New Version of Inverse Lindley Distribution and its Properties and Applications

Abdulkareem M. Basheer
Faculty of Administrative Sciences, Albaydha University, Albaydha, Yemen.
abdulkareembasheer@baydaauniv.net

Abstract
The inverse Lindley distribution has been generalized by many authors in recent years. Here, we introduce a new generalization called modified alpha power transformed inverse Lindley (MAPTIL) distribution that provides better fits than the inverse Lindley distribution and some of its known distributions. Various probability properties of the proposed distribution, including hazard rate function, moments, conditional moments, mean residual lifetime, entropy, stress strength model and order statistics are derived. The unknown parameters of the proposed distribution are estimated by the maximum likelihood estimation method. A simulation study is carried out to examine the mean square error and bias of maximum likelihood estimators. Finally, two applications data are utilized to illustrate the model's flexibility.
Keywords: *Modified alpha power transformation, Inverse Lindley distribution, Moments, Stress strength model, Maximum likelihood estimation, Simulation.*

نسخة جديدة من توزيع لندلي العكسي وخصائصه وتطبيقاته

الملخص
تم تعميم توزيع لندلي العكسي من قبل العديد من المؤلفين في السنوات الأخيرة. هنا، نقدم تعميمًا جديدًا يسمى توزيع لندلي العكسي المعدل باستخدام القوة ألفا (MAPTIL) الذي يوفر ملاءمة أفضل من توزيع لندلي العكسي وبعض توزيعاته المعروفة. تم اشتقاق مختلف الخواص الاحتمالية للتوزيع المقترح، بما في ذلك دالة معدل الخطر، والعزوم، والعزوم المشروطة، ومتوسط العمر المتبقي، والانتروبي، ونموذج القوة والاجهاد، والإحصاءات الترتيبية. تم تقدير المعلمات غير المعروفة للتوزيع المقترح باستخدام طريقة تقدير الإمكان الأعظم. تم إجراء دراسة محاكاة لفحص متوسط الخطأ المربعي والتحيز لمقدرات الإمكان الأعظم. وأخيراً، تم استخدام بيانات تطبيقين لتوضيح مرونة النموذج.
الكلمات الافتتاحية: تحويل قوة الفا المعدل، توزيع لندلي العكسي، العزوم، نموذج القوة – الاجهاد، تقدير الإمكان الأعظم، المحاكاه.

1 Introduction

Sharma et al. [18] proposed a one-parameter lifetime distribution, known as inverse Lindley (IL) distribution. The cumulative distribution function (cdf) and probability density function (pdf) of IL distribution are provided, respectively, as

$$G(y) = [1 + \frac{\theta}{(\theta+1)y}]e^{-\frac{\theta}{y}}; y > 0, \theta > 0, \tag{1.1}$$

and

$$g(y) = \frac{\theta^2}{\theta+1}(\frac{y+1}{y^3})e^{-\frac{\theta}{y}}; y > 0, \theta > 0. \tag{1.2}$$

The inverse Lindley distribution (IL) is a two component mixture of special case of the inverse gamma $(2, \theta)$ and the inverse exponential (θ) distributions with the mixing proportions $\frac{1}{\theta+1}$ and $\frac{\theta}{\theta+1}$.

In the recent past, many generalizations of inverse Lindley distribution have been studied by authors such as: three parameters inverse Lindley by Alkarni [1], the generalized inverse Lindley by Sharma et al. [19], extension of inverse Lindley by Sharma et al. [20], power inverse Lindley by Barco et al. [4], alpha power transformed inverse Lindley by Dey et al. [10] and logarithmic inverse Lindley by Eltehiwy [12].

On the other hand, Mahdavi et al. [13] suggested a new approach to present an additional parameter to a class of distributions for more additional flexibility. The offered technique is named alpha power transformation (APT). They studied the main properties of the APT method and introduced an extension to the exponential distribution using the new approach. Many authors used the same technique to introduce new generalizations of some well-known distribution. For example, alpha power Weibull distribution by Nassar et al. [15], alpha power inverse Weibull distribution by Basheer et al. [5, 6, 7, 8], alpha power transformed inverse Lindley distribution by Dey [10] and alpha power Gompertz distribution by Eghwerido [11]. To add more flexibility to the APT method, Alotaibi et al.[2] proposed a new form of the APT method which is called the modified APT (MAPT) method. The CDF and the PDF of the MAPT method are, respectively, given by

$$F_{_{MAPT}}(y) = \begin{cases} \frac{\alpha^{G(y)}-1}{(\alpha-1)(1+\alpha-\alpha^{G(y)})}, & \alpha > 0, \alpha \neq 1 \\ \\ G(y), & \alpha = 1, \end{cases} \tag{1.3}$$

and the corresponding probability density function (PDF) as

$$f_{_{MAPT}}(y) = \begin{cases} \frac{\alpha \log(\alpha) g(y) \alpha^{G(y)}}{(\alpha-1)(1+\alpha-\alpha^{G(y)})^2}, & \alpha > 0, \alpha \neq 1 \\ \\ g(y), & \alpha = 1. \end{cases} \tag{1.4}$$

Since then, many scholars have introduced the APT extended models to enhance the lifetime data sets modelings and investigated their mathematical properties. Some examples are given as follows. Basheer et al. [9] studied modified alpha power Kumaraswamy, Alotaibi et al.[3] studied modified alpha power Weibull.

The main objectives of this study are to contribute to the statistical literature and address some issues about the components for various applications of the new model of the MAPT family. The following reasons are sufficient justification for doing so:

1- Introducing the modified alpha power transformed inverse Lindley distribution as a novel two-parameter model based on the MAPT family of distributions.

2- The new suggested model is very flexible.

3- It is possible to compute several statistical features, including the moments, moment generating function, incomplete moments, conditional moment, mean residual life and stress strength model, and so on.

4- The parameters of the MAPTIL distribution can be estimated by utilizing maximum likelihood estimation method.

The rest of the paper is organized as follows: In Section 2, we introduce MAPTIL distribution. In Section 3, some useful probability properties of MAPTIL distribution are discussed. The performance of maximum likelihood estimation method is discussed through simulation study in Section 4. The MAPTIL distribution is applied on real data set and the results are given in Section 5.

2 Modified Alpha Power Transformed Inverse Lindley Distribution

By inserting the CDF of IL distribution given by (1.1) in the CDF of MAPT distribution given by (1.3), we get the CDF of a new distribution denoted as MAPTIL $(y;\alpha,\theta)$ distribution given by

$$F_{MAPTIL}(y)=\begin{cases}\dfrac{\alpha^{[1+\frac{\theta}{(\theta+1)y}]e^{-\frac{\theta}{y}}}-1}{(\alpha-1)\left(1+\alpha-\alpha^{[1+\frac{\theta}{(\theta+1)y}]e^{-\frac{\theta}{y}}}\right)}, & y>0,\alpha>0,\alpha\neq1,\theta>0,\\[2ex] (1+\frac{\theta}{(\theta+1)y})e^{-\frac{\theta}{y}}, & y>0,\alpha=1,\theta>0,\end{cases} \tag{2.1}$$

its corresponding PDF is given by

$$f_{MAPTIL}(y)=\begin{cases}\dfrac{\alpha\log(\alpha)(\frac{\theta^2}{\theta+1})(\frac{1+y}{y^3})e^{-\frac{\theta}{y}}\alpha^{[1+\frac{\theta}{(\theta+1)y}]e^{-\frac{\theta}{y}}}}{(\alpha-1)\left(1+\alpha-\alpha^{[1+\frac{\theta}{(\theta+1)y}]e^{-\frac{\theta}{y}}}\right)^2}, & y>0,\alpha>0,\alpha\neq1,\theta>0,\\[2ex] (\frac{\theta^2}{\theta+1})(\frac{1+y}{y^3})e^{-\frac{\theta}{y}}, & y>0,\alpha=1,\theta>0.\end{cases} \tag{2.2}$$

By using the binomial expansion and the power series, we obtain a useful linear representation for the PDF (if $\alpha>0$, $\alpha\neq1$) as

$$f_{MAPTIL}(y)=\sum_{j,k=0}^{\infty}\sum_{m=0}^{k}\Upsilon_{j,k,m}\,y^{-(m+3)}(1+y)e^{-\frac{\theta(k+1)}{y}}, \tag{2.3}$$

where

$$\Upsilon_{j,k,m}=\frac{\alpha(\log\alpha)^{k+1}(j+1)^{k+1}\binom{k}{m}\theta^{m+2}}{(\alpha-1)(\alpha+1)^{2+j}(k)!(\theta+1)^{m+1}}.$$

When α =1, the MAPTIL distribution of (2.1) reduces to the IL distribution of (1.1).
Figure 1 gives graphical representation of PDF for different values of α and θ.

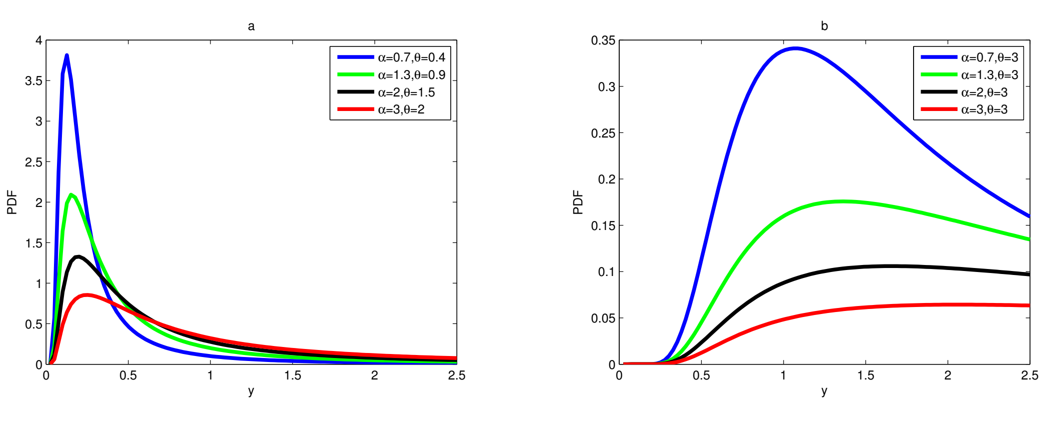

Figure 1: (a) and (b) Plots of the PDF of the MAPTIL distribution for some different values of parameters.

The reliability function of MAPTIL distribution is given by

$$R(y) = \frac{\alpha(\alpha - \alpha^{[1+\frac{\theta}{(\theta+1)y}]e^{-\frac{\theta}{y}}})}{(\alpha-1)(1+\alpha-\alpha^{[1+\frac{\theta}{(\theta+1)y}]e^{-\frac{\theta}{y}}})}. \tag{2.4}$$

The hazard rate function of MAPTIL distribution is given by

$$h(y) = \frac{(\frac{\theta^2}{\theta+1})(\frac{1+y}{y^3})e^{-\frac{\theta}{y}}\alpha^{[1+\frac{\theta}{(\theta+1)y}]e^{-\frac{\theta}{y}}}\log(\alpha)}{(1+\alpha-\alpha^{[1+\frac{\theta}{(\theta+1)y}]e^{-\frac{\theta}{y}}})(\alpha-\alpha^{[1+\frac{\theta}{(\theta+1)y}]e^{-\frac{\theta}{y}}})}. \tag{2.5}$$

Figure 2 gives graphical representation of HRF for different values of α and θ.

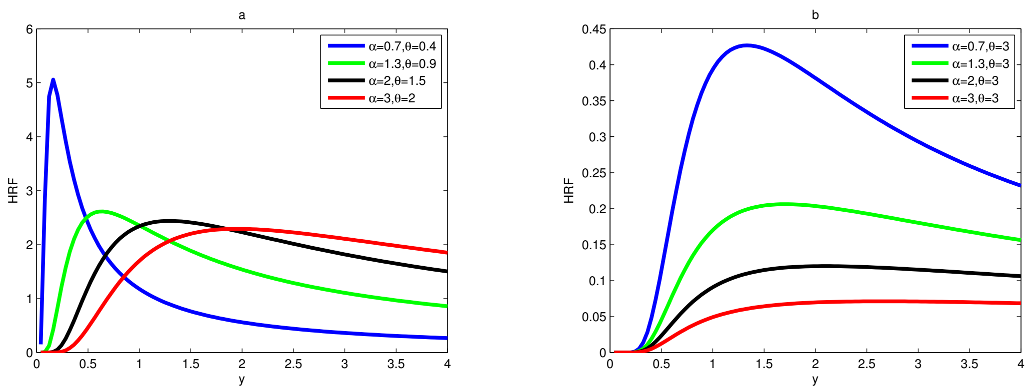

Figure 2: (a) and (b) Plots of the HRF of the MAPTIL distribution for some different values of parameters.

3 Probability Properties

We study some essential properties of the MAPTIL distribution in this section, such as moments, mean residual life, Rényi entropy, stress-strength model, and order statistics.

3.1 Moments

The r^{th} moments of the MAPTIL distribution is given by

$$\begin{aligned}
\acute{\mu_r} = E(Y^r) &= \int_0^\infty y^r f(y;\alpha,\theta)dy \\
&= \sum_{j,k=0}^{\infty}\sum_{m=0}^{k}\Upsilon_{j,k,m}\int_0^\infty y^r * y^{-(m+3)}(1+y)e^{-\frac{\theta(k+1)}{y}}dy \\
&= \sum_{j,k=0}^{\infty}\sum_{m=0}^{k}\Upsilon_{j,k,m}\int_0^\infty y^{r-m-3}(1+y)e^{-\frac{\theta(k+1)}{y}}dy,
\end{aligned}$$

to simplify the integration, we put $z = \frac{\theta(k+1)}{y}$

$$\begin{aligned}
E(Y^r) &= \sum_{j,k=0}^{\infty}\sum_{m=0}^{k}\Upsilon_{j,k,m}(\theta(k+1))^{r-m-2}\int_0^\infty z^{m+1-r}(1+\theta(k+1)z^{-1})e^{-z}dz \\
&= \sum_{j,k=0}^{\infty}\sum_{m=0}^{k}\Upsilon_{j,k,m}(\theta(k+1))^{r-m-2}(1+m-r+\theta(k+1))(m-r)!, \qquad m \geq r.
\end{aligned} \tag{3.1}$$

In particular, the first two moments can be obtained as

$$\acute{\mu}_1 = E(Y) = \sum_{j,k=0}^{\infty} \sum_{m=0}^{k} \Upsilon_{j,k,m} (\theta(k+1))^{-m-1} (m + \theta(k+1))(m-1)!,$$

and

$$\acute{\mu}_2 = E(Y^2) = \sum_{j,k=0}^{\infty} \sum_{m=0}^{k} \Upsilon_{j,k,m} (\theta(k+1))^{-m} (m - 1 + \theta(k+1))(m-2)!.$$

The following formulae can also be used to obtain the mean, variance, skewness and kurtosis:

$$Mean = \acute{\mu}_1 = \mu,$$

$$variance = \acute{\mu}_2 - \mu^2,$$

$$skewness = \frac{\acute{\mu}_3 - 3\acute{\mu}_2\mu + 2\mu^3}{(\acute{\mu}_2 - \mu^2)^{\frac{3}{2}}},$$

and

$$kurtosis = \frac{\acute{\mu}_4 - 4\acute{\mu}_3\mu + 6\acute{\mu}_2\mu^2 - 3\mu^4}{(\acute{\mu}_2 - \mu^2)^2}.$$

The behaviors of moments can be easily viewed through Figure 3.

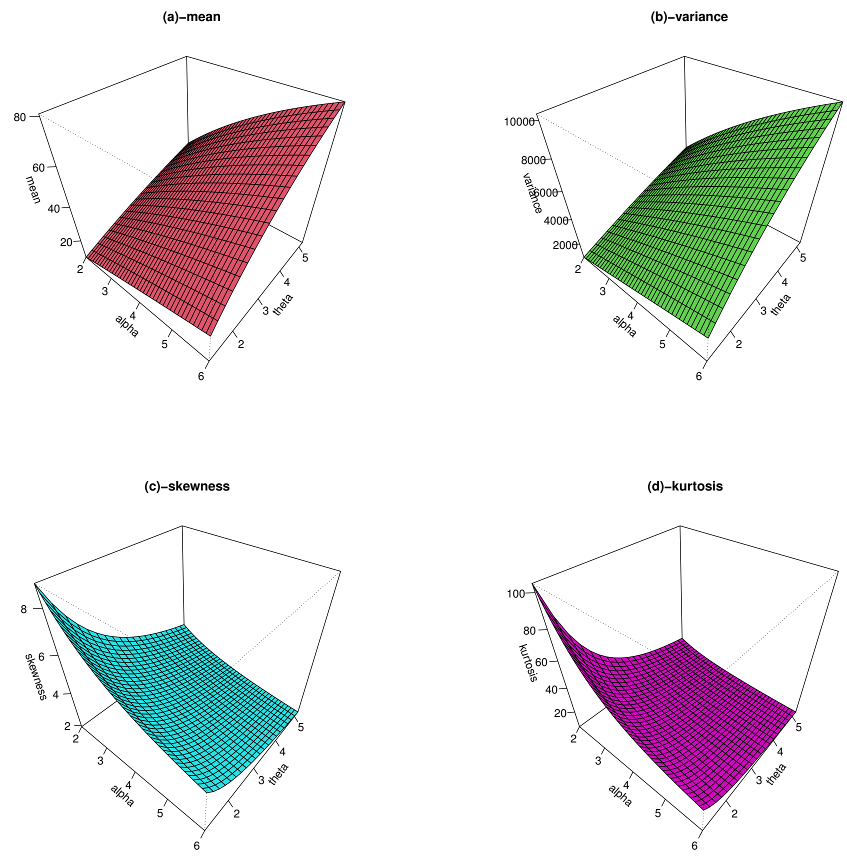

Figure 3: (a), (b), (c), and (d) plots of moments for the MAPTIL distribution.

The inverse moment (IM) of the MAPTIL distribution is obtained by replacing (r) with $(-r)$ in the (3.1).

The moment generating function of the any distribution is given by

$$M_Y(t)=\int_0^\infty e^{ty}f(y)dy=\sum_{r=0}^{\infty}\frac{t^rE(Y^r)}{r!}, \tag{3.2}$$

where $E(Y^r)$ is the moments of MAPTIL distribution given in (3.1).
The characteristics function of the MAPTIL distribution is obtained by replacing (t) with (it) in the (3.2).

$$M_Y(it)=\sum_{r=0}^{\infty}\frac{(it)^rE(Y^r)}{r!}.$$

The incomplete moments (IMs) of the MAPTIL distribution is given by

$$\begin{aligned}E(Y^r|Y\leq t)=\phi_r(t)=\int_0^t y^rf(y)dy&=\sum_{j,k=0}^{\infty}\sum_{m=0}^{k}\Upsilon_{j,k,m}(\theta(k+1))^{r-m-2}\\&\times[\theta(k+1)\Gamma(1+m-r,\frac{\theta(k+1)}{t})+\Gamma(2+m-r,\frac{\theta(k+1)}{t})].\end{aligned} \tag{3.3}$$

The conditional moments (CMs) of the MAPTIL distribution is given by

$$\begin{aligned}E(Y^r|Y>t)&=\frac{1}{R(t)}\int_t^\infty y^rf(y)dy=\frac{1}{R(t)}(E(Y^r)-\phi_r(t))\\&=\frac{1}{R(t)}\sum_{j,k=0}^{\infty}\sum_{m=0}^{k}\Upsilon_{j,k,m}(\theta(k+1))^{r-m-2}\Big[(1+m-r+\theta(k+1))\Gamma(1+m-r)\\&-\theta(k+1)\Gamma(1+m-r,\frac{\theta(k+1)}{t})+\Gamma(2+m-r,\frac{\theta(k+1)}{t})\Big].\end{aligned} \tag{3.4}$$

The mean residual life (MRL) function of MAPTIL distribution is defined as follows

$$\begin{aligned}\mu(t)=E(Y-t|Y>t)&=\frac{1}{R(t)}\int_t^\infty(y-t)f(y)dy\\&=\frac{1}{R(t)}\int_t^\infty yf(y)dy-t\\&=\frac{1}{R(t)}[E(Y)-\phi_1(t)]-t,\end{aligned} \tag{3.5}$$

where

$$\phi_1(t)=\sum_{j,k=0}^{\infty}\sum_{m=0}^{k}\Upsilon_{j,k,m}(\theta(k+1))^{-(m+1)}[\theta(k+1)\Gamma(m,\frac{\theta(k+1)}{t})+\Gamma(m+1,\frac{\theta(k+1)}{t})].$$

The mean inactivity time (MIT) function of MAPTIL distribution is defined as follows,

$$m(t)=E(t-Y|Y\leq t)=t-\frac{1}{F(t)}\int_0^t yf(y)dy=t-\frac{\phi_1(t)}{F(t)}. \tag{3.6}$$

3.2 Rényi Entropy

The Rényi entropy [17] of order δ is given by

$$H_\delta=\frac{1}{1-\delta}\log\left(\int_{-\infty}^{\infty}(f(y))^\delta dy\right),\quad \delta>0,\quad \delta\neq 1.$$

Thus, the Rényi entropy of MAPTIL is given by

$$H_\delta = \frac{1}{1-\delta}\log\left(\int_{-0}^{\infty}(f(y))^\delta dy\right)$$
$$= \frac{1}{1-\delta}\log\left(\int_{-0}^{\infty}\left(\frac{\alpha\log(\alpha)(\frac{\theta^2}{\theta+1})(\frac{1+y}{y^3})e^{-\frac{\theta}{y}}\alpha^{[1+\frac{\theta}{(\theta+1)y}]e^{-\frac{\theta}{y}}}}{(\alpha-1)(1+\alpha-\alpha^{[1+\frac{\theta}{(\theta+1)y}]e^{-\frac{\theta}{y}}})^2}\right)^\delta dy\right).$$

Applying the binomial expansion and power series expansion, we get

$$H_\delta = \frac{1}{1-\delta}\log\left(\left[\frac{\alpha\theta^2\log\alpha}{(\alpha-1)(\alpha+1)^2(\theta+1)}\right]^\delta \sum_{k,j=0}^{\infty}\sum_{m=0}^{j}\sum_{s=0}^{\delta}\frac{\binom{j}{m}\binom{\delta}{s}\Gamma(2\delta+k)(\delta+k)^j(\log\alpha)^j}{\Gamma(2\delta)k!j!(\alpha+1)^k}\right.$$
$$\left.\times\frac{\theta^m}{(\theta+1)^m}\int_{-0}^{\infty}y^{s-3\delta-m}e^{-\frac{(j+\delta)\theta}{y}}dy\right).$$

To simplify the integration, we put $z=\frac{(j+\delta)\theta}{y}$ and using gamma function

$$H_\delta = \frac{1}{1-\delta}\log\left(\left[\frac{\alpha\theta^2\log\alpha}{(\alpha-1)(\alpha+1)^2(\theta+1)}\right]^\delta \sum_{k,j=0}^{\infty}\sum_{m=0}^{j}\sum_{s=0}^{\delta}\frac{\binom{j}{m}\binom{\delta}{s}\Gamma(2\delta+k)(\delta+k)^j(\log\alpha)^j}{\Gamma(2\delta)k!j!(\alpha+1)^k}\right.$$
$$\left.\times\frac{\theta^m}{(\theta+1)^m}[(j+\delta)\theta]^{s-m-3\delta+1}\Gamma(m+3\delta-s-1)\right). \tag{3.7}$$

The behavior of Rényi entropy can be easily viewed through Figure 4.

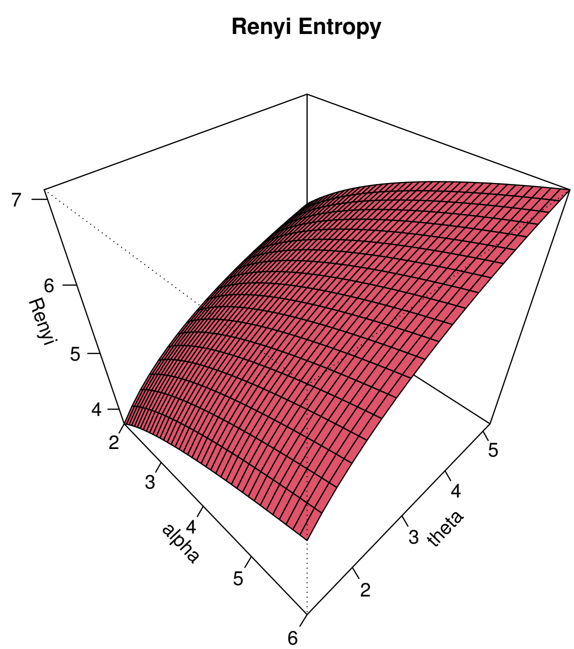

Figure 4: Renyi entropy for the MAPTIL distribution.

3.3 Stress-Strength model

Let $Z_1 \backsim$MAPTIL$(\alpha_1,\ \theta_1)$ and $Z_2 \backsim$ MAPTIL$(\alpha_2,\ \theta_2)$. If Z_1 describes the stress and Z_2 describes the strength, then the MAPTIL distribution's stress-strength parameter, indicated by R.

$$R = P(Z_2 > Z_1) = \int_0^\infty F_1(z)f_2(z)dz$$

$$= \int_0^\infty \left[\frac{\alpha_1^{[1+\frac{\theta_1}{(\theta_1+1)z}]e^{-\frac{\theta_1}{z}}} - 1}{(\alpha_1-1)(1+\alpha_1-\alpha_1^{[1+\frac{\theta_1}{(\theta_1+1)z}]e^{-\frac{\theta_1}{z}}})}\right]\left[\frac{\alpha_2\log(\alpha_2)(\frac{\theta_2^2}{\theta_2+1})(\frac{1+z}{z^3})e^{-\frac{\theta_2}{z}}\alpha_2^{[1+\frac{\theta_2}{(\theta_2+1)z}]e^{-\frac{\theta_2}{z}}}}{(\alpha_2-1)(1+\alpha_2-\alpha_2^{[1+\frac{\theta_2}{(\theta_2+1)z}]e^{-\frac{\theta_2}{z}}})^2}\right]dz$$

$$= \frac{\alpha_2\log(\alpha_2) * \frac{\theta_2^2}{(\theta_2+1)}}{(\alpha_1-1)(\alpha_2-1)(\alpha_1+1)(\alpha_2+1)^2}\int_0^\infty(\frac{1+z}{z^3})e^{-\frac{\theta_2}{z}}(\alpha_1^{[1+\frac{\theta_1}{(\theta_1+1)z}]e^{-\frac{\theta_1}{z}}} - 1)\alpha_2^{[1+\frac{\theta_2}{(\theta_2+1)z}]e^{-\frac{\theta_2}{z}}}$$

$$\times\left(1-\frac{\alpha_1^{[1+\frac{\theta_1}{(\theta_1+1)z}]e^{-\frac{\theta_1}{z}}}}{\alpha_1+1}\right)^{-1}\left(1-\frac{\alpha_2^{[1+\frac{\theta_2}{(\theta_2+1)z}]e^{-\frac{\theta_2}{z}}}}{\alpha_2+1}\right)^{-2}dz.$$

By using the power series and generalized binomial expansion, we get

$$R = \frac{\alpha_2\log(\alpha_2)}{(\alpha_1-1)(\alpha_2-1)(\alpha_1+1)(\alpha_2+1)^2}\times(\frac{\theta_2^2}{\theta_2+1})\sum_{h_1,h_2=0}^{\infty}\sum_{s_1,s_2=0}^{\infty}\sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_2}\binom{s_1}{k_1}\binom{s_2}{k_2}$$

$$\times\frac{(h_1+1)(\log\alpha_1)^{s_1}(\log\alpha_2)^{s_2}}{(\alpha_1+1)^{h_1}(\alpha_2+1)^{h_2}s_1!s_2!}[(h_1+1)^{s_1}-h_1^{s_1}](h_2+1)^{s_2}(\frac{\theta_1}{\theta_1+1})^{k_1}(\frac{\theta_2}{\theta_2+1})^{k_2}$$

$$\times\int_0^\infty\frac{(1+z)}{z^{3+k_1+k_2}}e^{-\frac{(-s_1\theta 1+(s_2+1)\theta_2)}{z}}dz. \tag{3.8}$$

3.4 Order statistics

The order statistics of a random sample $Y_1, ..., Y_n$ are the sample values placed in ascending order. They are denoted by $Y_{1:n}, ..., Y_{n:n}$. The PDF of the j^{th} order statistics $Y_{j:n}$ is given by

$$f_{j:n}(y) = \frac{n!}{(j-1)!(n-j)!}f(y)[F(y)]^{j-1}[1-F(y)]^{n-j}$$
$$= \frac{n!}{(j-1)!(n-j)!}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}f(y)[F(y)]^{k+j-1}.$$

Using (2.1), (2.2) and after some simplifications, we can write

$$f_{j:n}(y) = \frac{n!}{(j-1)!(n-j)!}\sum_{k=0}^{n-j}\sum_{m=0}^{k+j-1}\sum_{s,h=0}^{\infty}\sum_{b=0}^{h}(-1)^{m+j-1}\binom{n-j}{k}\binom{k+j-1}{m}\binom{h}{b}$$

$$\times\frac{\alpha(\log\alpha)^{h+1}(m+s+1)^h\Gamma(s+k+j+1)}{(\alpha-1)^{k+j}(\alpha+1)^{k+j+s+1}\Gamma(k+j+1)s!h!}\times\frac{\theta^{2+b}}{(\theta+1)^b}\times\frac{(1+y)}{y^{3+b}}e^{-\frac{(h+1)\theta}{y}}. \tag{3.9}$$

The r^{th} moments of $Y_{j:n}$ may be calculated as follows using (3.9)

$$\begin{aligned}
E(Y_{j:n}^r) &= \int_0^\infty y^r f_{j:n}(y)dy \\
&= \frac{n!}{(j-1)!(n-j)!}\sum_{k=0}^{n-j}\sum_{m=0}^{k+j-1}\sum_{s,h=0}^{\infty}\sum_{b=0}^{h}(-1)^{m+j-1}\binom{n-j}{k}\binom{k+j-1}{m}\binom{h}{b} \\
&\times \frac{\alpha(\log\alpha)^{h+1}(m+s+1)^h\Gamma(s+k+j+1)}{(\alpha-1)^{k+j}(\alpha+1)^{k+j+s+1}\Gamma(k+j+1)s!h!}\times\frac{\theta^{2+b}}{(\theta+1)^b} \\
&\times[(h+1)\theta]^{r-b-2}(b-r+(h+1)\theta+1)\Gamma(b-r+1), \quad b\geq r. \qquad (3.10)
\end{aligned}$$

4 Parameter Estimation

In this section, we consider the maximum likelihood approach to estimate the unknown parameters of the MAPTIL distribution.

4.1 Maximum likelihood estimation

This subsection discusses the maximum likelihood estimation (MLE) for the parameters $\Theta=(\alpha,\theta)$ of the MAPTIL distribution. Let $y_1, y_2, ..., y_n$ be a complete random sample of size n from the MAPTIL distribution. The likelihood function is given by:

$$\ell(y_1,...,y_n|\Theta)=\prod_{i=1}^n f(y_i;\alpha,\theta)=\frac{\left(\frac{\alpha\theta^2\log\alpha}{(\alpha-1)(\theta+1)}\right)^n\left(\prod_{i=1}^n\frac{y_i+1}{y_i^3}\right)e^{-\sum_{i=1}^n\frac{\theta}{y_i}}\alpha^{\sum_{i=1}^n(1+\frac{\theta}{(\theta+1)y_i})e^{\frac{-\theta}{y_i}}}}{\prod_{i=1}^n\left(1+\alpha-\alpha^{(1+\frac{\theta}{(\theta+1)y_i})e^{-\frac{\theta}{y_i}}}\right)^2}.$$

Then, the logarithm of the likelihood function is given by

$$\begin{aligned}
L(y_1,...,y_n|\Theta) &= n\log\left(\frac{\alpha\theta^2\log\alpha}{(\alpha-1)(\theta+1)}\right)+\sum_{i=1}^n\log\frac{y_i+1}{y_i^3}-\theta\sum_{i=1}^n\frac{1}{y_i} \\
&+\log\alpha\sum_{i=1}^n(1+\frac{\theta}{(\theta+1)y_i})e^{\frac{-\theta}{y_i}}-2\sum_{i=1}^n\log(1+\alpha-\alpha^{(1+\frac{\theta}{(\theta+1)y_i})e^{-\frac{\theta}{y_i}}}). \qquad (4.1)
\end{aligned}$$

By taking the first partial derivatives of the equation (4.1) with respect to the parameters α and θ in Θ

$$\begin{aligned}
\frac{\partial L}{\partial\alpha} &= \frac{n}{\alpha}-\frac{n}{\alpha-1}+\frac{n}{\alpha\log\alpha}+\frac{1}{\alpha}\sum_{i=1}^n(1+\frac{\theta}{(\theta+1)y_i})e^{\frac{-\theta}{y_i}} \\
&-2\sum_{i=1}^n\frac{1-(1+\frac{\theta}{(\theta+1)y_i})e^{-\frac{\theta}{y_i}}\alpha^{-1+(1+\frac{\theta}{(\theta+1)y_i})e^{-\frac{\theta}{y_i}}}}{1+\alpha-\alpha^{(1+\frac{\theta}{(\theta+1)y_i})e^{-\frac{\theta}{y_i}}}}, \qquad (4.2)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial L}{\partial\theta} &= \frac{2n}{\theta}-\frac{n}{\theta+1}-\sum_{i=1}^n\frac{1}{y_i}+\log\alpha\sum_{i=1}^n\frac{(\theta(\theta+1)(1+y_i)+\theta y_i)}{(\theta+1)^2y_i^2}e^{-\frac{\theta}{y_i}} \\
&-2\sum_{i=1}^n\frac{(\log\alpha)e^{-\frac{\theta}{y_i}}\alpha^{(1+\frac{\theta}{(\theta+1)y_i})e^{-\frac{\theta}{y_i}}}}{1+\alpha-\alpha^{(1+\frac{\theta}{(\theta+1)y_i})e^{-\frac{\theta}{y_i}}}}\times\frac{(\theta(\theta+1)(1+y_i)+\theta y_i)}{(\theta+1)^2y_i^2}. \qquad (4.3)
\end{aligned}$$

Equations (4.2) and (4.3) cannot be solved explicitly, so numerical methods can be used to compute the MLEs of $\Theta=(\alpha,\theta)$.

4.2 Simulation

In this subsection, the behavior of MLEs, $\hat{\alpha}$ and $\hat{\theta}$ are investigated through Monte Carlo simulation. The parameter inputs for simulation study include $\alpha(= 0.2, 0.7, 0.4, 0.3)$, $\theta(= 0.3, 0.5, 0.6, 0.1)$, and sample size n(=20, 60, 100, 140). Given each pairs, (α, θ) and sample size n, shown in Table 1, simulation was run 1000 times to produce 1000 pairs of MLE, $\hat{\Theta}_i = (\hat{\alpha}_i, \hat{\theta}_i)$ for $i = 1, 2, 3, \ldots, 1000$ by using Mathematica. Therefore, each computed MSE and Bias were calculated by

$$\text{MSE}(\hat{\Theta}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\Theta}_i - \Theta)^2 \quad \text{and} \quad \text{Bias}(\hat{\Theta}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\Theta}_i - \Theta), \tag{4.4}$$

where Θ indicates α or θ and $\hat{\Theta}$ is the MLE of Θ. Table 1 shows the mean square error (MSE) and bias of MLEs, $\hat{\alpha}$ and $\hat{\theta}$. Reading Table 1, one can observe the MSE and Bias for the MLEs of α and θ are decreasing when the sample size n is increasing. We use the inverse CDF method to generate the random sample from equation (2.1). For the MAPTIL distribution, the inverse CDF can not be obtained in explicit form. For this reason, we can apply Newton's method by Mathematica program to solve the equation $(F(y) - u(0, 1) = 0)$.

Table 1: MLE of parameters α and θ.

MAPTIL(α, θ)	n	MSE$(\hat{\alpha})$	Bias$(\hat{\alpha})$	MSE$(\hat{\theta})$	Bias$(\hat{\theta})$
MAPTIL(0.2, 0.3)	20	0.063808	0.194795	0.001459	-0.0238386
	60	0.046081	0.169144	0.000888	-0.0202983
	100	0.039736	0.155658	0.000764	-0.0181203
	140	0.037783	0.145357	0.000745	-0.0168378
MAPTIL(0.7, 0.5)	20	0.024586	0.130458	0.004059	0.0083612
	60	0.018224	0.115556	0.001194	0.0031316
	100	0.016042	0.103836	0.000577	0.0007927
	140	0.013356	0.088095	0.000339	-0.0004120
MAPTIL(0.4, 0.6)	20	0.103428	0.251786	0.009899	-0.073115
	60	0.047568	0.067936	0.002493	-0.016219
	100	0.037586	0.006819	0.001067	-0.002349
	140	0.035482	0.001637	0.000818	0.001758
MAPTIL(0.3, 0.1)	20	0.082676	0.163658	0.000522	-0.008593
	60	0.034956	0.042614	0.000237	0.003141
	100	0.023211	0.013604	0.000179	0.002825
	140	0.013211	-0.011316	0.000126	0.001105

5 Real Data

In this section, we analyze real data to illustrate that the MAPTIL can be a good lifetime model compared with many known distributions such as alpha power inverse Lindley (APIL), alpha power inverse Lomax (APILo), inverse Lomax (ILo) and inverse Lindley (IL) distributions. The pdfs of those competitive distributions are listed as,

- APIL: $f(y; \alpha, \theta) = \frac{\log(\alpha)}{\alpha-1}(\frac{\theta^2}{\theta+1})(\frac{y+1}{y^3})e^{-\frac{\theta}{y}}\alpha^{[1+\frac{\theta}{(\theta+1)y}]e^{-\frac{\theta}{y}}}, \quad y > 0, \alpha > 0, \theta > 0.$
- APILo: $f(y; \alpha, \lambda, \theta) = \frac{\log(\alpha)}{\alpha-1}(\frac{\lambda\theta}{y^2})(1 + \frac{\lambda}{y})^{-(\theta+1)}\alpha^{(1+\frac{\lambda}{y})^{-\theta}}, \quad y > 0, \alpha, \lambda, \theta > 0.$
- ILo: $f(y; \lambda, \theta) = (\frac{\lambda\theta}{y^2})(1 + \frac{\lambda}{y})^{-(\theta+1)}, \quad y > 0, \lambda > 0, \theta > 0.$
- IL: $f(y; \theta) = \frac{\theta^2}{\theta+1}(\frac{y+1}{y^3})e^{-\frac{\theta}{y}}; y > 0, \theta > 0.$

The first data set describes the mortality rates due to the COVID-19 pandemic in the United Kingdom for 76 days, from 15 April to 30 June 2020. The first data is originally investigated by Mubarak et al. [14]. The second data set consists of the times of breakdown of a sample of 25 devices at 180°C and given by Pham [16]. The first data and second data are displayed in Table 2.

Table 2: The first data and second data.

First data	0.0587	0.0863	0.1165	0.1247	0.1277	0.1303	0.1652	0.2079	0.2395
	0.2751	0.2845	0.2992	0.3188	0.3317	0.3446	0.3553	0.3622	0.3926
	0.3926	0.4110	0.4633	0.4690	0.4954	0.5139	0.5696	0.5837	0.6197
	0.6365	0.7096	0.7193	0.7444	0.8590	1.0438	1.0602	1.1305	1.1468
	1.1533	1.2260	1.2707	1.3423	1.4149	1.5709	1.6017	1.6083	1.6324
	1.6998	1.8164	1.8392	1.8721	1.9844	2.1360	2.3987	2.4153	2.5225
	2.7087	2.7946	3.3609	3.3715	3.7840	3.9042	4.1969	4.3451	4.4627
	4.6477	5.3664	5.4500	5.7522	6.4241	7.0657	7.4456	8.2307	9.6315
	10.187	11.1429	11.2019	11.4584					
Second data	112	260	298	327	379	487	593	658	701
	720	734	736	775	915	974	1023	1157	1227
	1293	1335	1472	1529	1545	2029	4568		

The MLEs of the MAPTIL distribution and some other competitive distributions are presented in Tables 3 and 4 for first data and second data, respectively. In addition, some goodness of fit statistics such as, Akaike information criterion (AIC), Bayesian information criterion (BIC), Cramer-von Mises (W*), Andrson-Darling (A*) and Kolmogorov-Smirnov (K-S) (with the corresponding p-value) are also displayed in Tables 3 and 4 for first data and second data, respectively. It is noted from these Tables that the MAPTIL distribution performed better than all other competitive distributions. As a result, the MAPTIL distribution is determined to be the best model to fit the given two data sets. Figures 5 and 6 display the fitted PDFs, RFs and probability-probability (P-P) plots of the MAPTIL distribution for first data and second data, respectively. These figures demonstrate that the MAPTIL distribution can furnish a close fit to the given data sets.

Table 3: MLEs and different statistics of MAPTIL for first data.

Distribution	Estimates	-2L	AIC	BIC	W^*	A^*	K-S	p-value
MAPTIL	α=10.544	285.847	289.847	294.508	0.110	0.685	0.0706	0.842
	θ=0.18401							
APTIL	α=22.106	297.303	301.303	305.964	0.183	1.142	0.1498	0.0658
	θ=0.4869							
APTILo	α=4.8905	288.130	294.13	301.12	0.1260	0.8505	0.1128	0.2880
	λ=0.0044							
	θ=80.992							
ILo	λ=0.4195	285.48	289.48	294.14	0.0935	0.6173	0.0741	0.7969
	θ=2.1439							
IL	θ=0.80158	316.844	318.844	321.175	0.2819	1.8001	0.2406	0.0003

Table 4: MLEs and different statistics of MAPTIL for second data.

Distribution	Estimates	-2L	AIC	BIC	W^*	A^*	K-S	p-value
MAPTIL	α=0.6133	398.505	402.505	404.943	0.1956	1.0394	0.1802	0.391
	θ=850.98							
APTIL	α=7912910	399.685	403.685	406.122	0.1424	0.7387	0.245	0.097
	θ=39.806							
APTILo	α=11826.4	399.94	405.94	409.60	0.1136	0.6681	0.2532	0.066
	λ=0.0146							
	θ=4661.4							
ILo	λ=3.6074	399.57	403.57	406.00	0.1414	0.8349	0.2359	0.104
	θ=162.84							
IL	θ=584.84	399.57	401.57	402.79	0.1601	0.8452	0.2346	0.128

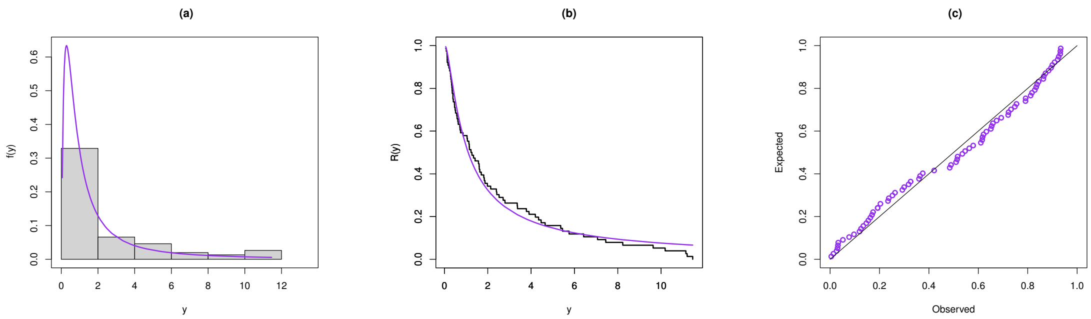

Figure 5: (a), (b), and (c) plots histogram and fitted PDF, estimated and empirical RF, and P-P plot of MAPTIL distribution for first data, respectively.

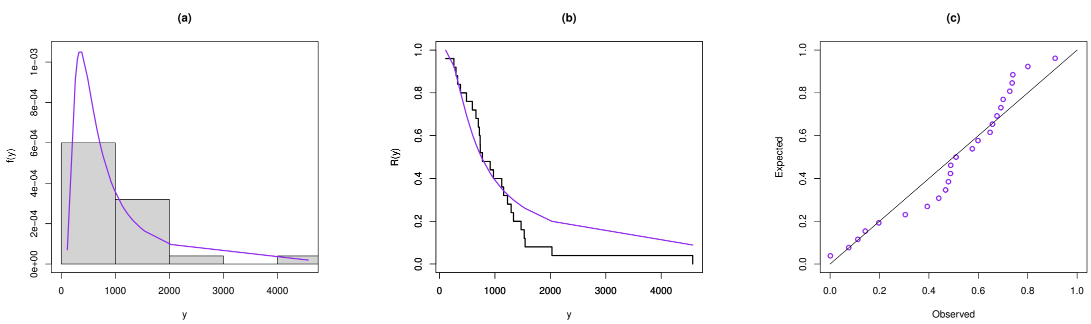

Figure 6: (a), (b), and (c) plots histogram and fitted PDF, estimated and empirical RF, and P-P plot of MAPTIL distribution for second data, respectively.

These figures show how the given data sets can be closely fitted using the MAPTIL distribution.

6 Conclusion

Current research work has contributed a new extended distribution through the modified alpha power (MAP) transformation on inverse Lindley (IL) distribution. This distribution is a generalization of the inverse Lindley distribution and called the MAPTIL distribution. The reliability functions and probability properties of MAPTIL distribution has been derived; such as, hazard rate function, moments, entropy, stress strength model and order statistics. The method of maximum likelihood estimation was also proposed to estimate the parameters of the MAPTIL distribution and the results of simulation study support that MLE method provide competitive MSE and Bias for estimating the parameters of the MAPTIL distribution. The goodness-of-fit of the MAPTIL distribution was demonstrated with real data set and the results from the data modelling show that the MAPTIL distribution offers a better fit to the data set than the other competing distributions.

References

[1] Alkarni, S.H. (2015). Extended inverse lindley distribution: properties and application. Springer- Plus, 4:1–13.

[2] Alotaibi, R., Okasha, H., Rezk, H., and Nassar, M. (2021). A new weighted version of alpha power transformation method: Properties and applications to COVID-19 and software reliability data. Physica Scripta, 96:125-221.

[3] Alotaibi, R., Okasha, H., Nassar, M., and Elshahhat, A. (2023), A novel modified alpha power transformed Weibull distribution and its engineering applications. Computer Modeling in Engineering & Sciences, 135:2065-2089.

[4] Barco, V. K., Mazucheli, J., and Janeiro, V. (2017). The inverse power Lindley distribution. Commun. Statistics - Simulation Computation 46(8).

[5] Basheer, A. M. (2019). Alpha power inverse Weibull distribution with reliability application. Journal of Taibah University for Science, 13:423–432.

[6] Basheer, A. M., Almetwally, E. M., and Okasha, H. M. (2021). Marshall Olkin alpha power inverse Weibull distribution: Non Bayesian and Bayesian estimations. Journal of Statistics Applications & Probability, 10:327–345.

[7] Basheer, A. M. (2022). Marshall-Olkin alpha power inverse exponential distribution: properties and applications. Annals of data science, 9:301-313.

[8] Basheer, A. M. (2024). Alpha power inverted exponentiated Weibull distribution with data applications. Journal of Mathematics and Statistics, 20(1):63-73.

[9] Basheer, A. M., Al-Garash, S., and Modhesh, A. A. (2024). On the modified alpha power transformed kumaraswamy distribution: properties and applications. Journal of Statistics Applications & Probability Letters, 11(3):167-182.

[10] Dey, S., Nassar, M., and Kumar, D. (2019). Alpha power transformed inverse lindley distribution: A distribution with an upside-down bathtub-shaped hazard function. Journal of Computational and Applied Mathematics, 348:130–145.

[11] Eghwerido, J. T., Nzei, L. C., and Agu, F. I. (2021). The alpha power gompertz distribution: Characterization, properties, and applications. Sankhya A, 83(1):449–475.

[12] Eltehiwy, M. (2020). Logarithmic inverse Lindley distribution: Model, properties and applications. Journal of King Saud University – Science, 32:136–144.

[13] Mahdavi, A. and Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. Communications in Statistics - Theory and Methods, 46:6543–6557.

[14] Mubarak, A. E., and Al-Metwally, E. M. (2021). A New Extension Exponential Distribution with Applications of COVID-19 Data.22(1):444-460.

[15] Nassar, M., Alzaatreh, A., Mead, M., and Abo-Kasem, O. (2017). Alpha power Weibull distribution: properties and applications. Communications in Statistics - Theory and Methods, 46:10236–10252.

[16] Pham, H. (2003). *Accelerated life testing*. In: Hoang Pham (Ed.). Handbook of reliability engineering, vol. 1. London: Springer.

[17] Renyi, A. (1961). On measures of entropy and information, In Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics, 4:547–562.

[18] Sharma, V. K., Singh, S. K., Singh, U., and Agarwal, V. (2015). The inverse Lindley distribution: a stress-strength reliability model with application to head and neck cancer data. Journal of Industrial and Production Engineering, 32(3):162–173.

[19] Sharma, V. K., Singh, S. K., Singh, U., and Merovci, F. (2016). The Generalized Inverse Lindley Distribution: A New Inverse Statistical Model for the Study of Upside-down Bathtub Data. Communications in Statistics-Theory and Methods, 45(19):5709–5729.

[20] Sharma, V. K., and Khandelwal, P. (2017). On the extension of inverse Lindley distribution. Journal of Data Science, 15:205-220.