n-Dual Square Free Ring

Abdul-Qawe Kaed
Mathematics Department, Faculty of Applied Science, Thamar University, Yemen.
abdulqawe.dabouan@tu.edu.ye

Abstract
In this work we introduce and study : In the part three (main result) which is called n- dual square free ring (n-DSF-ring for a short), in the part four some important results of n-dual square free rings with some kinds rings, in the part five some algebra operations on n-dual square free rings and we give example, properties, propositions, theorems and corollaries about them.
Keywords: *n-Dual Square-Free Ring*

حلقة حرة ثنائية المربعاتn-

الملخص
نقدم في هذا العمل وندرس: في الجزء الثالث (النتيجة الرئيسية) والتي تسمى-n حلقة حرة ثنائية المربعات (n-DSF-ring باختصار)، وفي الجزء الرابع بعض النتائج المهمة لحلقات حرة ثنائية المربعات n مع بعض أنواع الحلقات، وفي الجزء الخامس بعض العمليات الجبرية على-n حلقة حرة ثنائية المربعات ونعطي أمثلة وخصائص ومقترحات ونظريات ومشتقات عنها.
الكلمات المفتاحية: حلقة حرة-ثنائية المربعاتn-.

1 Introduction

Throughout this work all rings are noncommutative with unit unless otherwise noted and all R-modules are left R.

In [22], the concept of a square free ring was introduced and studied. It was defined as : A ring R is called a square-free if R is a square free R-module. The notion of square-freeness is very important in a ring theory and it is used as a tool to establish many useful and significant results in the research most notably.

A dual notion to square-free ring was introduced most recently in [14], and it was defined as : A ring R is called a dual square free ring (a DSF- ring for a short), if R is a dual square free R-module (i.e it was defined as : A ring R is called a dual square free (a DSF- module for a short), if R has no proper ideals A and B with $R = A + B$ with $R/A \cong R/B$. This work consists of five parts. We introduce and study in part one the introduction of the work, in part two some preliminaries, in part three the concept of n-dual square free ring (main result) as the generalized of the concept of a dual square free ring and it is defined as: A ring R is called n-daul square free ring (n-DSF-ring for a short), if R is n-dual square free R-module (i.e For a ring R and a positive integer n, a ring R is called n-dual-square (n-DSF for a short), if R has no proper ideal $I_1, I_2, \ldots, I_n$ with $R = I_1 + I_2 + \ldots + I_n$ with $R/I_i \cong R/I_j$ and $i, j = 1, 2, \ldots, n$ with $i \neq j$ and $n \geq 2$) 3.1, where the concept of R n-DSF-R module is defined as: For a ring R and a positive integer n. An R-module M is called n-dual-square free R-module n-DSF R-module, if M has no proper submodules $M_1, M_2, \ldots, M_n$ with $M = M_1 + M_2 + \ldots + M_n$ and $M/M_i \cong M/M_j$ and $i, j = 1, 2, \ldots, n$, $i \neq j$ and $n \geq 2$. **??**,.

In 3.3, we give some important remarks as the following : Let R be a ring. R is a DSF ring this equivalent R is 2-DSF ring. R is called a distributive ring if $I \cap (I_1 + I_2) = I \cap I_1 + I \cap I_2$ for all I, I_1 and $I_2 \in R$. R is distributive ring if and only if every ideal of R is DSF [22],. R is called $n-$ DSF distributive ring if $I \cap (I_1 + I_2 + ... + I_n) = I \cap I_1 + I \cap I_2 + ... + I \cap I_n$ for all $I, I_1, I_2, ..., I_n \in R$. R is $n-$ DSF distributive ring if and only if every ideals of R is n-DSF. An ideal J of a ring R is called a proper if $J \subsetneqq R$ and finally it is obviously that every n-DSF ring R is k-DSF R for every positive integer $k \leq n$. In 3.4, we give characterization of n-DSF ring as: R is n-DSF ring if and only if R is n-DSF as R-module,. In 3.5, explain that if R is a distributive n-DSF ring if and only if R is n^2 -DSF ring. In 3.6, R is a DSF-ring if and only if R is $n-$ DSF ring. In 3.7, we give the important characterization of n-DSF-ring as: R is n-DSF ring if and only if R is a finitely generated ring. In 3.11, if R is $n-$ DSF ring, then every ideal of R is n-DSF and R/I is $n-$ DSF ring, where I is an ideal of R. In 3.12, we give characterization of n DSF ring as : Let R be a ring the following are equivalents: R is n-DSF ring if and only if R is finitely generated ring if and only if R is Noetherian ring if and only if R is finitely presented ring if and only if R is a coherent ring. In 3.13, iet R be a ring then the following are holds : Every ideal of finitely generated ring R is n-DSF, every ideal of Neotherian ring R is n-DSF, every ideal of finitely presented ring R is n-DSF and every ideal of a a coherent ring R is n-DSF.

In 3.14, we study the behavior of n-DSF rings on the short exact sequence of a homomorphism rings such that $0 \longrightarrow S \xrightarrow{f} R \xrightarrow{g} T \longrightarrow 0$, then we have the following cases : R is n-DSF if and only if S and T are n-DSF. S and R are n-DSF, then T is n-DSF. R and T are n-DSF, then S is n-DSF. R is n-DSF if and only if $S \oplus T$ is n-DSF. In 3.15, Let R be a ring. I, J and L be ideals of R. Let $0 \longrightarrow I \xrightarrow{f} L \longrightarrow J \longrightarrow 0$ be a short exact sequence of homomorphism rings, then we have the following cases : L is n-DSF ideal if and only if I and J are n-DSF ideals. I and L are n-DSF ideals, then J is n-DSF ideal. L and J are n-DSF ideals, then I is n- DSF ideal. I and J n-DSF ideals if and only if $I \oplus J$ is n-DSF ideal.

In part four we introduce and study some important results of n-dual square free rings with some Kinds Rings as the following: Let R be a ring and if R a maximal 4.1, a finitely generated 4.2, a local (A ring R is called a local if it has a unique maximal ideal) 4.3, a semilocal (A ring R is called a semilocal if it has finitely many maximals ideals) 4.4, Notherian 4.6, pricepal ideal domain 4.7, and prime ideal 4.8, rings, then R is n-DSF ring in every case. In 4.5, let R be a ring and I be an ideal of R, if I is $n-$ DSF ideal and R/I is $n-$ DSF ring, then R is also n-DSF ring. In 4.9, we gave an example and proved $\mathbb{Z}$ is a n-DSF ring.

In part five we introduce and study some algebra operations on n DSF ring such that in 3.7, let R and S be n-DSF rings, then we have the following cases : $R+S$, $R \cap S$ and $\times$ are Sn-DSF rings. In 5.2, we generalized 5.1,. Also in 5.2, let R_i be n-DSF rings, then we have the following cases : $\Sigma_{i=1}^{n} R_i$, $\cap_{i=1}^{n} R_i$ and $\prod_{i=1}^{n} R_i$ are n-DSF rings. In 5.3, let I and J be ideals of a ring R and if R is distributive ring, then we have the following cases: $I + J$, $I \cap J$ and $I \times J$ are n-DSF rings. In 5.4, we generalized

5.3, such that I_i are n-DSF ideals, $i = 1, 2, ..., n$, then $\Sigma_{i=1}^{n} I_i$, $\cap_{i=1}^{n} I_i$ and $\prod_{i=1}^{n} I_i$ are n-DSF ideals.

2 Preliminaries

Definition 2.1. *[14] For a ring R and a positive integer n. An R-module M is called n-dual-square free R-module n-DSF R-module, if M has no proper submodules $M_1, M_2, \ldots, M_n$ with $M = M_1 + M_2 + \ldots + M_n$ and $M / M_i \cong M / M_j$ and $i, j = 1, 2, \ldots, n$, $i \neq j$ and $n \geq 2$.*

Definition 2.2. *For a ring R, an R-module R is called finitely generated (f. g for short), for every family $\{T_i\}_{i \in I}$ of ideals of R with $R = \Sigma_{i \in I} T_i$, there is a finite subset $J \subset I$ such that $R = \Sigma_{i \in J} T_j$.*

Definition 2.3. *An ideal I in a ring R is called a maximal if $I \neq R$ and I is such that if J is an ideal with $I \subseteq J \subseteq R$, then $I = J$ or $J = R$.*

3 n-Dual Square Free ring

In this part we introduce and study the concept of n-dual square free ring (n-DSF ring for short) and some results about it.

Definition 3.1. *A ring R is called n-dual square free ring (n-DSF for a short), if R is n-DSF R module. (i.e For a ring R and a positive integer n, a ring R is called n-dual-square (n-DSF for a short), if R has no proper ideals $I_1, I_2, \ldots, I_n$ with $R = I_1 + I_2 + \ldots + I_n$ with $R/I_i \cong R/I_j$ and $i, j = 1, 2, \ldots, n$ with $i \neq j$ and $n \geq 2$)*

Definition 3.2. *Let R be a ring.*

1. *A ring R is called a dual square free (a DSF for a short) if it is a DSF R module. [14]*
2. *A ring R is called finitely generated if it is finitely generated R module. [1]*
3. *A ring R is called Noetherian if it is Noetherian R module. [13]*
4. *A ring R is called finitely presented if it is finitely presented R module. [27]*
5. *A ring R is called coherent if it is coherent R module. [27]*

Remark 3.3. *Let R be a ring:*

1. *A ring R is called distributive n-DSF ring, if R is distributive n- DSF R-module, i.e $J \cap (J_1 + J_2 + ... + J_n) = J \cap J_1 + J \cap J_2 + ... + J \cap J_n$, for all $J, J_1, J_2, ..., J_n$ are R-submodules of R.*
2. *Let R be a distributive ring, then R is a DSF RING this equivalent R is 2-DSF ring.*
3. *Let R be distributive ring, then this equivalent R is 2-DSF ring. [14]*
4. *Distributive $n-$ DSF ring if and only if every ideal of R is $n-$ DSF as R-submodule of R. [22]*
5. *An ideal I of a n-DSF ring R is called a proper if $I \subsetneq R$.*
6. *Obviously, every n-DSF ring is k-DSF for every positive integer $k \leq n$.*

Lemma 3.4. *A ring R is n-DSF if and only if R is n-DSF R-module.*

Proof. Suppose that R is n-DSF ring. It is clear that R is an R-module and every ideal of a ring is R-submodules of R as R-module. From the hypothesis R has no proper submodules $I_1, I_2, ..., I_n$ with $R = I_1 + I_2 + ... + I_n$. It is easily to prove that $R/I_i \cong R/I_j, i = 1, 2, ..., n$ with $i \neq j$. Hence R is $n-$ DSF as R-module.

Conversely, let R be $n-$ DSF as R-module, then by 3.1, R is n-DSF ring.

Proposition 3.5. *Let R be a distributive ring. R is n-DSF if and only if R is n^2-DSF.*

Proof. Let R be a distributive ring, then R is a distributive R-module. Suppose that [12], R is n-DSF ring if and only if by 3.4, R is n-DSF R-module if and only if by 3.5, R is n^2-DSF R-module if and only if R is n^2-DSF ring.

Proposition 3.6. *Let R be a ring distributive. R is a DSF if and only if R is $n-$ DSF.*

Proof. Let R be a ring distributive ring, then R is a distributive R-module. Suppose that R is a DSF ring this is equivalent that by 3.4, R is a DSF R-module and by [12], and 3.3, R is $n-$ DSF R-module if and only if by ref175t, R is $n-$ DSF ring.

Theorem 3.7. *Let R be an n-DSF distributive ring. R is a n-DSF ring if and only if R is finitely generated ring.*

Proof. Suppose that R is n-DSF distributive ring, then R is n-DSF distributive R-module. that is equivalent that by 3.1, R is n-DSF R-module that is equivalent that by [12], R is finitely generated R module that is equivalent that by 3.2, R is finitely generated ring.

Theorem 3.8. *Let R be a ring. R is finitely generated if and only if R is n-DSF ring.*

Proof. Let R be finitely generated ring this equivalent that by 3.2 R is finitely generated R-module if and only if by [12], R is n-DSF R-module if and only if by 3.4, R is n-DSF ring.

Proposition 3.9. *Let R be n-DSF distributive ring, then every ideal of R is n-DSF.*

Proof. Let R be $n-$ DSF distributive ring this equivalent that R be $n-$ DSF distributive module and by 3.7, 3.8, R is finitely generated R-module and let J be an ideal of R this implies that J is finitely generated R submodule of R that implies that by [14], J is n-DSF R submodule of R, hence by 3.4, J is n-DSF ideal.

Proposition 3.10. *Let R be n-DSF distributive ring, then R/I is $n-$ DSF ring, where I is an ideal of R.*

Proof. Let R be n-DSF distributive ring this equivalent that R be $n-$ DSF distributive module and let I be an ideal of a ring R, then I is $R-$ submodule of R_R module, then R/I is the quotient R-module and by 3.7, 3.8, R is finitely generated R-module. By 3.7, this implies that R/I is also finitely generated R-module and hence by 3.7, R/I is n-DSF $R-$ module that is equivalent that 3.4, R/I is n-DSF ring.

Proposition 3.11. *Let R be finitely generated ring and let I be an ideal of a ring R, then*

1. *I is n-DSF ideal of a ring R.*

2. *R/I is n-DSF ring.*

Proof.

1. Let R be finitely generated ring, then by 3.2, R is finitely generated R-module this implies that R is n-DSF R-module, then I is finitely generated R-submodule of R this equivalent that I is n-DSF as R-submodule of R is $R-$ubmodule, hence by 3.4, I is n-DSF ideal.

2. Let R be finitely generated ring, then by **??**, R is finitely generated R-module this implies that, R/I is finitely generated R-module and by 3.7, R/I is $n-$ DSF R-module, hence by 3.4 R/I is n-DSF ring.

Theorem 3.12. *Let R be a ring. The following are equivalents :*

1. *R is n-DSF ring.*

2. *R is finitely generated ring.*

3. *R is Noetherian ring.*

4. *R is finitely presented ring.*

5. *R is a coherent ring.*

Proof.

1. (1) $\Longrightarrow$ (2). Suppose that R is n-DSF ring that is equivalent that R is n-DSF R-module this is equivalent that R is finitely generated module and hence by 3.6 R is finitely generated ring.

2. (2) $\Longrightarrow$ (3). Assume that R is finitely generated ring that is equivalent that R is finitely generated R-module by [?] R is Noetherian R-module and hence R is Noetherian ring.

3. (3) $\Longrightarrow$ (4). Assume that R is Noetherian ring this is equivalent that R is Noetherian R-module then by [13] R is finitely generated R-module, then R is finitely presented R module and hence R is finitely presented ring.

4. (4) $\Longrightarrow$ (5). Suppose that R is is finitely presented ring that is equivalent that R is is finitely presented R-module this equivalent that R is a coherent R-module and hence R is a coherent ring.

5. (5) $\Longrightarrow$ (1). Suppose that R is a coherent ring this equivalent that R is a coherent R-module, then R is finitely presented R module this equivalent that R is finitely generated R module this equivalent that R is n-DSF R-module and hence R is n-DSF ring.

Corollary 3.13. *Let R be a ring and R be an R-module, the following are holds :*

1. *Every ideal of a finitely generated ring is n-DSF ideal.*

2. *Every ideal of a Neotherian ring is n-DSF ideal.*

3. *Every ideal of a finitely presented ring is n-DSF ideal.*

4. Every ideal of a coherent R ring is n-DSF ideal.

Proof. 1. Let R be finitely generated ring, then R is finitely generated R-module this implies that by 3.12, R is n-DSF R-module and by [12], I is n-DSF R-submodule of R, hence by 3.4, I is n-DSF ideal.

2. Let R be Neotherian R-module ring, then R is Neotherian R-module this implies that by 3.12, R is n-DSF R-module and by [12], I is n-DSF R-submodule of R, hence by 3.4, I is n-DSF ideal.

3. Let R be finitely presented ring, then R is finitely presented R-module this implies that by 3.12, R is n-DSF R-module and by [12], I is n-DSF R-submodule of R, hence by 3.4, I is n-DSF ideal.

4. Let R be a coherent ring, then R is coherent R-module this implies that by 3.12, R is n-DSF R-module and by [12], I is n-DSF R-submodule of R, hence by 3.4, I is n-DSF ideal.

Theorem 3.14. *Let R, S, T be rings and $0 \longrightarrow S \longrightarrow R \longrightarrow T \longrightarrow 0$ be a short exact sequence of R-homomorphism rings, then we have the following cases :*

1. *R is n-DSF ring if and only if S and T are n-DSF rings.*
2. *S and R are n-DSF ring, then T is n-DSF ring.*
3. *R and T are n-DSF ring, then S is n- DSF ring.*
4. *T and S n-DSF ring if and only if $R \bigoplus T$ is n-DSF ring.*

Proof. 1. Suppose that R is n-DSF ring this equivalent that R is n-DSF R-module this equivalent that R finitely generated R-module this equivalent that S and T are finitely generated R-modules this equivalent that S and T are $n-$ DSF R-modules and from 3.4 this equivalent that S and T are $n-$ DSF rings.

Conversely, Suppose that S and T are n-DSF rings and by 3.6 this equivalent that R finitely generated R-module and this equivalent this equivalent that R is n-DSF R-module and hence this equivalent that R is n-DSF ring.

2. Suppose that S and R are n-DSF rings, then they are finitely generated R-modules and T must be finitely generated R-module because if it is not this implies that R is not finitely generated module. Hence T is finitely generated ring and it is n-DSF R-module, then by 3.6 T is n-DSF ring.

3. Suppose that T and R are n-DSF rings, then they are n-DSF-modules and this equivalent that T and R are finitely generated R-modules and so S finitely generated R-module this equivalent that S is $n-$ DSF R-module and from 3.6 this equivalent that S is $n-$ DSF ring.

4. Assume that the short exact sequence of R-modules: $0 \longrightarrow S\ longrightarrowR = S + T \longrightarrow T\ longrightarrow0$ We Apply (1) S and T are n-DSF R-modules, then $R = S \bigoplus T$ is n-DSF R-module.

Corollary 3.15. *Let R be a ring. I, J and L be ideals. Let $0 \longrightarrow I \xrightarrow{f} L \longrightarrow J \longrightarrow 0$ be a short exact sequence of homomorphism rings, then we have the following cases :*

1. *L is n-DSF ideal if and only if I and J are n-DSF ideals.*
2. *I and L are n-DSF ideals, then J is n-DSF ideal.*
3. *L and J are n-DSF ideals, then I is n- DSF ideals.*
4. *I and J n-DSF ideals if and only if $I \oplus J$ is n-DSF ideal .*

Proof. 1. Let I and J be ideals of a ring R, then by 3.4, I, J and L are R-modules and from [12], in 3.1.3. Suppose that I, J are n-DSF R-modules that is equivalent that L is n-DSF R-module.

2. Let I and L be ideals of a ring R, then by 3.4, I, L are R-modules and from [12], in 3.1.3. Suppose that I, L are n-DSF R-modules, then J is n-DSF R-module.

3. Let J and L be ideals of a ring R, then by 3.4, J, L are R-modules and from [12], in 3.1.3. Suppose that J, L are n-DSF R-modules, then I is n-DSF R-module.

4. Let I, J and L be ideals of a ring R, then by 3.4, I, J and L are R-modules and from [12], in 3.1.3. Suppose that I, J are n-DSF R-modules that is equivalent that $L = I \oplus J$ is n-DSF R-module.

4 Some results of n-Dual Square Free Rings with some Kinds Rings

Theorem 4.1. *Let R be a ring and I be a maximal ideal of R, then R is n-DSF ring.*

Proof. Let R be a ring and I be a maximal ideal. Let $x_i \in R$ and x_i does not belong to I, $i = 1, 2, ..., n$. Hence by [13], R has no proper R-submodules $J_1 = x_1R + I, J_2 = x_2R + I, ..., J_n = x_nR + I$, with $J_i \subsetneqq R$ and $R = J_1 + J_2 +, ..., +J_n$. It is easily to prove that $R/J_i \cong R/J_j$ with $i, j = 1, 2, ..., n$, and $i \neq j$. Hence by [14], R is n-DSF is R-module and from 3.4, R is n-DSF ring.

Corollary 4.2. *Let R be a finitely generated ring R, then R is n-DSF ring.*

Proof. Let R be a a finitely generated ring, then R has at least one maximal ideal. Suppose that this is a maximal ideal of a ring R is I and and from the 4.1, hence R R is n-DSF ring.

Theorem 4.3. *Let R be a local ring, then R is n-DSF ring.*

Proof. Let R be a local ring that implies that R has a unique maximal ideal I. Let $x_i \in R$ and x_i does not belong to I, $i = 1, 2, ..., n$. Hence by [13], R has no proper R-submodules $J_1 = x_1R+I, J_2 = x_2R+I, ..., J_n = x_nR+I$, with $J_i \subsetneqq R$ and $R = J_1 + J_2 +, ..., +J_n$. It is easily to prove that $R/J_i \cong R/J_j$ with $i, j = 1, 2, ..., n$, and $i \neq j$. Hence by [12], R is n-DSF is R-module and from 3.4, R is n-DSF ring.

Theorem 4.4. *Let R be a semilocal ring, then R is n-DSF ring.*

Proof. Let R be a semilocal ring that implies that R has many finitely maximal ideals $I_1, I_2, ..., I_n$. Let $x \in R$ and x does not belong to any maximal I_i, $i = 1, 2, ..., n$. Hence by [13], R has no proper R-submodules $J_1 = xR + I_1, J_2 = xR + I_2, ..., J_n = xR + I_n$, with $J_i \subsetneqq R$ and $R = J_1 + J_2 +, ..., +J_n$. It is easily to prove that $R/J_i \cong R/J_j$ with $i, j = 1, 2, ..., n$, and $i \neq j$. Hence by [12], R is n-DSF as R-module and from 3.4, R is n-DSF ring.

Proposition 4.5. *Let R be a ring and I be an ideal, if I and R/I are $n-$ DSF ring, then R is n-DSF ring.*

Proof. Let R be a ring and I be an ideal of R, then we have $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ be a short exact sequence of a homomorphism ring. If I is $n-$ DSF ideal and R/I is a $n-$ DSF ring, then from 3.1, I and R/I are $n-$ DSF R- modules that is implies that from 3.7, they are finitely generated R-modules this equivalent that R is finitely generated R-module, then R is n-DSF R-module, hence from 3.4, R is n-DSF ring.

Proposition 4.6. *Let R be a ring, if R is Noetherian ring, then R is n-DSF ring.*

Proof. Let R be Noetherian ring, then R is Noetherian R-module and from [13], R is finitely generated R-module, hence by 3.4, R is $n-$ DSF R-module and it is $n-$ DSF R ring.

Theorem 4.7. *Let R be a PID ring, then R is n-DSF ring.*

Proof. Let R be a PID ring and It is a known that every ideal in a PID R is a finitely generated R submodule of R as R- module and also any finitely generated R-module has at least a maximal ideal. Suppose that R has maximal ideals as R submodules I_i and $i = 1, 2, ..., n$. and $I_i \subseteq R$. let $x \in R$ and x does not belong to any maximal ideal I_i and $i = 1, 2, ..., n$. Hence R has no proper ideals as R-submodules $J_1 = x_1R + I_1, J_2 = x_2R + I_2, ..., J_n = x_nR + I_n$. with $J_i \subsetneqq R$. and $R = J_1 + J_2 +, ..., +J_n$. And it is clear that $R/J_i \cong R/J_j$ and $i, j = 1, 2, ..., n, i \neq j$. Hence R is n-DSF R-module and by 3.4, R is n-DSF ring.

Theorem 4.8. *Let R be a PID ring with identity. Let $I_1, I_2, ..., I_n$ be prime ideals of R, then R is n-DSF ring.*

Proof. Let R be Let R be a PID ring with identity. Let $I_1, I_2, ..., I_n$ be prime ideals of R this equivalent by [1] $I_1, I_2, ..., I_n$ are maximal ideals of R and $I_i \subsetneqq R$ and $i = 1, 2, ..., n,$. Let $x \in R$ and x does not belong to any maximal ideal of R. It is clear that every ideal of R is R-submodule of R as R- module, then by [13] R has no proper R-submodules $J_1 = xR + I_1, J_2 = xR + I_2, ..., J_n = xR + I_n$. and $J_i \subsetneqq R$ with $R = J_1 + J_2 +, ..., +J_n$. And it is easily prove that $R/J_i \cong R/J_j$ and $i, j = 1, 2, ..., n$, and $i \neq j$. and hence R is n-DSF R-module and from 3.4, R is n-DSF ring.

Example 4.9. *Let $\mathbb{Z}$ be the integer ring, then $\mathbb{Z}$ is a n-DSF ring.*

Proof. It is clear that $\mathbb{Z}$ is $\mathbb{Z}$-module and $\mathbb{Z}$ is a PID, then every ideal of $\mathbb{Z}$ is a finitely generated $\mathbb{Z}$-submodule of $\mathbb{Z}$ as $\mathbb{Z}$- module. Let p_i and $i = 1, 2, ..., n$ be prime numbers, then are maximal ideals and $p_i\mathbb{Z} \subsetneqq \mathbb{Z}$. hence $\mathbb{Z}$ has no proper $\mathbb{Z}$-submodules $J_1 = x\mathbb{Z} + p_1\mathbb{Z}, J_2 = x\mathbb{Z} + p_2\mathbb{Z}, ..., J_n = x\mathbb{Z} + p_n\mathbb{Z}$. and $J_i \subseteqq \mathbb{Z}$ with $\mathbb{Z} = J_1 + J_2 +, ..., +J_n$. And it is easily prove that $\mathbb{Z}/J_i \cong \mathbb{Z}/J_j$ and $i, j = 1, 2, ..., n$, and $i \neq j$. and hence $\mathbb{Z}$ is n-DSF $\mathbb{Z}$-module and from 3.4, $\mathbb{Z}$ is n-DSF ring.

5 Algebra of n-Dual Square Free Rings and ideals of them

In this part we study some algebra operations as: the intersection, addition and multiplication of n-dual square free ring and generalized them.

Proposition 5.1. *Let R and S be n-DSF rings, if R and S are distributive rings, then*

1. *R and S are n-DSF rings if and only if $R + S$ is n-DSF ring.*
2. *R and S are n-DSF rings if and only if $R \cap S$ is n-DSF ring.*
3. *R and S are n-DSF rings if and only if $R \times S$ is n-DSF ring.*

Proof.

1. Let R and S be n-DSF rings. We put $T = R + S$, then T is a ring and hence $T = R + S$ is a T-module. If R and S are distributive rings, then by 3.4, R is n-DSF-R-module and also S is n-DSF-S-module. By [12], $T = R + S$ is $n-$ DSF $T = R + S$-module, hence 3.4, $T = R + S$ is $n-$ DSF ring.

   Conversely, suppose that $R + S$ is n-DSF ring, then by 3.4, $R + S$ is $n-$ DSF R-module. Hence [12], R is n-DSF-R-submodule and also S is n- DSF-S-module, hence 3.4, R is n-DSF-R ring and also S is n- DSF-S ring.

2. Let R and S be n-DSF rings, then $R \cap S$ are rings that implies that R is n-DSF as R-module and S is n-DSF as S-module. If R and S are distributive rings, then by 3.4, $R \cap S$ is n-DSF-R-submodule of R and also S- is n-DSF-S-submodule of S, hence by [12], $R \cap S$ is $n-$ DSF ring.

   Conversely, suppose that $R \cap S$ is n-DSF ring, then by 3.4, $R \cap S$ is $n-$ DSF $R \cap S$-module. By [12] R is n-DSF-$R \cap S$-module and also S is n- DSF-$R \cap S$-module, R is n-DSF-

3. Let R and S are rings, then $R \times S$ is a rings that implies that $R \times S$ is $R \times S$-module. Let R and S be n-DSF rings. If R and S are distributive rings, then by 3.4, R is n-DSF-R-module and also S- is n-DSF-S-module, hence by [12], $R \times S$ is $n-$ DSF $R \times S$-module.

   Conversely, suppose that $R \times S$ is n-DSF ring, then by 3.4, then $R \times S$ is $n-$ DSF $R \times S$-module. Hence by [12] R is n-DSF-$R \times S$-submodule and also S is n- DSF-$R \times S$-submodule of $R + S$.

Proposition 5.2. *Let R_i be n-DSF rings, $i = 1, 2, ..., n$, then*

1. *R_i be n-DSF rings, $i = 1, 2, ..., n$ if and only if $R = \Sigma_{i=1}^{n} R_i$ is n-DSF ring.*
2. *R_i be n-DSF rings, $i = 1, 2, ..., n$ if and only if $\cap_{i=1}^{n} R_i$ is n-DSF ring.*
3. *R_i be n-DSF rings, $i = 1, 2, ..., n$ if and only if $\prod_{i=1}^{n} R_i$ is n-DSF ring.*

Proof. We will prove it by induction mathematical.

1. For $n = 2$, then from 5.1 $R_1 + R_2$ is n-DSF ring.

   Suppose that the assertion is true for n, (i.e $R = \Sigma_{i=1}^{n} R_i$ is n-DSF ring, then by 3.4, $R = \Sigma_{i=1}^{n} R_i$ is n-DSF R-module if and only if by [12] R_i are n-DSF R_i-modules, hence R_i are n-DSF rings, $i = 1, 2, ...n$). ....(*)

   We prove it is true for $n + 1$. Let $R_1, R_2, ..., R_n, R_{n+1}$ be n-DSF rings, then by 3.4, R_i are n-DSF R_i-modules, $i = 1, 2, ...n, n + 1$. And from [12], $\Sigma_{i=1}^{n+1} R_i = \Sigma_{i=1}^{n} R_i + R_{n+1}$ is n-DSF R modules, then 5.1, $\Sigma_{i=1}^{n+1} R_i = \Sigma_{i=1}^{n} R_i + R_{n+1}$ is n-DSF ring. And by ......(*) and R_{n+1} is n-DSF R-module, then $\Sigma_{i=1}^{n+1} R_i$ is a ring, Hence, $\Sigma_{i=1}^{n} R_i$ is n-DSF ring for every n.

   Conversely, suppose that $R = \Sigma_{i=1}^{n} R_i$ is n-DSF ring. then by 3.4, then R is R n-DSF modules, that implies that R_i is a submodules of R, hence R_i are n-DSF rings, $i = 1, 2, ...n$.

2. For $n=2$, then from 5.1 $R_1 \cap R_2$ is n-DSF ring.

   Suppose that the assertion is true for n, (i.e $R = \cap_{i=1}^{n} R_i$ is n-DSF ring, then by 3.4, $R = \cap_{i=1}^{n} R_i$ is n-DSF R-module if and only if R_i are n-DSF $\cap_{i=1}^{n} R_i$ modules, hence R_i are n-DSF rings, $i = 1, 2, ...n$). ....(*)

   We prove it is true for $n+1$. Let $R_1, R_2, ..., R_n, R_{n+1}$ be n-DSF rings, then by 3.4, R_i are n-DSF R_i-modules, $i = 1, 2, ...n, n+1$. And from [12], $\cap_{i=1}^{n+1} R_i = \cap_{i=1}^{n} R_i \cap R_{n+1}$ is n-DSF R modules, then 5.1, $\cap_{i=1}^{n+1} R_i = \cap_{i=1}^{n} R_i \cap R_{n+1}$ is n-DSF ring. And by ......(*) and R_{n+1} is n-DSF R-module, then $\cap_{i=1}^{n+1} R_i$ is ring, Hence, $\cap_{i=1}^{n} R_i$ is n-DSF ring for every n.

   Conversely, suppose that $R = \cap_{i=1}^{n} R_i$ is n-DSF ring. then by 3.4, R is n-DSF R-module, hence R_i is R n-DSF submodules, then R_i are n-DSF rings, $i = 1, 2, ...n$.

3. For $n=2$, then from 5.1 $R_1 \times R_2$ is n-DSF ring.

   Suppose that the assertion is true for n, (i.e $R = \prod_{i=1}^{n} R_i$ is n-DSF ring then, by 3.4, $R = \prod_{i=1}^{n} R_i$ is n-DSF R-module if and only if by [12] R_i are n-DSF R_i modules, $i = 1, 2, ...n$, hence by 3.4, R_i are n-DSF rings, $i = 1, 2, ...n$). ....(*)

   We prove it is true for $n+1$. Let $R_1, R_2, ..., R_n, R_{n+1}$ be n-DSF rings, then by 3.4, R_i are n-DSF R_i-modules, $i = 1, 2, ...n, n+1$. And from [12], $\prod_{i=1}^{n+1} R_i = \prod_{i=1}^{n} R_i \times R_{n+1}$ is n-DSF $\prod_{i=1}^{n} R_i \times R_{n+1}$ module then, **??**, $\prod_{i=1}^{n+1} R_i = \prod_{i=1}^{n} R_i \times R_{n+1}$ is n-DSF ring. And by ......(*) and R_{n+1} is n-DSF R-module Hence, $\prod_{i=1}^{n} R_i$ is n-DSF ring for every n.

   Conversely, suppose that $R = \prod_{i=1}^{n} R_i$ is n-DSF ring. then by 3.4, R is R n- module, then R_i are R_i submodule of $\prod_{i=1}^{n} R_i$, hence R_i are n-DSF rings, $i = 1, 2, ...n$.

Corollary 5.3. *Let R be n-DSF ring and let I and J be ideals of a ring R. If R is distributive ring, then*

1. *I and J are n-DSF ideals if and only if $I + J$ is n-DSF ideal.*
2. *I and J are n-DSF ideals if and only if $I \cap J$ is n-DSF ideal.*
3. *I and J are n-DSF ideals if and only if $I \times J$ is n-DSF ideal.*

Proof. 1. Let R be n-DSF ring, then R is n-DSF R-module. Let I, J be ideals of a ring R, then I, J are n-DSF R-submodules of R as R module. by [18], this implies that $I + J$ is an ideal and it is n-DSF R-submodule of R as n-DSF-R module, hence by 3.4, $I + J$ is n-DSF ideal.

   Conversely, suppose that $I + J$ is n-DSF ideal, then $I + J$ is n-DSF as R submodule of a ring R as n-DSF- R module, then by 3.4, then I, J are $n-$ DSF R modules. Hence by [12] I and J are n-DSF ideals.

2. Let R be n-DSF ring, then R is n-DSF R-module. I and J are ideals of R, then $I \cap J$ is an ideal of a ring R that implies that $I \cap J$ is R-submodule of a ring R as n-DSF- R module, then by 3.4, $I \cap J$ is n-DSF ideal.

   Conversely, suppose that $I \cap J$ is n-DSF ideal of a ring R, then by 3.4, $I \cap J$ is $n-$ DSF-R-submodule. Hence by [12] I is n-DSF-R-submodule and also J is n- DSF-R-submodule, hence by 3.4, I, J are n-DSF ideals.

3. Let R be n-DSF rings I and J be ideals of a ring R, then $I \times J$ is an ideal of a ring R that implies that $I \times J$ is R-submodule of a ring R as n-DSF- R module, then by 3.4, $I \times J$ is n-DSF ideal,

   Conversely, suppose that $I \times J$ is n-DSF ideal of a ring R, then by 3.4, $I \times J$ is $n-$ DSF-R-submodule of a ring R as n- DSF R module. Hence by [12], I is n-DSF-R-submodule and also J is n- DSF-R-submodule, hence by 3.4, I, J are n-DSF-R ideals.

Corollary 5.4. *Let I_i be n-DSF ideals of a ring R, $i = 1, 2, ..., n$. If R is distributive ring, then*

1. *I_i are n-DSF ideals, $i = 1, 2, ..., n$ if and only if $\Sigma_{i=1}^{n} I_i$ is n-DSF an ideal.*
2. *I_i are n-DSF ideals, $i = 1, 2, ..., n$ if and only if $\bigcap_{i=1}^{n} I_i$ is n-DSF an ideal.*
3. *I_i are n-DSF ideals, $i = 1, 2, ..., n$ if and only if $\prod_{i=1}^{n} I_i$ is n-DSF an ideal.*

Proof. 1. Let I_i be n-DSF ideals of n-DSF ring R, $i = 1, 2, ..., n$. If R is distributive ring, then I_i are n-DSF ideals then by [12], I_i are n-DSF R-submodules of R is n-DSF R-module and by [?], and 5.2, $\Sigma_{i=1}^{n} I_i$ is n-DSF an ideal.

Conversely, suppose that $\Sigma_{i=1}^{n} I_i$ is n-DSF ideal. then by 3.4, and [12], hence I_i are n-DSF ideals, $i = 1, 2, ...n$.

2. Let I_i be n-DSF ideals of n-DSF ring R, $i = 1, 2, ..., n$. If R is distributive ring, then I_i are n-DSF ideals then by [12], I_i are n-DSF R-submodules of R is n-DSF R-module and by [?], and 5.2, $\cap_{i=1}^{n} I_i$ is n-DSF ideal.

Conversely, suppose that $\cap_{i=1}^{n} I_i$ is n-DSF ideal. then by 3.4, and [12], hence I_i are n-DSF ideals, $i = 1, 2, ...n$.

3. Let I_i be n-DSF ideals of n-DSF ring R, $i = 1, 2, ..., n$. If R is distributive ring, then I_i are n-DSF ideals then, by [12], I_i are n-DSF R-submodules of R is n-DSF R-module and by [?], and 5.2, $\prod_{i=1}^{n} I_i$ is n-DSF an ideal.

Conversely, suppose that $\prod_{i=1}^{n} I_i$ is n-DSF ideal. then by 3.4, and [12], hence I_i are n-DSF ideals, $i = 1, 2, ...n$.

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