A Simplified and innovative solution to the exact differential equation and its comparison with the method of Arfken and koshlyakov

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Abstract

This research introduces a simplified boundary-fixing method for solving first-order exact differential equations, which reduces computational complexity while maintaining accuracy. The proposed approach is compared with classical techniques, particularly those developed by George B .Arfken and Nikolai G. koshlyakov, with both theoretical and pratical examples provided to demonstrate its efficacy . Through these comparisons, the paper aims to show the advantages of the new method in terms of computational efficiency and ease of application.

KEYWORDS : *Differential Equation , solving , Arfken , theoretical , Koshlyakov , accurate .*

Introduction

A differential equation is an equation which contains one ore more terms which involve the derivatives of one variable with respect to the other variable , There are generally two types of differential equations used in engineering analysis

1- Ordinary differential equation (ODE) : equation with functions that involve only one variable and with different orders of ordinary derivatives

2- Partial differential equation(PDE):equation with functions that involve more than one variable and with different orders of partial derivatives .

Exact differential equation:

Exact differential equations are a fundamental tool in various branches of mathematics and physics. They are used to describe numerous physical systems, from mechanics to electromagnetism .The classical methods, such as those developed by George B .Arfken in Mathematical Methods for physicists and Nikolai G. Koshlyakov in his seminal works on differential equations, have long been used to solve these equations through integration techniques .However, these methods , while reliable, can sometimes be cumbersome and involve multiple complicated expressions or boundary conditions. This paper presents a novel method for solving first-order exact differential equations, aiming to streamline the process by introducing a boundary-fixing technique that simplifies calculations. The new approach is compared with Afrken's and Koshlyakov's methods, both of which are foundational in the study of differential equations.

An exact differential equation is defined as [1]

$$\boldsymbol{M}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dx}+\boldsymbol{N}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy}=0\ ,\qquad \frac{\partial \mathbf{M}}{\partial \mathbf{y}}=\frac{\partial \mathbf{N}}{\partial \mathbf{x}}.$$

Classical Methods:

The Arfken method relies on introducing the definite integral of the function M(x,y) from x_0 to x and the function N(x,y) from y_0 to y ,as follows

$$\boldsymbol{\phi}(\mathbf{x,y})=\int_{\mathbf{x0}}^{\mathbf{x}}\mathbf{M}(\mathbf{x},\mathbf{y})\mathbf{dx}+\int_{\mathbf{y0}}^{\mathbf{y}}\mathbf{N}(\mathbf{x_0},\mathbf{y})\mathbf{dy}=\mathbf{C}$$

Example : solve the differential equation

$$(x+y)dx+xdy=0.$$

Solution:

M(x,y)= x+y , $\frac{\partial M}{\partial y}$= 1.

N(x,y) =x , $\frac{\partial N}{\partial x}$ =1 .

The equation is exact ,thus

$$\int_{x0}^{x}(x+y)dx+\int_{y0}^{y}x0\ dy=c$$

$$\frac{x^2}{2}+\frac{x_0^2}{2}+yx-yx_0+yx_0-y_0x_0=c$$

$$\frac{x^2}{2}+yx=c+\frac{x0^2}{2}+y_0x_0$$

$$\frac{x^2}{2}+yx=c$$

Koshlyakov's Method [2]

The Koshlyakov's Method is based on the following steps

$$\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y})=\int\boldsymbol{M}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dx}+\boldsymbol{g}(\boldsymbol{y})$$

$$\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y})=\int\boldsymbol{N}(\boldsymbol{x},\boldsymbol{y})\ \boldsymbol{dy}+\boldsymbol{g}(\boldsymbol{x})$$

$$F_X(x,y)=M(x,y)=\frac{\delta}{\delta x}\left(\int N(x,y)dx+g(x)\right)$$

$$F_y(x,y)=N(x,y)=\frac{\delta}{\delta y}\left(\int M(x,y)d\,y+g(y)\right)$$

Example (2): Solve the differential equation

$(2xy - 3x^2)dx + (x^2 - 2y)\,dy = 0$

Solution:

the given differential equation is exact because

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}[2xy-3x^2]=2x$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}[x^2-2y]=2x$$

The general solution $F(x,y) = \int M(x,y)dx + g(y)$

$F(x,y) = \int M(x,y)dx + g(y) = x^2y - x^3 + g(y)$

$F_y(x,y) = N(x,y) = \frac{\delta}{\delta y}(\int M(x,y)d\,y + g(y))$

$F_y(x,y) = \frac{\partial}{\partial y}[x^2y - x^3 + g(y)]$

$x^2 + \bar{g}(y) = x^2 - 2y$

Thus

$\bar{g}(y) = -2y$ and it follows

That

$g(y) = -y^2 + c1$. Therefore

$F(x,y) = x^2y - x^3 - y^2 = c$

and the general solution is

$\mathbf{x^2y - x^3 - y^2 = c}$

Methodology

The new method proposed in this paper simplifies the solution process by fixing one variable at a time and integrating with respect to the other variable .This approach not only reduces the complexity of the solution but also provides a clear pathway to general solution. The steps are as follows

1- Reformulate the Equation

Write the equation in the form

$\boldsymbol{M(x,y)dx + N(x,y)dy} = 0$

Ensuring that it satisfies the exactness condition

2- Boundary Fixing

Fix one of the variables (y = 0) and integrate the equation with respect to the other variable .This step reduces the dimensionality of the problem and allows for easier integration.

3-Iterative integration

After fixing the fierst variable , integrate the remaining terms with respect to the second variable . This step involves standard integration techniques.

4- Combine Resalts

Once both integrations are performed, the partial solutions can be combined to construct the potential function F(x,y) .This function satisfies the equation F(x,y) = C, where C is a constant of integration .

$\int \boldsymbol{M(x,0)dx} + \int \boldsymbol{N(x,y)dy} = C$

Comparative Analysis of Methods ,

Example (1):solve

$(2x + y)\, dx + (x + 2y)dy = 0$

1- Arfken Method

$\int_{x_0}^{x} (2x + y)\, dx + \int_{y_0}^{y} (x_0 + 2y)dy = C$

$\boldsymbol{x^2} - x_0^2 + yx - yx_0 + x_0y - x_0y_0 + y^2 - y_0^2 = C$

$x^2 + xy + y^2 - y_0x_0 + y_0^2 = C$

$x^2 + xy + y^2 = C$

2- koshlyakov Method

$F(x,y) = \int M(x,y)dx + g(y) = yx + x^2 + g(y)$

$F_y(x,y) = N(x,y)$

$= \frac{\delta}{\delta y}(yx + x^2 + g(y))$

$x + 2y = x + g^{-}(y)$

$g(y) = \int 2y = y^2$

$F(x,y) = yx + x^2 + y^2 = C$

3- Proposed Method

$(2x + y)\, dx + (x + 2y)dy = 0$

$\int(2x + 0)dx + \int(x + 2y)dy = c$

$x^2 + xy + y^2 = c$

Example (2): solve the equation

$(\boldsymbol{\cos x - x\sin x + y^2})\boldsymbol{dx + 2xydy = 0}$

We will solve the equation by 3 method

Method (1)

Arfken Method

$\int_{x0}^{x} cosx - xsinx + y^2)dx + \int_{y0}^{y} 2x_0\, ydy = c$

$$\sin x - sinx_0 + xcosx - x_0cosx_0 - sinx + sinx_0 + y^2 - y^2x_0 + x_0y^2 - x_0y_0^2 = c$$

Thus

$-x_0 y_0^2 + xy^2 + xcosx - x_0 cosx_0 = c$

$y^2 x + xcosx = c + x_0 cosx_0 + x_0 y_0^2$

$y^2 x + xcosx = c$

Method (2)

koshlyakov Method

$$F(x,y) = \int N(x,y)dy = \int 2xydy = x^2 y + g(x)$$

$$f_x(x,y) = \frac{\partial}{\partial x}[xy^2 + g(x) = y^2 + g^{-}(x)$$

$$f_x(x,y) = cosx - xsinx + y^2$$

$$g^{-}(x,y) = cosx - xsinx$$

$$g(x) = \int (cosx - xsinx)dx$$

$$= x\cos x + c_1$$

$$f(x,y) = xy^2 + xcosx + c_1$$

The general solution $xy^2 + xcosx = c$

Method(3)

Proposed Method

$$(\cos x - xsinx + y^2)dx + 2xydy = 0$$

Put y=0

$$\int (cosx - xsinx + 0)dx + \int 2xydy = c$$

$$\sin x - (-xcosx + sinx) + xy^2 = c$$

$$\sin x + xcosx - sinx + xy^2 = c$$

$$y^2 x + xcosx = c$$

Discussion

The new boundary-fixing method proves to be a more intuitive and simplified approach to solving exact differential equations. By fixing one boundary and integrating the remaining terms, the solution process is streamlined, reducing computational complexity . When compared with the classical methods by Arfken and Koshlyakov , The new method demonstrates similar accuracy while requiring fewer steps in some cases.

Conclusion

This paper presents a novel approach to solving first-order exact differential equations using a boundary-fixing method. By comparing the proposed technique with classical methods by Arfken and Koshlykov, we demonstrate its effectiveness in simplifying the solution process. Future work will explore extensions of this method to higher-order equations and its potential applications in various fields of science and engineering

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