

Some novel classes of G -open sets in G -metrizable spaces

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Abstract

This paper introduces and studies a novel class of G -open sets in G -metrizable spaces, known as $G(\Theta)$ -open sets. The connections between this form and the other known sets are explained. The concepts of the frontier, closure, and interior operators are provided via $G(\Theta)$ -open sets.

Keywords: novel classes- G -open sets- G -metrizable spaces

بعض الفئات الجديدة من المجموعات المفتوحة- G في الفضاءات القابلة للقياس- G

الملخص

تقدم هذه الورقة وتدرس فئة جديدة من المجموعات المفتوحة- G في الفضاءات القابلة للقياس- G ، والمعروفة باسم المجموعات المفتوحة- $G(\Theta)$. يتم شرح الارتباطات بين هذا الشكل والمجموعات الأخرى المعروفة. يتم توفير مفاهيم المشغلات الحدودية والإغلاقية والداخلية من خلال المجموعات المفتوحة- $G(\Theta)$.

الكلمات المفتاحية: الفئات الجديدة- المجموعات المفتوحة- G -الفضاءات القابلة للقياس- G .

1 Introduction

Many mathematical concepts that can be discussed in this space. The first attempt to generalize the ordinary distance function to a distance of three points was introduced by Gähler, [2, 3], in 1993. K. S. Ha, et al; [9], showed that a 2-metric is not a generalization of the usual notion of a metric. The concept of a metric space was introduced by Frechet in 1906, [1]. It has a very important basic role in mathematics and its application. It was mentioned by Gähler, [2], that the notion of a 2-metric is an extension of an idea of ordinary metric and geometrically $??(x, y, z)$ represents the area of a triangle formed by the points x, y and z in X as its vertices. But this is not always true. A. Sharma, [6], showed that $??(x, y, z) = 0$ for any three distinct points $x, y, z \in \mathbb{R}^2$. B. C. Dhage in 1963 introduced a new class of generalized metrics called D-metrics, [3]. However, several errors for fundamental topological properties in a D-metric space were found by Z. Mustafa and B. Sims, [4]. Due to these considerations, Z. Mustafa and B. Sims, [5], proposed a more appropriate notion of a generalized metric space, called $\mathcal{G}$ -metric space.

In this paper we introduce the concept of $\mathcal{G}(\Theta)$ -open sets by utilizing the $\mathcal{G}$ -open balls. Furthermore, the relationship with the other known sets will be studied. Next we introduce the concepts of the interior operator, the closure operator and frontier operator via $\mathcal{G}(\Theta)$ -open sets.

2 Preliminaries

Definition 2.1. [1] Let X be any nonempty set. A function $d : X \times X \rightarrow [0, \infty)$ is called a metric function on X if it satisfies the following three conditions for all $x, y, z \in X$:

1. (positive property) $d(x, y) \geq 0$ with equality if and only if $x = y$;
2. (symmetric property) $d(x, y) = d(y, x)$;
3. (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

A pair (X, d) , where d is a metric on X is called a metric space.

Definition 2.2. [4] Let X be a nonempty set and $\mathbb{R}$ be the set of real numbers. A function $G : X \times X \times X \rightarrow \mathbb{R}$ is called a G -metric function on X if it satisfies the following:

1. $G(x, x, y) > 0$ for all $x \neq y \in X$;
2. $G(x, y, z) = 0$ if and only if $x = y = z$;

3. $G(x, x, y) \leq G(x, y, z)$ for every $x, y, z \in X$ with $y \neq z$;
4. $G(x, y, z) = G(p(x, y, z))$ for every $x, y, z \in X$ and for any permutation p of x, y, z ;
5. $G(x, y, z) \leq G(x, u, u) + G(u, y, z)$ for every $x, y, z, u \in X$.

If G is a G -metric function on X , then the pair (X, G) is called a G -metric space.

Example 2.3. [5] Let $(\mathbb{R}, d)$ be the usual metric space. Define G_s by $G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z)$ for all $x, y, z \in \mathbb{R}$. Then it is clear that $(\mathbb{R}, G_s)$ is a G -metric space.

Example 2.4. [5] Let $X = \{a, b\}$. Define G on $X \times X \times X$ by $G(a, a, a) = G(b, b, b) = 0$, $G(a, a, b) = 1$, $G(a, b, b) = 2$.

Example 2.5. [5] Let $(\mathbb{R}, G)$ be $\mathcal{G}$ -metric space defined by $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$.

Definition 2.6. [7] Let (X, G) be a G -metric space, $x \in X$ and $A \subseteq X$. The open ball with center x and radius ϵ in metric space (X, G) is denoted by $B_G(x, \epsilon)$ and defined by

$$B_G(x, \epsilon) = \{y \in X \mid d(x, y, y) < \epsilon\}.$$

The closed ball with center x and radius ϵ in G -metric space (X, G) is denoted by $C_G(x, \epsilon)$ and defined by

$$C_G(x, \epsilon) = \{y \in X \mid d(x, y, y) \leq \epsilon\}.$$

The set A is called an open set in G -metric space (X, G) if for every $x \in A$, there is $\epsilon > 0$ such that $B_G(x, \epsilon) \subseteq A$. The set A is called closed set in metric space (X, G) if $X - A$ is an open set in G -metric space (X, G) .

Theorem 2.7. [7] Every G -open ball $B_G(x, \epsilon)$, $x \in X$, $\epsilon > 0$ is an open set in X .

Theorem 2.8. [5] Let (X, G) be a $\mathcal{G}$ -metric space, then for any $x \in X$ and $\epsilon > 0$, we have.

- (1) If $G(y, x, x) < \epsilon$ then $x, y \in B_G(x, \epsilon)$;
- (2) If $y \in B_G(x, \epsilon)$ then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(x, \epsilon)$.

Definition 2.9. [7] $Cl_G(A)$ is called the G -closure of A if it is the intersection of all G -closed sets containing A .

Definition 2.10. [7] A set U in a $\mathcal{G}$ -metric space X , is said to be closed if its complement $X - U$ is G -open.

3 $\mathcal{G}(\Theta)$ -open sets

Definition 3.1. Let (X, G) be a G -metric space and $A \subseteq X$. A point $x \in X$ is called a G -point of A in G -metric space (X, G) if there is $\delta > 0$ such that for every $y \in B_G(x, \delta)$,

$$B_G(y, \epsilon) \cap A \neq \emptyset \quad \forall \epsilon > 0.$$

$\mathcal{G}(\Theta)(A)$ denotes the set of all $\mathcal{G}(\Theta)$ -points of A in G -metric space (X, G)

Example 3.2. Let $(\mathbb{R}, G)$ be G -metric space defined by $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$. Let $A = (0, 2)$ and $B = Q$ be that set of rational numbers. Note that $\mathcal{G}(\Theta)(A) = (0, 2)$ and $\mathcal{G}(\Theta)(B) = \mathbb{R}$.

Theorem 3.3. Let (X, G) be any G -metric space and $A, B \subseteq X$. Then

1. $\mathcal{G}(\Theta)(\phi) = \phi$ and $\mathcal{G}(\Theta)(X) = X$;
2. if $A \subseteq B$ Then $\mathcal{G}(\Theta)(A) \cup \mathcal{G}(\Theta)(B)$;
3. $\mathcal{G}(\Theta)(A \cap B) \subseteq \mathcal{G}(\Theta)(A) \cap \mathcal{G}(\Theta)(B)$;
4. $\mathcal{G}(\Theta)(A) \cup \mathcal{G}(\Theta)(B) \subseteq \mathcal{G}(\Theta)(A \cup B)$.

Proof. 1. It is clear from the definition, we get that $\mathcal{G}(\Theta)(\phi) = \phi$ and $\mathcal{G}(\Theta)(X) = X$.

2. Let $A \subseteq B$ and $x \in \mathcal{G}(\Theta)(A)$. Then is $\delta > 0$ such that for every $y \in B_G(x, \delta) \cap A \neq \emptyset$, for all $\epsilon > 0$. Since $A \subseteq B$. Then $B_G(y, \epsilon) \cap B \neq \emptyset$, for all $\epsilon > 0$. That is, $x \in \mathcal{G}(\Theta)(B)$. Then $\mathcal{G}(\Theta)(A) \subseteq \mathcal{G}(\Theta)(B)$.

3. Since $A \cap B \subseteq A$. Then by part (2) $\mathcal{G}(\Theta)(A \cap B) \subseteq \mathcal{G}(\Theta)(A)$. Similar $\mathcal{G}(\Theta)(A \cap B) \subseteq \mathcal{G}(\Theta)(B)$. Then $\mathcal{G}(\Theta)(A \cap B) \subseteq \mathcal{G}(\Theta)(A) \cap \mathcal{G}(\Theta)(B)$.

4. Since $A \subseteq (A \cup B)$. Then by part (2) $\mathcal{G}(\Theta)(A) \subseteq \mathcal{G}(\Theta)(A \cup B)$. Similar $\mathcal{G}(\Theta)(B) \subseteq \mathcal{G}(\Theta)(A \cup B)$. Then $\mathcal{G}(\Theta)(A) \cup \mathcal{G}(\Theta)(B) \subseteq \mathcal{G}(\Theta)(A \cup B)$. $\square$

Definition 3.4. Let (X, G) be a G -metric space. A subset $A \subseteq X$ is called a $\mathcal{G}(\Theta)$ -open set in G -metric space (X, G) if for every $x \in A$,

$$B_G(x, \epsilon) \cap \mathcal{G}(\Theta)(A) \neq \emptyset \quad \forall \epsilon > 0.$$

A subset $A \in X$ is called a $\mathcal{G}(\Theta)$ -closed set in G -metric space (X, G) if $X - A$ is a $\mathcal{G}(\Theta)$ -open set in G -metric space (X, G) .

Example 3.5. In Example(3.2), the sets A and B are $\mathcal{G}(\Theta)$ -open sets. Note that any finite sub sets of $\mathbb{R}$ are not $\mathcal{G}(\Theta)$ -open set.

Theorem 3.6. Every G -open set is a $\mathcal{G}(\Theta)$ -open set.

Proof. Let A be any G -open set in G -metric space (X, G) . Let $x \in A$ be arbitrary point. Then there is $\delta > 0$ such that $B_G(x, \delta) \subseteq A$. For every $y \in B_G(x, \delta)$, $y \in B_G(x, \delta)(y)$ and $y \in A$ for every $\epsilon > 0$. That is, $B_G(y, \epsilon) \cap A \neq \emptyset$ for every $\epsilon > 0$. Hence A is $\mathcal{G}(\Theta)$ -open set. $\square$

The converse of above theorem need not be true.

Example 3.7. In Example(3.2), note that for the closed interval $A = [a, b]$, $\mathcal{G}(\Theta)(A) = (a, b)$. Then it is clear to check that A is a $\mathcal{G}(\Theta)$ -open set. Take $x = a$ or $x = b$. Note that $x \in A$ but there is no G -open ball with center x contained in A . That is, A is not G -open set in $(\mathbb{R}, G)$.

The intersection of two $\mathcal{G}(\Theta)$ -open sets no need to be $\mathcal{G}(\Theta)$ -open set. In Example(3.2), set of rational numbers Q is a $\mathcal{G}(\Theta)$ -open set but not G -open set in $(\mathbb{R}, G)$ and the set $IR \cup \{q\}$ is a $\mathcal{G}(\Theta)$ -open set in $(\mathbb{R}, G)$, where IR is the set of irrational numbers and q is any rational number, but $Q \cap (IR \cup \{q\}) = \{q\}$ is not $\mathcal{G}(\Theta)$ -open set. That is, the collection of all $\mathcal{G}(\Theta)$ -open sets in G -metric space (X, G) does not form topology on a set X .

The following theorem shows that the intersection of a G -open set and a $\mathcal{G}(\Theta)$ -open set is a $\mathcal{G}(\Theta)$ -open set.

Theorem 3.8. The intersection of a G -open set and a $\mathcal{G}(\Theta)$ -open set is a $\mathcal{G}(\Theta)$ -open set.

Proof. Let A be G -open set and B be $\mathcal{G}(\Theta)$ -open set in G -metric space in (X, G) . Let $x \in A \cap B$ be arbitrary point. Then $x \in A$ and $x \in B$. Then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that $B_G(x, \delta_1) \subseteq A$ and for every $y \in B_G(x, \delta_2)$, $B_G(y, \epsilon) \cap B \neq \emptyset$ for every $\epsilon > 0$. Take $\delta = \min\{\delta_1, \delta_2\} > 0$. Then $B_G(x, \delta) \subseteq A$ and for every $y \in B_G(x, \delta)$, $B_G(y, \epsilon) \cap B \neq \emptyset$ for every $\epsilon > 0$. Now for every $y \in B_G(x, \delta)$ and since A is G -open set, then there is $\epsilon_y > 0$ such that $B_G(y, \epsilon_y) \subseteq A$ and $B_G(y, \min\{\delta_1, \delta_2\}) \cap B \neq \emptyset$. Since $B_G(y, \min\{\delta_1, \delta_2\}) \cap B \subseteq B_G(y, \epsilon_y) \cap A \cap B$, then $B_G(y, \epsilon_y) \cap (A \cap B) \neq \emptyset$ for every $\epsilon > 0$. That is $A \cap B$ is G -open set. Hence $x \in \mathcal{G}(\Theta)(A \cap B)$. Then $B_G(y, \epsilon) \cap \mathcal{G}(\Theta)(A \cap B) \neq \emptyset$ for all $\epsilon > 0$. There for $A \cap B$ is $\mathcal{G}(\Theta)$ -open set. $\square$

Theorem 3.9. The union of any family of $\mathcal{G}(\Theta)$ -open sets is $\mathcal{G}(\Theta)$ -open set.

Proof. Let H_λ be a $\mathcal{G}(\Theta)$ -open in G -metric space (X, G) for all $\lambda \in \Delta$. Let $x \in \cup_{\lambda \in \Delta} H_\lambda$ be an arbitrary point. Then there is at least $\lambda_0 \in \Delta$ such that $x \in H_{\lambda_0}$. Since H_{λ_0} is a $\mathcal{G}(\Theta)$ -open set then

$B_G(x, \varepsilon) \cap \mathcal{G}(\Theta)(H_{\lambda_0}) \neq \emptyset$ for all $\varepsilon > 0$. Hence by Theorem (3.3), $\mathcal{G}(\Theta)(H_{\lambda_0}) \subseteq \mathcal{G}(\Theta)(\cup_{\lambda \in \Delta} H_\lambda)$. Hence $B_G(x, \varepsilon) \cap \mathcal{G}(\Theta)(\cup_{\lambda \in \Delta} H_\lambda) \neq \emptyset$ for all $\varepsilon > 0$. That is $\cup_{\lambda \in \Delta} H_\lambda$ is $\mathcal{G}(\Theta)$ -open set. $\square$

4 $\mathcal{G}(\Theta)$ -open operators

In this section, we define the interior operator, the closure operator and frontier operator via $\mathcal{G}(\Theta)$ -open sets.

Definition 4.1. Let (X, G) be a $\mathcal{G}$ -metric space and $A \subseteq X$. The G -closure operator of A is denoted by $Cl_G^\beta(A)$ and defined by

$$Cl_G^\beta(A) = \cap \{H \subseteq X : A \subseteq H \text{ and } H \text{ is } \mathcal{G}(\Theta)\text{-closed set}\}.$$

The G -interior functor of A is denoted by $Int_G^\beta(A)$ and defined by

$$Int_G^\beta(A) = \cup \{H \subseteq X : H \subseteq A \text{ and } H \text{ is } \mathcal{G}(\Theta)\text{-open set}\}.$$

Remark 4.2.

1. By Theorem(3.9), $Cl_G^\beta(A)$ is a $\mathcal{G}(\Theta)$ -closed set and $Int_G^\beta(A)$ is $\mathcal{G}(\Theta)$ -open set in $\mathcal{G}$ -metric space (X, G) .
2. For a $\mathcal{G}$ -metric space (X, G) and $A \subseteq X$, it is clear from the definition of $Cl_G^\beta(A)$ and $Int_G^\beta(A)$ that $A \subseteq Cl_G^\beta(A)$ and $Int_G^\beta(A) \subseteq A$.

Theorem 4.3. For a $\mathcal{G}$ -metric space (X, G) and $A \subseteq X$, $Cl_G^\beta(A) = A$ if and only if A is a $\mathcal{G}(\Theta)$ -closed set.

Proof. Let $Cl_G^\beta(A) = A$. Then from definition of $Cl_G^\beta(A)$ and Theorem(3.9), $Cl_G^\beta(A)$ is a $\mathcal{G}(\Theta)$ -closed set and A is a $\mathcal{G}(\Theta)$ -closed set. Conversely, we have $A \subseteq Cl_G^\beta(A)$ by Remark(4.2). Since A is a $\mathcal{G}(\Theta)$ -closed set, then it is clear from the definition of $Cl_G^\beta(A)$, $Cl_G^\beta(A) \subseteq A$. Hence $A = Cl_G^\beta(A)$. $\square$

Theorem 4.4. For a $\mathcal{G}$ -metric space (X, G) and $A \subseteq X$, and $Int_G^\beta(A) = A$ if and only if A is a $\mathcal{G}(\Theta)$ -open set.

Proof. Let A be $\mathcal{G}(\Theta)$ -open set. Then for all $x \in A$, we have $x \in A \subseteq A$. That is, $A \subseteq Int_G^\beta(A)$. Then $A = Int_G^\beta(A)$ from Remark(4.2). The converse is trivial. $\square$

Theorem 4.5. For a $\mathcal{G}$ -metric space (X, G) and $A \subseteq X$, $x \in Cl_G^\beta(A)$ if and only if for all $\mathcal{G}(\Theta)$ -open set B containing x , $B \cap A \neq \emptyset$.

Proof. Let $x \in Cl_G^\beta(A)$ and B be any $\mathcal{G}(\Theta)$ -open set containing x . If $B \cap A = \emptyset$ then $A \subseteq X - B$. Since $X - B$ is a $\mathcal{G}(\Theta)$ -closed set containing A , then $Cl_G^\beta(A) \subseteq X - B$ and so $x \in Cl_G^\beta(A) \subseteq X - B$. Hence this is contradiction, because $x \in B$. Therefore $B \cap A \neq \emptyset$.

Conversely, Let $x \notin Cl_G^\beta(A)$. Then $X - Cl_G^\beta(A)$ is a $\mathcal{G}$ -open set containing x . Hence by hypothesis, $[X - Cl_G^\beta(A)] \cap A \neq \emptyset$. But this is contradiction, because $X - Cl_G^\beta(A) \subseteq X - A$. $\square$

Theorem 4.6. For a $\mathcal{G}$ -metric space (X, G) and $A \subseteq X$, $x \in Int_G^\beta(A)$ if and only if there is $\mathcal{G}(\Theta)$ -open set B such that $x \in B \subseteq A$.

Proof. Let $x \in Int_G^\beta(A)$ and take $B = Int_G^\beta(A)$. Then by Theorem(4.5) and definition of $Int_G^\beta(A)$ we get that B is a $\mathcal{G}(\Theta)$ -open set and by Remark(4.2), $x \in B \subseteq A$. Conversely, let there is $\mathcal{G}(\Theta)$ -open set B such that $x \in B \subseteq A$. Then by definition of $Int_G^\beta(A)$, $x \in B \subseteq Int_G^\beta(A)$. $\square$

Theorem 4.7. For a $\mathcal{G}$ -metric space (X, G) and $A, B \subseteq X$, the following hold:

1. If $A \subseteq B$ then $Cl_G^\beta(A) \subseteq Cl_G^\beta(B)$;
2. $Cl_G^\beta(A) \cup Cl_G^\beta(B) \subseteq Cl_G^\beta(A \cup B)$;
3. $Cl_G^\beta(A \cap B) \subseteq Cl_G^\beta(A) \cap Cl_G^\beta(B)$;
4. $Cl_G^\beta(A) \subseteq Cl_\beta(A)$.

Proof. 1. Let $x \in Cl_G^\beta(A)$. Then by Theorem(4.5), for all $\mathcal{G}(\Theta)$ -open set C containing x , $C \cap A \neq \emptyset$. Since $A \subseteq B$ then $C \cap B \neq \emptyset$. Hence $x \in Cl_G^\beta(B)$. That is, $Cl_G^\beta(A) \subseteq Cl_G^\beta(B)$.

2. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then by part(1), $Cl_G^\beta(A) \subseteq Cl_G^\beta(A \cup B)$ and $Cl_G^\beta(B) \subseteq Cl_G^\beta(A \cup B)$. Hence $Cl_G^\beta(A) \cup Cl_G^\beta(B) \subseteq Cl_G^\beta(A \cup B)$.
3. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then by part(1), $Cl_G^\beta(A \cap B) \subseteq Cl_G^\beta(A)$ and $Cl_G^\beta(A \cap B) \subseteq Cl_G^\beta(B)$. Hence $Cl_G^\beta(A \cap B) \subseteq Cl_G^\beta(A) \cap Cl_G^\beta(B)$.
4. It is clear from Theorem(4.5) and from every $\mathcal{G}$ -open set is $\mathcal{G}(\Theta)$ -open set. $\square$

In the above theorem $Cl_G^\beta(A \cup B) \neq Cl_G^\beta(A) \cup Cl_G^\beta(B)$ as it is shown in the following example.

Example 4.8. Let $(\mathbb{R}, G)$ be $\mathcal{G}$ -metric space, where

$$G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$$

and $(\mathbb{R}, d)$ is usual metric space. Let $A = IR$ and $B = Q - \{\sqrt{2}\}$, where Q is the set of rational numbers, IR is the set of irrational numbers and $\sqrt{2}$ is any rational number. Since A and B are $\mathcal{G}(\Theta)$ -closed sets in $\mathbb{R}$. Then $Cl_G^\beta(A) \cup Cl_G^\beta(B) = A \cup B = \mathbb{R} - \{\sqrt{2}\}$. If $\mathbb{R} - \{\sqrt{2}\}$ is $\mathcal{G}(\Theta)$ -closed set in $\mathbb{R}$ then $\{\sqrt{2}\}$ is $\mathcal{G}(\Theta)$ -open set but $\{\sqrt{2}\}$ is not $\mathcal{G}(\Theta)$ -open set and this contradiction. Hence $\mathbb{R} - \{\sqrt{2}\}$ is not $\mathcal{G}(\Theta)$ -closed set in $\mathbb{R}$. Since $\mathbb{R} - \{\sqrt{2}\} \subseteq Cl_G^\beta(\mathbb{R} - \{\sqrt{2}\})$ then

$$Cl_G^\beta(A \cup B) = Cl_G^\beta(\mathbb{R} - \{\sqrt{2}\}) = \mathbb{R}.$$

Theorem 4.9. For a $\mathcal{G}$ -metric space (X, G) and $A, B \subseteq X$, the following hold:

1. If $A \subseteq B$ then $Int_G^\beta(A) \subseteq Int_G^\beta(B)$;
2. $Int_G^\beta(A) \cup Int_G^\beta(B) \subseteq Int_G^\beta(A \cup B)$;
3. $Int_G^\beta(A \cap B) \subseteq Int_G^\beta(B) \cap Int_G^\beta(A)$;
4. $Int_G^\beta(A) \subseteq Int_G^\beta(A)$.

Proof. 1. Let $x \in Int_G^\beta(A)$. Then by Theorem(4.6), there is $\mathcal{G}(\Theta)$ -open set C such that $x \in C \subseteq A$. Since $A \subseteq B$ then $x \in C \subseteq B$. Hence $x \in Int_G^\beta(B)$. That is, $Int_G^\beta(A) \subseteq Int_G^\beta(B)$.

2. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then by part(1), $Int_G^\beta(A) \subseteq Int_G^\beta(A \cup B)$ and $Int_G^\beta(B) \subseteq Int_G^\beta(A \cup B)$. Hence $Cl_G^\beta(A) \cup Cl_G^\beta(B) \subseteq Int_G^\beta(A \cup B)$.

3. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then by part(1), $Int_G^\beta(A \cap B) \subseteq Int_G^\beta(A)$ and $Int_G^\beta(A \cap B) \subseteq Int_G^\beta(B)$. Hence $Int_G^\beta(A \cap B) \subseteq Int_G^\beta(A) \cap Int_G^\beta(B)$.

4. It is clear from Theorem(4.5) and from every $\mathcal{G}$ -open set is $\mathcal{G}(\Theta)$ -open set. $\square$

In the last theorem $Int_G^\beta(A \cap B) \neq Int_G^\beta(A) \cap Int_G^\beta(B)$ as it is shown in the following example.

Example 4.10. In Example(4.8), take $A = Q \cup \{\sqrt{2}\}$ and $B = IR$, where Q is the set of rational numbers, IR is the set of irrational numbers and $\sqrt{2}$ is any irrational number. Since A and B are $\mathcal{G}(\Theta)$ -open sets in $\mathbb{R}$. Then $Int_G^\beta(A) \cap Int_G^\beta(B) = A \cap B = (Q \cup \{\sqrt{2}\}) \cap IR = \{\sqrt{2}\}$. Since $\{\sqrt{2}\}$ is not $\mathcal{G}(\Theta)$ -open set and $Int_G^\beta(\{\sqrt{2}\}) \subseteq \{\sqrt{2}\}$ then $Int_G^\beta(A \cap B) = Int_G^\beta(\{\sqrt{2}\}) = \emptyset$.

Theorem 4.11. For a $\mathcal{G}$ -metric space (X, G) and $G \subseteq X$, the following hold:

1. $Int_G^\beta(X - A) = X - Cl_G^\beta(A)$;
2. $Cl_G^\beta(X - A) = X - Int_G^\beta(A)$.

Proof. 1. Since $A \subseteq Cl_G^\beta(A)$, then $X - Cl_G^\beta(A) \subseteq X - A$. Since $Cl_G^\beta(A)$ is a $\mathcal{G}(\Theta)$ -closed set then $X - Cl_G^\beta(A)$ is a $\mathcal{G}$ -open set. Then

$$X - Cl_G^\beta(A) = Int_G^\beta[X - Cl_G^\beta(A)] \subseteq Int_G^\beta(X - A).$$

For the other side, let $x \in Int_G^\beta(X - A)$. Then there is $\mathcal{G}(\Theta)$ -open set C such that $x \in C \subseteq X - A$. Then $X - C$ is a $\mathcal{G}(\Theta)$ -closed set containing A and $x \notin X - C$. Hence $x \notin Cl_G^\beta(A)$, that is, $x \in X - Cl_G^\beta(A)$.

2. Since $Int_G^\beta(A) \subseteq A$, then $X - A \subseteq X - Int_G^\beta(A)$. Since $Int_G^\beta(A)$ is a $\mathcal{G}(\Theta)$ -open set then $X - Int_G^\beta(A)$ is a $\mathcal{G}(\Theta)$ -closed set. Then

$$Cl_G^\beta(X - A) = Cl_G^\beta[X - Int_G^\beta(A)] = X - Int_G^\beta(A).$$

For the other side, let $x \notin Cl_G^\beta(X - A)$. Then by Theorem(4.5), there is a $\mathcal{G}(\Theta)$ -open set C containing x such that $C \cap (X - A) = \emptyset$. Then $x \in C \subseteq A$, that is, $x \in Int_G^\beta(A)$. Hence $x \notin X - Int_G^\beta(A)$. Therefore $X - Int_G^\beta(A) \subseteq Cl_G^\beta(X - A)$. $\square$

Theorem 4.12. For a subset $A \subseteq X$ of $\mathcal{G}$ -metric space (X, G) the following hold:

1. If B is a $\mathcal{G}$ -open set in X then $Cl_G^\beta(A) \cap B \subseteq Cl_G^\beta(A \cap B)$;
2. If B is a $\mathcal{G}$ -closed set in X then $Int_G^\beta(A \cup B) \subseteq Int_G^\beta(A) \cup B$.

Proof. 1. Let $x \in Cl_G^\beta(A) \cap B$. Then $x \in Cl_G^\beta(A)$ and $x \in B$. Let D be any $\mathcal{G}(\Theta)$ -open set in (X, G) containing x . By Theorem(3.8), $D \cap B$ is $\mathcal{G}(\Theta)$ -open set containing x . Since $x \in Cl_G^\beta(A)$ then by Theorem(4.5), $(D \cap B) \cap A \neq \emptyset$. This implies, $D \cap (B \cap A) \neq \emptyset$. Hence by Theorem(4.5), $x \in Cl_G^\beta(A \cap B)$. That is, $Cl_G^\beta(A) \cap B \subseteq Cl_G^\beta(A \cap B)$.

2. Since B is a G -closed set X then by the part(1) and Theorem(4.11),

$$\begin{aligned} X - [Int_G^\beta(A) \cup B] &= [X - Int_G^\beta(A)] \cap [X - B] \\ &= [Cl_G^\beta(X - A)] \cap [X - B] \\ &\subseteq Cl_G^\beta[(X - A) \cap (X - B)] \\ &= Cl_G^\beta(X - (A \cup B)) \\ &= X - (Int_G^\beta(A \cup B)). \end{aligned}$$

$$\text{Hence } Int_G^\beta(A \cup B) \subseteq Int_G^\beta(A) \cup B.$$

□

Theorem 4.13. For a $\mathcal{G}$ -metric space (X, G) and $A \subseteq X$, $x \in Cl_G(A)$ if and only if for all $\varepsilon > 0$, $B_G(x, \varepsilon) \cap A \neq \emptyset$.

Proof. Let $x \in Cl_G(A)$ and $\varepsilon > 0$. If $B_G(x, \varepsilon) \cap A = \emptyset$ then $A \subseteq X - B_G(x, \varepsilon)$. Since $X - B_G(x, \varepsilon)$ is a G -closed set containing A , then $Cl_G(A) \subseteq X - B_G(x, \varepsilon)$ and $x \in Cl_G(A) \subseteq X - B_G(x, \varepsilon)$. Hence this is contradiction, because $x \in B_G(x, \varepsilon)$. Therefore $B_G(x, \varepsilon) \cap A \neq \emptyset$.

Conversely, Let $x \notin Cl_G(A)$. Then $X - Cl_G(A)$ is a $\mathcal{G}$ -open set containing x . Then there is $\varepsilon > 0$ such that $B_G(x, \varepsilon) \subseteq X - Cl_G(A)$. Hence by hypothesis, $B_G(x, \varepsilon) \cap A \neq \emptyset$. But this is contradiction, because $B_G(x, \varepsilon) \subseteq X - Cl_G(A) \subseteq X - A$. □

For a subset A of $\mathcal{G}$ -metric space (X, G) the G -frontier operator of A is defined by

$$\Gamma_G^\beta(A) = Cl_G^\beta(A) - Int_G^\beta(A).$$

Theorem 4.14. For a subset $A \subseteq X$ of $\mathcal{G}$ -metric space (X, G) , the following hold:

1. $Cl_G^\beta(A) = \Gamma_G^\beta(A) \cup Int_G^\beta(A)$;
2. $\Gamma_G^\beta(A) \cap Int_G^\beta(A) = \emptyset$;
3. $\Gamma_G^\beta(A) = Cl_G^\beta(A) \cap Cl_G^\beta(X - A)$.

Proof. 1. Note that

$$\begin{aligned} \Gamma_G^\beta(A) \cup Int_G^\beta(A) &= (Cl_G^\beta(A) - Int_G^\beta(A)) \cup Int_G^\beta(A) \\ &= [Cl_G^\beta(A) \cap (X - Int_G^\beta(A))] \cup Int_G^\beta(A) \\ &= [Cl_G^\beta(A) \cup Int_G^\beta(A)] \cap [(X - Int_G^\beta(A)) \cup Int_G^\beta(A)] \\ &= Cl_G^\beta(A) \cap X = Cl_G^\beta(A). \end{aligned}$$

2. It is clear from the definition of $\Gamma_G^\beta(A)$.

3. By Theorem(4.11),

$$\begin{aligned} \Gamma_G^\beta(A) &= Cl_G^\beta(A) - Int_G^\beta(A) \\ &= Cl_G^\beta(A) \cap (X - Int_G^\beta(A)) \\ &= Cl_G^\beta(A) \cap Cl_G^\beta(X - A). \end{aligned}$$

□

Corollary 4.15. For a subset $A \subseteq X$ of $\mathcal{G}$ -metric space (X, G) , $\Gamma_G^\beta(A)$ is $\mathcal{G}(\Theta)$ -closed set in (X, G) .

Proof.

Proof. By Theorem(4.9) and the part(3) of the last theorem. □

Theorem 4.16. For a subset $A \subseteq X$ of $\mathcal{G}$ -metric space (X, G) , the following hold:

1. A is a $\mathcal{G}(\Theta)$ -open set if and only if $\Gamma_G^\beta(A) \cap A = \emptyset$;
2. A is a $\mathcal{G}(\Theta)$ -closed set if and only if $\Gamma_G^\beta(A) \subseteq A$;
3. A is both $\mathcal{G}(\Theta)$ -open set and $\mathcal{G}(\Theta)$ -closed set if and only if $\Gamma_G^\beta(A) = \emptyset$.

Proof. 1. Let A be a $\mathcal{G}(\Theta)$ -open set. Then $Int_G^\beta(A) = A$. Then by Theorem(4.14),

$$\Gamma_G^\beta(A) \cap A = \Gamma_G^\beta(A) \cap Int_G^\beta(A) = \emptyset$$

Conversely, suppose that $\Gamma_G^\beta(A) \cap A = \emptyset$. Then

$$\begin{aligned} A - Int_G^\beta(A) &= [A \cap Cl_G^\beta(A)] - [G \cap Int_G^\beta(A)] \\ &= G \cap (Cl_G^\beta(A) - Int_G^\beta(A)) \\ &= A \cap \Gamma_G^\beta(A) = \emptyset. \end{aligned}$$

That is, $Int_G^\beta(A) = A$. Hence A is a $\mathcal{G}(\Theta)$ -open set.

2. Let A be a $\mathcal{G}(\Theta)$ -closed set. Then $Cl_G^\beta(A) = A$. Then

$$\Gamma_G^\beta(A) = Cl_G^\beta(A) - Int_G^\beta(A) = A - Int_G^\beta(A) \subseteq A.$$

Conversely, suppose that $\Gamma_G^\beta(A) \subseteq A$. Then by Theorem(4.14),

$$Cl_G^\beta(A) = Int_G^\beta(A) \cup \Gamma_G^\beta(A) \subseteq Int_G^\beta(A) \cup A \subseteq A.$$

That is, $Cl_G^\beta(A) = A$. Hence A is $\mathcal{G}(\Theta)$ -closed set.

3. Let A be both $\mathcal{G}(\Theta)$ -closed set and $\mathcal{G}(\Theta)$ -open set. Then $Cl_G^\beta(A) = A = Int_G^\beta(A)$. Then

$$\Gamma_G^\beta(A) = Cl_G^\beta(A) - Int_G^\beta(A) = A - A = \emptyset.$$

Conversely, suppose that $\Gamma_G^\beta(A) = \emptyset$. Then $Cl_G^\beta(A) - Int_G^\beta(A) = \emptyset$. Since $Int_G^\beta(A) \subseteq Cl_G^\beta(A)$ then $Cl_G^\beta(A) = Int_G^\beta(A)$. Since $Int_G^\beta(A) \subseteq A \subseteq Cl_G^\beta(A)$ then

$$Cl_G^\beta(A) = A = Int_G^\beta(A).$$

That is, $Cl_G^\beta(A) = A$. Hence A is both $\mathcal{G}(\Theta)$ -closed set and $\mathcal{G}(\Theta)$ -open set. □

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