal of Engineering and Information Systems, 3(5)(2019) 34-42.

[15] S. Alfaqeih and T. Ozis, *Note on triple Aboodh transform and its application*, International Journal of Engineering and Information Systems, 3(3)(2019) 41-50.

[16] S. Alfaqeih and T. Ozis, *Note on double Aboodh transform of fractional order and its properties*, OMJ, 1(1) 1-14.

[17] S. Alkaleeli, A. Mtawal and M. Hmad, *Triple Shehu transform and its properties with applications*, African Journal of Mathematics and Computer, 14(1)(2021) 4-12.

[18] R. M. H. Alshmary, *Applications of partial differentail equations*, Journal of Physics: Conference Series, 1591(2020) 1–10.

[19] G. A. Articolo, Partial Differential Equations and Boundary Value Problems with Maple, Academic Press is an imprint of Elsevier, 2^{nd} ed., USA, 2009.

[20] N. H. Asmar, Partial Differential Equations with Fourier Series and Boundary Value Problems, 2^{nd} ed., Pearson Education, Inc., 2005.

[21] A. Atangana, *A note on the triple Laplace transform and its applications to some kind of third-order differential equation*, Abstract and Applied Analysis, Hindawi, (2013) 1-10.

[22] F. B. M. Belgacem and A. A. Karaballi, Sumudu Transform Fundamental Properties Investigations and Applications, Journal of Applied Mathematics and Stochastic Analysis, (2006).

[23] A. Bokharia, D. Baleanub and R. Belgacema, *Application of Shehu transform to Atangana-Baleanu derivatives*, Journal of Mathematics and Computer Science, 20 (2020) 101-107.

[24] H. Brezis and F. Browder, *Partial differential equations in the 20th Century*, Advances in Mathematics, 135(1998) 76-144.

[25] D. Caratelli, P. E. Ricci and J. Gielis, *The Robin problem for the Laplace equation in a three-dimensional starlike domain*, Applied Mathematics and Computation, 218(3)(2011) 713-719.

[26] R. Chaudhary, S. D. Sharma, N. Kumar and S. Aggarwal, *Connections between Aboodh transform and some effective integral transforms*, International Journal of Innovative Technology and Exploring Coefficients Engineering, 9(1)(2019) 1465-1470.

[27] D. Cioranescu, P. Donato and M. P. Roque, An Introduction to second Order Partial Differential Equations Classical and Variational Solutions, University of CaliforniaWorld Scientific Publishing Co. Pte. Ltd., USA, 2018.

[28] C. Constanda, Solution Techniques for Elementary Partial Differential Equations, The University of Tulsas, 3^{rd} ed., USA, 2016.

[29] C. Constanda, *Localized method of fundamental solutions for solving two-dimensional Laplace and biharmonic equations*, Engineering Analysis with Boundary Elements, 101(2019) 188-197.

[30] M. P. Coleman, An Introduction to Partial Differential Equations with MATLAB, 2^{nd} ed., Taylor and Francis Group, USA, 2013.

[31] L. Debnath, *The double Laplace transforms and their properties with applications to functional, integral and partial differential equations*, Int. J. Appl. Comput. Math., 2(2016) 223-241.

[32] R. Dhunde and G. Waghmare, *Double Laplace transform method in mathematical physics*, International Journal of Theoretical and Mathematical Physics, 7(1)(2017) 14-20.

[33] H. Eltayeb and A. Kilicman, *A note on solutions of wave, Laplace's and heat equations with convolution terms by using a double Laplace transform*, Applied Mathematics Letters, 21(12)(2008) 1324-1329.

[34] L. C. Evance, Partial Differential Equations, American Mathematical Society, USA, 1997.

[35] S. J. Farlow, Partial Differential Equations for Scientists and Engineers, The university of Maine, New York, 1993.

[36] V. Grigoryan, Partial Differential Equations, University of California, USA, 2010.

[37] Q. Han, A Basic Course in Partial Differential Equations, American Mathematical Society, USA, 2011.

[38] Q. Han and F. Lin, Elliptic Partial Differential Equations, $2^{nd}ed.$, American Mathematical Society, USA, 2011.

[39] E. I. Hassan, *Solution of Laplace's equation using numerical methods*, American Scientific Research Journal for Engineering, Technology, and Sciences, 16(1)(2016) 178-192.

[40] J. Hoskins and M. Rachh , *On the discretization of Laplace's equation with Neumann boundary conditions on polygonal domains*, Journal of Computational Physics, 8(2020) 1-24.

[41] F. J. Jing, H. Ling and Y. Shi-Jie, *Solutions of Laplace equation in n-dimensional spaces*, Communications in Theoretical Physics, 56(4)(2011) 623-625.

[42] S. Kahlaf, A. Mhassin, *Numerical Solution of a Two-Dimensional Laplace Equation with Dirichlet Boundary Conditions*, Iraq Academics Syndicate International Conference for Pure and Applied Sciences,1818(2021) 012193.

[43] K. Khader, Laplace's Equation, The Nonlinear Poisson Equation and the Effects of Gaussian White Noise on The Boundary, Ph.D. Thesies in Math., Utah Univ., USA, 2010.

[44] R. F. Khalaf and F. B. M. Belgacem, Extraction of the Laplace, Fourier, and Mellin Transforms from the Sumudu transform, AIP Puplishing, 2014.

[45] S. B. Kiwne, S. M. Sonawane, *On the Applications of Laplace - Aboodh Transforms in Engineering Field and Comparison with Sumudu Transform*, Global Journal of Pure and Applied Mathematics, 15 (6)(2019) 1079-1101.

[46] M. Kumar and G. Mishra, *An introduction to numerical methods for the solutions of partial differential equations*, Applied Mathematics, 2(2011) 1327-1338.

[47] K. Lakmon and Y. Mensah, *The Plancherel formula for the Shehu transform*, Annals of Pure and Applied Mathematical Sciences, 1(1)(2021) 14-19.

[48] T. Lin, W. Hub and C. Misbahc, A direct Poisson solver in spherical geometry with an application to diffusiophoretic problems, Journal of Computational Physics, Elsevier, 409, (2020).

[49] S. Maitama and W. Zhao, *New integral transfrom: Shehu transform ageneralizattion of Sumudu and Laplace transfrom for solving differential equation*, International Journal of Analysis and Applications, 17(2)(2019) 167-190.

[50] B. Mebrate and P. R. Koya, *Numerical Solution of a Two Dimensional Poisson Equation with Dirichlet Boundary Conditions*, American Journal of Applied Mathematics, 3(6)(2015) 297-304.

[51] M. S. Mechee and A.J. Naeemah, A Study of Triple Sumudu Transform for Solving Partial Differential Equations with Some Applications, Multidisciplinary European Academic Journal, 2(2)(2020) 1-15.

[52] M. S. Mechee and A. J. Naeemah, *A study of double Sumudu transform for solving differential equations with some applications*, International Journal of Engineering and Information Systems, 4(1)(2020) 20-27.

[53] M. Mohand, K. S. Aboodh and A. Abdelbagy, *On the solution of srdinary differential equation with variable coefficients using Aboodh transform*, Advances in Theoretical and Applied Mathematics, 11(4)(2016) 383-389.

[54] O. D. Momoh, M. N. Sadiku and C. M. Akujuobi,*Solution of axisymmetric potential problem in spherical coordinates using exodus method*, PIERS Proceedings, Cambridge, USA,(2014) 1-6.

[55] S. Mottin, The Solution of the Laplace Equation with The Robin Boundary Conditions: Applications to Inverse Problem, University of Lyon, France, 2004.

[56] T. Myint-U and L. Debnath, Linear Partial Differential Equation for Scientists and Engineers, Birkh$\ddot{a}$user, 4^{th}ed., 2007.

[57] R. K. Nagle, E. B. Saff and A. D. Snider, Fundamentals of Differential Equations and Boundary Value Problems, 6^{th} ed., Pearson Education, Inc., USA, 2012.

[58] G. Nhawu, P. Mafuta and J. Mushanyu, *The adomian decomposition method for numerical solution of first-order differential equations*, J. Math. Comput. Sci., 6(3)(2016) 307-314.

[59] P. J. Olever, Introduction to Partial Differential Equations, Springer International Publishing Switzerland, Switzerland, 2014.

[60] M. A. Pinsky, *Partial Differential Equations and Boundary Value Problems with Applications*, 3^{rd}., American Mathematical Society, USA, 2011.

[61] D. Poltem and S. Srimongkol, *A note on computational method for Shehu transform by Adomian decomposition method*, Advances in Mathematics: Scientific Journal, 10(2)(2021) 713-721.

[62] R. Precup, *Linear and Semilinear Partial Differential Equations An Introduction*, Walter de Gruyter GmbH, Germany, 2013.

[63] Z. Qian, C. Fu and X. Xiong, *Fourth-order modified method for the Cauchy problem for the Laplace equation*, Journal of Computational and Applied Mathematics, 192(2006) 205-218.

[64] A. Raeli, Solution of the variable coefficients Poisson equation on Cartesian hierarchical meshes in parallel : applications to phase changing materials, Ph.D.Thesis in math., France, 2017.

[65] K. Rao, *Introduction to Partial Differential Equations*, The University of Tulsa, PHI Learning Private Limited, 3^{rd} ed., New Delhi, 2011.

[66] P. Redhu, Partial Differential Equations, M.Sc. Thesis in math., Maharshi Dayanand Univ., India, 2021.

[67] Z. Ridha, A. ALbukhuttar and H. Kadhim, *Differentiation and integration of Shehu transform*, Applied Mathematics, 144(2020) 54553-54560.

[68] P. Rockstroh, Boundary Value Problems for the Laplace Equation on Bounded Convex Domains with Analytic Boundary, Ph.D.Thesis in math., Cambridge Univ. UK, 2018.

[69] H. Schlichtkrull, Boundary Value Problems for Partial Differential Equations, University of American Mathematical Society, USA, 2013.

[70] A. Sedeeg and M. Mahgoub, Comparison of new integral transform Aboodh transform and adomian decomposition method, Int. J. Math. And App., 4(2)(2016) 127-135.

[71] Y. Shao and O. Faltinsen, *A harmonic polynomial cell (HPC) method for 3D Laplace equation with application in marine hydrodynamics*, Journal of Computational Physics, 274(2014) 312-332.

[72] H. Singh, Study on Elliptic Partial Differential Equations, M.Sc. Thesis in math., National Institute of Technology, Rourkela, 2015.

[73] I. N. Sneddon, Elements of Partial Differential Equations, McGraw-Hill, USA, 2006.

[74] A. D. Snider, Fundamentals of Differential Equations and Boundary Value Problems, 6^{th} ed., Pearson Education, Inc., 2012.

[75] I. P. Stavroulakis and S. A. Tersian, Partial Differential Equations: An Introduction with Mathematica and Maple, 2^{nd} ed., World Scientific Publishing Co. Re. Ltd, USA, 2004.

[76] A. K. Thakur and S. Panda, *Some properties of triple Laplace transforms*, Journal of Mathematics and Computer Applications Research, 2(1)(2015) 23-28.

[77] T. G. Thange and A. R. Gade, *On Aboodh transform for fractional differential operator*, Malaya Journal of Matematik, 8(1)(2020) 225-229.

[78] W. F. Trench, Elementary Differential Equations with Boundary Value Problems, Trinity University, USA, 2013.

[79] J. Vashi and M. G. Timol, *Laplace and Sumudu Transforms and Their Application*, Iternational Journal of Innovative Science, Engineering and Techology, 3(8)(2016) 538-542.

[80] D. Verma, *Applications of Laplace transformation for solving various differential equations with variable*, International Journal for Innovative Research in Science and Technology, 4(11)(2018) 124-127.

[81] A. Wazwaz, Partial Differential Equations and Solitary Waves Theory, Springer, USA, 2009.

[82] X. Yu, *Research on Separation Variable Method in Mathematical Physics Equation*, Advances in Engineering Research, 123(2017) 62-65.

[83] H. L. Zhu, *General solutions of the Laplace equation*, Elsevier, (2022) 1-13.

List of Publications

1. M. Hunaiber and A. Al-Aati, *New double transform to solve some partial differential equations*, Albaydha University Journal, 3(3)(2021) 181-190.

2. M. Hunaiber and A. Al-Aati, *The double Laplace-Aboodh transform and their properties with applications*, global scientific journals, 10(3)(2022) 2055-2065.

3. M. Hunaiber, *The new integral transform "Hunaiber transform"*, International Journal of Innovation Scientific Research and Review, 4(4)(2022) 2619-2619.

4. M. Hunaiber and A. Al-Aati, *Some fundamental properties of Hunaiber transform and its applications to partial differential equations*, International Journal of Mathematics and Computer Research, 10(7)(2022) 2808-2811.

5. A. Al-Aati, M. Hunaiber and Y. Ouideen, *On triple Laplace-Aboodh-Sumudu transform and its properties with pplications*, Journal of Applied Mathematics and Computation, 6(3)(2022) 290-309.

6. M. Hunaiber, A. Al-Aati, *Double Sumudu-Kamal transform with applications*, in: The third scientific conference of Al Baydha university, Albaydha University Journal, Yemen, 2022, pp. 1053-1062.