Solving Hermit Differential Equation by Adomian Decomposition Method

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Abstract
In this paper, we investigate the application of the Adomian decomposition method for solving Hermit differential equations. The Adomian decomposition method is a powerful technique that allows us to solve non-linear and linear differential equations by decomposing the solution into a series of operators. In this paper, we present the theoretical basis of the Adomian decomposition method and demonstrate its applicability in solving Hermit differential equations through various numerical examples.
Keyword: *Adomian decomposition method, Hermit differential equation, Hermit polynomial.*

حل معادلة هيرمت التفاضلية بطريقة التحليل لأدوميان.

الملخص
في هذه الورقة، ندرس تطبيق طريقة التحليل لأدوميان لحل معادلة هيرميت التفاضلية. طريقة التحليل لأدوميان هي تقنية قوية تسمح لنا بحل المعادلة التفاضلية غير الخطية والخطية من خلال تحليل الحل إلى سلسلة من العوامل. في هذه الورقة، نقدم الأساس النظري لطريقة التحليل لأدوميان ونوضح إمكانية تطبيقها في حل معادلة هيرميت التفاضلية من خلال أمثلة عددية مختلفة.
الكلمات المفتاحية: طريقة التحلل الآدومي-معادلة هيرميث التفاضلية-حدود هيرميت.

1-Introduction:

Hermit differential equations are a class of differential equations that have important applications in various fields such as quantum mechanics, optics, and signal processing. To solve these equations,

Different approaches have been proposed, including numerical and analytical methods. One such method is the Adomian decomposition method, which has been successfully applied to solve various types of differential equations, including Hermit differential equations[1,2,3,4,7,11,16,17]. The Hermit differential equation is given by:

$H''(x) - 2xH'(x) + 2nH(x) = 0 .$ (1)

Where n is a constant. The exact solution for the Hermit differential equation is given by the Hermit polynomial, denoted as $H_n(x)$. $H_n(x)$ is a polynomial of degree n, and its explicit form can be obtained using various methods, such as generating functions, recurrence relations, or Rodrigues' formula. One of the most common ways to express the

Hermit polynomials is through the Rodrigues' formula:

$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$, where $\frac{d^n}{dx^n}$ represents the nth derivative [5,6,8,9,10]. The Hermit polynomials have specific values for different values of n. Here are a few examples:

For n = 0

$H_0(x) = 1 .$

For n = 1

$H_1(x) = 2x .$

For n =2

. $H_2(x) = 4x^2 - 2.$

For n =3

$H_n(x) = 8x^3 - 12x.$

For n=4

$H_4 = 16x^4 - 48x^2 + 12$.

The Hermit Polynomial satisfy the orthogonally property, which makes them useful in various applications, including quantum mechanics, statistical mechanics, and harmonic oscillators [12,13,14,15]. Therefore, the corresponding Hermit gives the exact solution to the Hermit differential equation Polynomial for the given value of n.

2- Analysis of method

Consider the Hermit's differential equation as follows

$H''(x) - 2xH'(x) + 2nH(x) = 0$. (2)

Now, based on the standard Adomian decomposition method, the Hermit's equation given above is rewrite as follows

$H''(x) = 2xH'(x) - 2nH(x)$. (3)

So, we consider the right-hand side of the equation non-homo-geneous term, where the differential operator L defined by $\frac{d^2}{dx^2}$. More So, we consider the inverse operatorL^{-1} as two-integral operator defined $\iint_0^x (.)dxdx$.

Applying the inverse operator L^{-1} to both sides of Equation (3), on gets $L^{-1}[2xH'(x) - 2nH(x)]$

$$H = \varphi(x)$$

Such that

$$L\varphi(x) = 0$$

Thus, based on the ADM, the solution H(x) is introduced through an infinite summation of components $H_n(x)$ [18,19,20] . Hence, the recursive solution of the equation is obtain as follows

$H_0(x) = \varphi(x) = H(0) + H'(0)x,$

(4)

$H_{n+1}(x) = L^{-1}[2xH'(x) - 2nH(x)], n \geq 0$

Where the overall solution $H(x)$ follows immediately by summing the above components as follows

$H(x) = \lim_{n\to\infty} \omega(x) = \sum_{j=0}^{\infty} H_j(x).$

(5)

3- Applications

We will introduce of some Example of solve the Hermit differential Equation by ADM.

Example 3.1

Let us consider an IVP featuring Hermit differential Equation as follows

1) Case 1: n=0

$$\mathrm{H}''(\mathrm{x}) - 2\mathrm{xH}'(\mathrm{x}) + 2\mathrm{nH}(\mathrm{x}) = 0, \quad (6)$$

$$\mathrm{H}(0) = 1, H'(0) = 0.$$

We will using standard ADM, from Eq.(6) becomes

$LH = 2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})$,

and further apply the inverse operator L^{-1} to the both sides of the latter equation to obtain

$H = 1 + L^{-1}[2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})]$.

Hence, the application of the standard ADM gives the general relation for the model as follows

$$H_0(x) = 1,$$

$H_{n+1}(x) = L^{-1}[2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})] = 0$, n≥0

Which gives upon summing the above iterates the following exact solution

$H(x) = 1$. (7)

In fact, this is a well-Known exact analytical solution for the Hermit's differential equation when n=0.

Case2: n=1

$H(0) = 0, H'(0) = 2$

$LH = 2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})$,

and further apply the inverse operator L^{-1} to the both sides of the latter equation to obtain

$H = 2x + L^{-1}[2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})]$.

Hence, the application of the standard ADM gives the general relation for the model as follows

$$H_0(x) = 2x,$$

$H_{n+1}(x) = L^{-1}[2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})] = 0$, n≥0

Which gives upon summing the above iterates the following exact solution

$H(x) = 2x$. (8)

In fact, this is a well-Known exact analytical solution for the Hermit's differential equation when n=1.

Case3: n=2

$H(0) = -2, H'(0) = 0$,

$LH = 2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})$,

and further apply the inverse operator L^{-1} to the both sides of the latter equation to obtain

$H(x) = -2 + L^{-1}[2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})]$.

Hence, the application of the standard ADM gives the general relation for the model as follows

$$H_0(x) = -2,$$

$H_{n+1}(x) = L^{-1}[2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})] =$, n≥0

if n=0

$$H_1(x) = L^{-1}[2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})] = L^{-1}[8] = 4x^2$$

If n=1

$H_2(x) = L^{-1}[2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})] = 0$.

This implies

$H(x) = H_0(x) + H_1(x) + H_2(x)$.

Which gives upon summing the above iterates the following exact solution

$H(x) = 4x^2 - 2$. (9)

In fact, this is a well-Known exact analytical solution for the Hermit's differential equation when n=2.

Case4: n=3

$H(0) = 0, H'(0) = -12$,

$LH = 2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})$,

and further apply the inverse operator L^{-1} to the both sides of the latter equation to obtain

$H(x) = -12x + L^{-1}[2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})]$.

Hence, the application of the standard ADM gives the general relation for the model as follows

$$H_0(x) = -12x,$$

$H_{n+1}(x) = L^{-1}[2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})]$, n≥0

if n=0

$$H_1(x) = L^{-1}[2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})] = L^{-1}[48\mathrm{x}] = 8x^3,$$

If n=1

$H_2(x) = L^{-1}[2\mathrm{xH}'(\mathrm{x}) - 2\mathrm{nH}(\mathrm{x})] = 0$.

This implies

$H(x) = H_0(x) + H_1(x) + H_2(x)$.

Which gives upon summing the above iterates the following exact solution

$H(x) = 8x^3 - 12x$. (10)

In fact, this is a well-Known exact analytical solution for the Hermit's differential equation when n=3.

Case5: n=4

$H(0) = 12, H'(0) = 0$,

$LH = 2xH'(x) - 2nH(x)$,

and further apply the inverse operator L^{-1} to the both sides of the latter equation to obtain

$H(x) = 12 + L^{-1}[2xH'(x) - 2nH(x)]$.

Hence, the application of the standard ADM gives the general relation for the model as follows

$$H_0(x) = 12,$$

$$H_{n+1}(x) = L^{-1}[2xH'(x) - 2nH(x)], n \geq 0$$

if n=0

$H_1(x) = L^{-1}[2xH'(x) - 2nH(x)] = L^{-1}[-96] = -48x^2$,

If n=1

$H_2(x) = L^{-1}[2xH'(x) - 2nH(x)] = L^{-1}[192x^2] = 16x^4$.

If n=2

$H_3(x) = L^{-1}[2xH'(x) - 2nH(x)] = 0$.

This implies

$H(x) = H_0(x) + H_1(x) + H_2(x) + H_3(x)$.

Which gives upon summing the above iterates the following exact solution

$H(x) = 16x^4 - 48x^2 + 12$. (11)

In fact, this is a well-Known exact analytical solution for the Hermit's differential equation when n=4.

In the same way, we compute the cases n=5, n=6,…….

From four cases we observation that, the exact solution is easily obtained by Adomian decomposition method. It's same Hermit polynomial

$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

$H_0(x) = 1$,

$H_1(x) = 2x$,

$H_2(x) = 4x^2 - 2$,

4- Adomian Modification Method for Hermit's differential equation In this section, we will using Adomian modification method, we introduce new differential operator.

Example 4.1. Let us considered for the solution of this form of Hermit's differential equation

$H''(x) - 2xH' = -2nH(x)$. (12)

We propose the new differential operator as below

$L(.) = e^{x^2} \frac{d}{dx} e^{-x^2} \frac{d}{dx} (.)$. (13)

The inverse operator $L^{-1}(.)$ is therefore consider a two-fold integral operator, as below

$L^{-1}(.) = \int_0^x e^{x^2} \int_0^x e^{-x^2} (H''(x) - 2xH') dx dx$. (14)

Case 1: n=0 the Eq. (12) becomes

$H''(x) - 2xH' = 0$ (15)

$H(0) = 1, H'(0) = 0$.

The differential operator, as below

$L(.) = e^{x^2} \frac{d}{dx} e^{-x^2} \frac{d}{dx} (.)$.

So

$L^{-1}(.) = \int_0^x e^{x^2} \int_0^x e^{-x^2} (H''(x) - 2xH') dx dx$.

In an operator form (14) becomes

$LH = 0$. (16)

Applying L^{-1} on both sides of (16)

$L^{-1}LH = 0$, (17)

Applying L^{-1} of the two terms $H''(x) - 2xH'$ of Eq. (15) we find

$$L^{-1}(H''(x) - 2xH') = \int_0^x e^{x^2} \int_0^x e^{-x^2} (H''(x) - 2xH') dx dx$$

$= \int_0^x H' - e^{x^2} H'(0) = H(x) - H(0) - H'(0) \int_0^x e^{x^2} dx dx$.

It implies,

$H(x) - H(0) = 0$.

So,

$H(x) = 1$. (18)

The exact solution of Hermit's equation for n=0 is $\mathrm{H_0(x) = 1}$. Then the exact solution is easily obtain by Adomian decomposition method.

Case 2: if n=1 the Eq. (14) becomes

$\mathrm{H''(x) - 2xH' = -2H(x)}$. (19)

$\mathrm{H(0) = 0 , H'(0) = 2}$

The differential operator, as below

$L(.) = e^{x^2} \frac{d}{dx} e^{-x^2} \frac{d}{dx} (.)$.

So

$L^{-1}(.) = \int_0^x e^{x^2} \int_0^x e^{-x^2} (\mathrm{H''(x) - 2xH'}) dxdx$

In an operator form (14) becomes

$LH = -2\mathrm{H(x)}$. (20)

Applying L^{-1} on both sides of (16)

$L^{-1}LH = -L^{-1}2\mathrm{H(x)}$. (21)

Applying L^{-1} of the two terms

$\mathrm{H''(x) - 2xH'}$ of Eq. (15) we find

$L^{-1}(\mathrm{H''(x) - 2xH'}) =$

$\int_0^x e^{x^2} \int_0^x e^{-x^2} (\mathrm{H''(x) - 2xH'}) dxdx =$

$-L^{-1}2\mathrm{H(x)}$.

$= \int_0^x H' - e^{x^2} H'(0) = H(x) - H(0) -$

$H'(0) \int_0^x e^{x^2} dx$,

It implies,

$H(x) - H(0) - H'(0) \int_0^x e^{x^2} dx = -L^{-1}2\mathrm{H(x)}$

So,

$H(x) = H'(0) \int_0^x e^{x^2} dx - L^{-1}2\mathrm{H(x)}$,

$H(x) = 2 \int_0^x e^{x^2} dx - L^{-1}2\mathrm{H(x)}$.

We make of the Taylor's expansion on e^{x^2} , we have

$H(x) = 2x + \frac{2x^3}{3} + \frac{x^5}{5} + \cdots - 2L^{-1}\mathrm{H(x)}$

(22)

It implies,

$H_0(x) = 2x + \frac{2x^3}{3} + \frac{x^5}{5} + \cdots,$

$H_{n+1}(x) = -2L^{-1}H_n(\mathrm{x})$,

$H_1(x) = -2L^{-1}H_0(\mathrm{x}) = -2x^2 - \frac{x^4}{3} - \frac{x^6}{15} + \cdots$

It implies,

$H(x) = H_0(x) + H_1(x) + \cdots = 2x - 2x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} - \frac{x^6}{15} + \cdots$ (23)

The exact solution of Hermit's equation for n=1 is $\mathrm{H_1(x) = 2x}$. Then the exact solution is easily obtain by Adomian decomposition method.

Example 4.2. In this Example, we will using Hermit's differential equation which n negative such as

$\mathrm{H''(x) - 2xH'(x) + 2nH(x) = 0}$ (24)

If n =-1 the Eq. (23) becomes

$\mathrm{H''(x) - 2xH'(x) - 2H(x) = 0}$ (25)

$\mathrm{H(0) = 1, H'(0) = 0}$.

We will assume a new differential operator such as,

$L(.) = \frac{d}{dx} e^{-1+x^2} \frac{d}{dx} e^{1-x^2} (.)$.

The inverse operator as below,

$$L^{-1}(.) = e^{-(1-x^2)} \int_0^x e^{1-x^2} \int_0^x (\mathrm{H''(x)} - 2\mathrm{xH'} - 2\mathrm{H(x)}) dxdx$$

$$= e^{-(1-x^2)} \int_0^x e^{1-x^2} (\mathrm{H'}(x) - 2\mathrm{xH(x)} - \mathrm{H'(0)}) dx$$

$$= e^{-(1-x^2)} (e^{1-x^2} H(x) - e\mathrm{H(0)} - \mathrm{H'(0)} \int_0^x e^{1-x^2} dx$$

$= H(x) - e^{x^2}\mathrm{H(0)} - \mathrm{H'(0)} e^{-(1-x^2)} \int_0^x e^{1-x^2} dx$

.

This implies,

$H(x) = e^{x^2}\mathrm{H(0)} = e^{x^2}$. (26)

In Hermit's differential equation n is non-negative constant. In this Example we assume n is non-negative constant, we have solution for Eq. (25) by Adomian decomposition method.

5. Conclusion

In this paper, we offered a new differential operator for solving Hermit differential equation. The examples presented in this paper illustrated the quality of the Adomian decomposition method for finding the solution. The Adomian decomposition method is an effective analytical method used in solving a wide range of differential equations, including Hermit differential equations. The ADM decomposes the differential equation into simpler parts, allowing for the use of simpler analytical techniques to solve the equation. The accuracy of the ADM solution can be improved by increasing the number of Adomian polynomials used in the expansion. The ADM applied to Hermit differential equations allows for the accurate description of complex physical systems, such as those found in quantum mechanics, signal processing, and optics.

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