Solving 3D Helmholtz Equation Using Triple Laplace-Aboodh-Sumudu Transform

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Abstract
This paper introduces a novel integral transform, termed the triple Laplace-Aboodh-Sumudu transform (TLAST), by combining the Laplace transform, Aboodh transform, and Sumudu transform. The primary objective of this research is to utilize TLAST for solving the Helmholtz equation in three dimensions. The TLAST provides a powerful tool for transforming complex partial differential equations into algebraic equations, simplifying their solution. The paper presents the mathematical formulation of TLAST and demonstrates its application in solving the Helmholtz equation. By employing TLAST, the solution to the Helmholtz equation is obtained efficiently, highlighting the potential of this new integral transform in addressing various problems in science, engineering, and other fields.

Keywords: *Triple Laplace-Aboodh-Sumudu transform, Helmholtz equation, Integral transform, Laplace transform, Aboodh transform, Sumudu transform.*

1. Introduction

The origin of the integral transforms can be traced back to the work of P. S. Laplace in 1780s and Joseph Fourier in 1822 [10]. One of the most significant applications of the integral transforms is capability to be used in solving initial boundary problems [7,9].

In the literature, numerous different transforms are presented and applied to solve ordinary and partial differential equations such as Laplace transform, Sumudu transform, Aboodh transform, nature transform, Hunaiber transform, Shehu transform and so on, see [1,3,4,8,10,11]. Integral transform has played an important role in solving differential ordinary and partial equations and integral equations, through transform these equations to algebraic equations [5].

In 2022, the authors in [2] introduced a new triple integral transform called triple Laplace-Aboodh-Sumudu transform and applied it to solve the heat equation, wave equation, Laplace equation and Poisson equation.

The aim of this work is to solve the Helmholtz equation with initial and boundary conditions by triple Laplace-Aboodh-Sumudu transform.

1.1 Definition. Let u be a continuous function of three variables say $x, y, t > 0$; then, the triple Laplace-Aboodh-Sumudu transform of $u(x,y,t)$ is denoted by the operator $L_x A_y S_t[u(x,y,t)] = U(\rho,q,r)$ and defined by

$$L_x A_y S_t[u(x,y,t)] = U(\rho,q,r) = \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+qy+\frac{t}{r}\right)}u(x,y,t)dxdydt. \qquad (1)$$

Provided the integral exists.

The inverse triple Laplace-Aboodh-Sumudu transform is defined by

$$L_x^{-1}A_y^{-1}S_t^{-1}[U(\rho,q,r)] = u(x,y,t)$$
$$= \frac{1}{(2\pi i)^3}\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\rho x}\left\{\int_{\lambda-i\infty}^{\lambda+i\infty} qe^{qy}\left\{\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{r}e^{\frac{t}{r}}\, U(\rho,q,r)dr\right\}dq\right\}d\rho, \qquad (2)$$

where κ, λ and ω are real constants [2].

1.2 Theorem. Let $U(\rho,q,r)$ be the triple Laplace-Aboodh-Sumudu transform of the continuous function $u(x,y,t)$, then

(1). $L_x A_y S_t\left[\frac{\partial u(x,y,t)}{\partial x}\right] = \rho U(\rho,q,r) - A_y S_t[u(0,y,t)]$.

Proof:

$$L_x A_y S_t\left[\frac{\partial u(x,y,t)}{\partial x}\right] = \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+qy+\frac{t}{r}\right)}\frac{\partial u(x,y,t)}{\partial x}dxdydt$$
$$= \frac{1}{qr}\int_0^\infty e^{-qy}dy\int_0^\infty e^{-\frac{t}{r}}\,dt\left\{\int_0^\infty e^{-\rho x}\,u_x(x,y,t)dx\right\}.$$

Using integration by parts, let $\xi = e^{-\rho x}$, $d\eta = u_x(x,y,t)dx$, then we obtain

$$L_x A_y S_t\left[\frac{\partial u(x,y,t)}{\partial x}\right] = \frac{1}{qr}\int_0^\infty e^{-qy}dy\int_0^\infty e^{-\frac{t}{r}}\,dt\left\{-u(0,y,t) + \rho\int_0^\infty e^{-\rho x}\,u(x,y,t)dx\right\}$$
$$= \rho U(\rho,q,r) - A_y S_t[u(0,y,t)].$$

(2). $L_x A_y S_t\left[\frac{\partial u(x,y,t)}{\partial y}\right] = qU(\rho,q,r) - \frac{1}{q}L_x S_t[u(x,0,t)]$

(3). $L_x A_y S_t\left[\frac{\partial u(x,y,t)}{\partial t}\right] = \frac{1}{r}\left[U(\rho,q,r) - L_x A_y[u(x,y,0)]\right]$

(4). $L_xA_yS_t\left[\frac{\partial^2 u(x,y,t)}{\partial x^2}\right] = \rho^2 U(\rho,q,r) - \rho A_yS_t[u(0,y,t)] - A_yS_t[u_x(0,y,t)].$

(5). $L_xA_yS_t\left[\frac{\partial^2 u(x,y,t)}{\partial y^2}\right] = q^2 U(\rho,q,r) - L_xS_t[u(x,0,t)] - \frac{1}{q}L_xS_t[u_y(x,0,t)].$

(6). $L_xA_yS_t\left[\frac{\partial^2 u(x,y,t)}{\partial t^2}\right] = \frac{1}{r^2}U(\rho,q,r) - \frac{1}{r^2}L_xA_y[u(x,y,0)] - \frac{1}{r}L_xA_y[u_t(x,y,0)].$

(7). $L_xA_yS_t\left[\frac{\partial^2 u(x,y,t)}{\partial x\partial y}\right] = \rho q U(\rho,q,r) - \frac{\rho}{q}L_xS_t[u(x,0,t)] - A_yS_t[u_y(0,y,t)].$

(8). $L_xA_yS_t\left[\frac{\partial^2 u(x,y,t)}{\partial x\partial t}\right] = \frac{\rho}{r}U(\rho,q,r) - \frac{\rho}{r}L_xA_y[u(x,y,0)] - A_yS_t[u_t(0,y,t)].$

(9). $L_xA_yS_t\left[\frac{\partial^2 u(x,y,t)}{\partial y\partial t}\right] = \frac{q}{r}U(\rho,q,r) - \frac{q}{r}L_xA_y[u(x,y,0)] - \frac{1}{q}L_xS_t[u_t(x,0,t)].$

By the same proof (1), we can prove (2)-(9) [2].

2. Triple Laplace-Aboodh-Sumudu Transform of Some Basic Functions

(1). $L_xA_yS_t[1] = \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+qy+\frac{t}{r}\right)}\,dxdydt = \frac{1}{\rho q^2}.$

(2). $L_xA_yS_t[xyt] = \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+qy+\frac{t}{r}\right)}\,xyt\,dxdydt = \frac{r}{\rho^2 q^3}.$

(3). $L_xA_yS_t[x^n y^m t^k] = \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+qy+\frac{t}{r}\right)}\,x^n y^m t^k dxdydt = \frac{n!}{\rho^{n+1}}\frac{m!}{q^{m+2}}k!\,r^k,$

where $n,m,k = 0,1,2,\dots$.

(4). $L_xA_yS_t[x^\alpha y^\beta t^\lambda]$

$$= \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+qy+\frac{t}{r}\right)}\,x^\alpha y^\beta t^\lambda dxdydt = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\rho^{\alpha+1}q^{\beta+2}}\Gamma(\lambda+1)r^\lambda,$$

where $n,m,k \geq -1$.

(5). $L_xA_yS_t[e^{(\alpha x+\beta y+\lambda t)}] = \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+qy+\frac{t}{r}\right)}\,e^{(\alpha x+\beta y+\lambda t)}dxdydt$

$$= \frac{1}{q(\rho-\alpha)(q-\beta)(r-\lambda r)}.$$

(6). $L_xA_yS_t[\sin(\alpha x+\beta y+\lambda t)] = \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+qy+\frac{t}{r}\right)}\,\sin(\alpha x+\beta y+\lambda t)\,dxdydt$

$$= \frac{\lambda\rho qr+\beta\rho+\alpha q-\alpha\beta\lambda r}{q(\rho^2+\alpha^2)(q^2+\beta^2)(1+\lambda^2r^2)}.$$

(7). $L_xA_yS_t[\cos(\alpha x+\beta y+\lambda t)] = \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+qy+\frac{t}{r}\right)}\,\cos(\alpha x+\beta y+\lambda t)\,dxdydt$

$$= \frac{\rho q-\beta\lambda\rho r-\alpha\lambda qr-\alpha\beta}{q(\rho^2+\alpha^2)(q^2+\beta^2)(1+\lambda^2r^2)}.$$

(8). $L_xA_yS_t[\sinh(\alpha x+\beta y+\lambda t)] = \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+qy+\frac{t}{r}\right)}\,\sinh(\alpha x+\beta y+\lambda t)\,dxdydt$

$$= \frac{\beta\rho+\alpha q+\lambda\rho qr+\alpha\beta\lambda r}{q(\rho^2-\alpha^2)(q^2-\beta^2)(1-\lambda^2r^2)}.$$

$$(9).\, L_xA_yS_t[cosh(\alpha x+\beta y+\lambda t)]=\frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+qy+\frac{t}{r}\right)}cosh(\alpha x+\beta y+\lambda t)\,dxdydt$$
$$=\frac{\rho q+\alpha\beta+\beta\lambda\rho r+\alpha\lambda qr}{q(\rho^2-\alpha^2)(q^2-\beta^2)(1-\lambda^2r^2)}.$$

3. Application of The Triple Laplace-Aboodh-Sumudu Transform for The Helmholtz Equation

In this section, we applying the triple Laplace-Aboodh-Sumudu transform to solve the Helmholtz equation. The Helmholtz equation is the basic partial differential equation of acoustics [6]. Let the nonhomogeneous Helmholtz equation in three independent variables (x,y,t) be in the form

$$u_{xx}(x,y,t)+u_{yy}(x,y,t)+u_{tt}(x,y,t)+u(x,y,t)=g(x,y,t), \quad (3)$$

with the initial and boundary conditions

$$u(0,y,t)=\hbar_1(y,t), \qquad u_x(0,y,t)=\hbar_2(y,t), \quad (4)$$
$$u(x,0,t)=\hbar_3(x,t), \qquad u_y(x,0,t)=\hbar_4(x,t), \quad (5)$$
$$u(x,y,0)=\hbar_5(x,y), \qquad u_t(x,y,0)=\hbar_6(x,y), \quad (6)$$

where $g(x,y,t)$ is the source term. Using the property of partial derivative of triple Laplace-Aboodh-Sumudu transform for Eq.(3), double Aboodh-Sumudu transform for Eq.(4), double Laplace-Sumudu transform for Eq.(5) and double Laplace-Aboodh transform for Eq.(6), then

$$\rho^2U(\rho,q,r)-\rho A_yS_t\{\hbar_1(y,t)\}-A_yS_t\{\hbar_2(y,t)\}+q^2U(\rho,q,r)-L_xS_t\{\hbar_3(x,t)\}-\frac{1}{q}L_xS_t\{\hbar_4(x,t)\}$$
$$+\frac{1}{r^2}U(\rho,q,r)-\frac{1}{r^2}L_xA_y\{\hbar_5(x,y)\}-\frac{1}{r}L_xA_y\{\hbar_6(x,y)\}+U(\rho,q,r)=G(\rho,q,r).$$

By simplifying, we get

$$U(\rho,q,r)=\frac{1}{\rho^2+q^2+\frac{1}{r^2}+1}\left[\rho\hbar_1(q,r)+\hbar_2(q,r)+\hbar_3(\rho,r)+\frac{1}{q}\hbar_4(\rho,r)+\frac{1}{r^2}\hbar_5(\rho,q)+\frac{1}{r}\hbar_6(\rho,q)+G(\rho,q,r)\right], (7)$$

where $L_xA_yS_t[g(x,y,t)]=G(\rho,q,r)$.

Solving this algebraic equation in $U(\rho,q,r)$ and taking the inverse triple Laplace-Aboodh-Sumudu transform on both sides of Eq.(7), yields

$$u(x,y,t)=L_x^{-1}A_y^{-1}S_t^{-1}\left\{\frac{1}{\rho^2+q^2+\frac{1}{r^2}+1}\left[\rho\hbar_1(q,r)+\hbar_2(q,r)+\hbar_3(\rho,r)+\frac{1}{q}\hbar_4(\rho,r)+\frac{1}{r^2}\hbar_5(\rho,q)+\frac{1}{r}\hbar_6(\rho,q)+G(\rho,q,r)\right]\right\}, \quad (8)$$

which represent the general formula for the solution of Eq.(3) by triple Laplace-Aboodh-Sumudu transform method.

Example 3.1. Consider the following Helmholtz equation

$$u_{xx}(x,y,t)+u_{yy}(x,y,t)+u_{tt}(x,y,t)=7e^{2x-y+t}-u(x,y,t), \quad (9)$$

with the initial and boundary conditions

$$u(0,y,t)=e^{-y+t}, \qquad u_x(0,y,t)=2e^{-y+t}, \quad (10)$$
$$u(x,0,t)=e^{2x+t}, \qquad u_y(x,0,t)=-e^{2x+t}, \quad (11)$$
$$u(x,y,0)=e^{2x-y}, \qquad u_t(x,y,0)=e^{2x-y}. \quad (12)$$

Solution:

Substituting

$$\hbar_1(q,r)=\frac{1}{q(q+1)(1-r)}, \qquad \hbar_2(q,r)=\frac{2}{q(q+1)(1-r)},$$

$$\hbar_3(\rho,r)=\frac{1}{(\rho-2)(1-r)},\qquad \hbar_4(\rho,r)=\frac{-1}{(\rho-2)(1-r)},$$
$$\hbar_5(\rho,q)=\frac{1}{q(\rho-2)(q+1)},\qquad \hbar_6(\rho,q)=\frac{1}{q(\rho-2)(q+1)},$$
$$G(\rho,q,r)=\frac{7}{q(\rho-2)(q+1)(1-r)}.$$

in Eq.(7), we obtain

$$U(\rho,q,r)=\frac{1}{\rho^2+q^2+\frac{1}{r^2}+1}\left[\frac{\rho}{q(q+1)(1-r)}+\frac{2}{q(q+1)(1-r)}+\frac{1}{(\rho-2)(1-r)}\right.$$
$$-\frac{1}{q(\rho-2)(1-r)}+\frac{1}{qr^2(\rho-2)(q+1)}+\frac{1}{qr(\rho-2)(q+1)}$$
$$\left.+\frac{7}{q(\rho-2)(q+1)(1-r)}\right].$$
$$=\frac{r^2}{\rho^2r^2+q^2r^2+r^2+1}\left[\frac{\rho^2r^2+q^2r^2+r^2+1}{qr^2(\rho-2)(q+1)(1-r)}\right].$$

By simplifying, we obtain

$$U(\rho,q,r)=\frac{1}{q(\rho-2)(q+1)(1-r)}. \tag{13}$$

Taking the inverse triple Laplace-Aboodh-Sumudu transform of Eq.(13), we get a solution of Eq.(9)

$$u(x,y,t)=e^{2x-y+t}. \tag{14}$$

Example 3.2. Consider the following Helmholtz equation

$$u_{xx}(x,y,t)+u_{yy}(x,y,t)+u_{tt}(x,y,t)+u(x,y,t)=0, \tag{15}$$

with the initial and boundary conditions

$$u(0,y,t)=e^{-t}siny,\qquad u_x(0,y,t)=e^{-t}cosy, \tag{16}$$
$$u(x,0,t)=e^{-t}sinx,\qquad u_y(x,0,t)=e^{-t}cosx, \tag{17}$$
$$u(x,y,0)=sin(x+y),\qquad u_t(x,y,0)=\cos(x+y). \tag{18}$$

Solution:

Substituting

$$\hbar_1(q,r)=\frac{1}{q(q^2+1)(1+r)},\qquad \hbar_2(q,r)=\frac{1}{(q^2+1)(1+r)},$$
$$\hbar_3(\rho,r)=\frac{1}{(\rho^2+1)(1+r)},\qquad \hbar_4(\rho,r)=\frac{\rho}{(\rho^2+1)(1+r)},$$
$$\hbar_5(\rho,q)=\frac{\rho+q}{q(\rho^2+1)(q^2+1)},\qquad \hbar_6(\rho,q)=-\frac{\rho+q}{q(\rho^2+1)(q^2+1)},$$

in Eq.(7), we obtain

$$U(\rho,q,r)=\frac{1}{\rho^2+q^2+\frac{1}{r^2}+1}\left[\frac{\rho}{q(q^2+1)(1+r)}+\frac{1}{(q^2+1)(1+r)}+\frac{1}{(\rho^2+1)(1+r)}\right.$$
$$\left.+\frac{\rho}{q\,(\rho^2+1)(1+r)}+\frac{\rho+q}{qr^2(\rho^2+1)(q^2+1)}-\frac{\rho+q}{qr(\rho^2+1)(q^2+1)}\right]$$

$$= \frac{r^2}{\rho^2 r^2 + q^2 r^2 + r^2 + 1}\left[\frac{\rho^3 r^2 + \rho^2 q r^2 + q^3 r^2 + q r^2 + \rho q^2 r^2 + \rho r^2 + \rho + q}{q r^2(\rho^2+1)(q^2+1)(1+r)}\right].$$

By simplifying, we get

$$U(\rho,q,r) = \frac{1}{\rho^2 r^2 + q^2 r^2 + r^2 + 1}\left[\frac{(\rho+q)(\rho^2 r^2 + q^2 r^2 + r^2 + 1)}{q(\rho^2+1)(q^2+1)(1+r)}\right]$$

$$= \frac{\rho+q}{q(\rho^2+1)(q^2+1)(1+r)}. \quad (19)$$

Taking the inverse triple Laplace-Aboodh-Sumudu transform of Eq.(19), we get a solution of Eq.(15)

$$u(x,y,t) = e^{-t} sin(x+y). \quad (20)$$

Conclusion

In Conclusion, the triple Laplace-Aboodh-Sumudu transform is a significant transform among all the integral transforms of exponential sort kernels. The Helmholtz equation solved simply and easily by this triple transform. We will apply the triple Laplace-Aboodh-Sumudu transform to more partial differential equations and integral equations in future studies.

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