On Double Kamal-Shehu Transform and its Properties with Applications

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Abstract
The primary purpose of this paper is to present the combination of Kamal transform and Shehu transform to produce a new integral transform. This integral transform called the double Kamal-Shehu transform to solve partial differential equations, some main properties and several theorems related to the Kamal-Shehu transform have been established. Furthermore, we use this transform for solving some linear partial differential equations and finding some functions.
Keywords*: Double Kamal-Shehu transform, Kamal transform, Shehu transform, Partial differential equations.*

حول تحويل كمال شيخو الثنائي وخصائصه مع التطبيقات

الملخص
الغرض الرئيسي من هذا البحث هو الجمع بين تحويل كمال المفرد وتحويل شيخو المفرد للحصول على تحويل تكاملي جديد، يسمى هذا التحويل التكاملي تحويل كمال – شيخو الثنائي لحل المعادلات التفاضلية الجزئية، وقد تم دراسة بعض الخصائص الرئيسية والعديد من النظريات والخصائص المتعلقة بتحويل كمال-شيخو. علاوة على ذلك، يتم تطبيق هذا التحويل الجديد لحل بعض المعادلات التفاضلية الجزئية الخطية.
الكلمات المفتاحية: تحويل كمال – شيخو الثنائي، تحويل كمال، تحويل شيخو، المعادلات التفاضلية الجزئية.

1. Introduction

The solutions of initial and boundary value problems are given by several Integral transforms methods. In fact, there are many different types of integral transforms methods such as Laplace, Kamal, Shehu transforms, and so on [1, 2, 12]. These kinds of integral transforms have many applications in various fields of mathematical sciences and engineering such as physics, mechanics, chemistry, acoustic, etc, [1, 11, 12]. They play an important role in solving integral equations or partial differential equations describing physical phenomena [4, 5, 9, 10, 12]. Recently, double integral transforms has got the considerable attention of researchers due to its numerous applications in various fields of science and engineering, see for example [3, 8, 13]. Alfaqeih and Misirli in [6] dealt with double Shehu transform to get the solution of initial and boundary value problems in different areas of real life science and engineering.
Motivated by [3], we applied new double Kamal-Shehu transform to solve some partial differential equations subject to the initial and boundary conditions, through the derivation of general formula for solutions of these equations, or by applying the double Kamal-Shehu transform directly to the given equation.
In this paper, we use double Kamal-Shehu transform to solve the partial differential equations subject to the initial and boundary conditions.

Definition 1.1 ([1]). *Let $\mathcal{A}$ be a function set defined by*

$$\mathcal{A} = \Big\{u(x) : \exists M, \gamma_1, \gamma_2 > 0, |u(x)| < Me^{\frac{|x|}{\gamma_j}},\ x \in (-1)^j \times [0, \infty),\ j = 1, 2\Big\},$$

where M is a constant and γ_1, γ_2 are finite constants or infinite.
For a function of exponential order, the single Kamal integral transform of the real continuous function $u(x)$ is defined as follows;

$$K[u(x)] = \int_0^\infty e^{-\frac{x}{\gamma}} u(x)dx = F(\gamma), \tag{1.1}$$

where $e^{-\frac{x}{\gamma}}$ is the kernel function, and $K[.]$ is the Kamal transform operator.
The inverse Kamal transform is defined by

$$u(x) = K^{-1}[F(\gamma)] = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\frac{x}{\gamma}} F(\frac{1}{\gamma})d\gamma,\ \ \kappa \geq 0. \tag{1.2}$$

Definition 1.2 ([7]). *Over the set of functions,*

$$\mathcal{M} = \Big\{u(t) : \exists N, \kappa_1, \kappa_2 > 0, |u(t)| < Ne^{\frac{|t|}{\kappa_i}},\ \ for\ \ t \in (-1)^i \times [0, \infty),\ \ i = 1, 2\Big\},$$

the Shehu transform is defined by

$$S[u(t)] = \int_0^\infty e^{-\frac{\delta}{\mu}t} u(t)dt = F(\delta, \mu),\ \ \delta, \mu > 0, \tag{1.3}$$

where $e^{-\frac{\delta}{\mu}}$ is the kernel function and S is the operator of Shehu transform.
The inverse Shehu transform is defined by

$$u(t) = S^{-1}[F(\delta, \mu)] = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{1}{\mu} e^{\frac{\delta}{\mu}t} F(\delta, \mu)d\delta,\ \ \omega \geq 0. \tag{1.4}$$

In the next definition, we introduce the double Kamal-Shehu transform.

Definition 1.3. *The double Kamal-Shehu transform of the function $u(x,t)$ of two variables $x>0$ and $t>0$ is denoted by* $K_xS_t[u(x,t)] = F(\gamma,\delta,\mu)$ *and defined as*

$$\begin{aligned} K_xS_t[u(x,t)] &= F(\gamma,\delta,\mu) = \int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}u(x,t)dxdt \\ &= \lim_{\alpha\to\infty,\beta\to\infty}\int_0^\alpha\int_0^\beta e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}u(x,t)dxdt. \end{aligned} \tag{1.5}$$

It converges if the limit of the integral exists, and diverges if not.

Definition 1.4. *The inverse double Kamal-Shehu transform of a function $F(\gamma,\delta,\mu)$ is given by*

$$K_x^{-1}S_t^{-1}[F(\gamma,\delta,\mu)] = u(x,t).$$

Equivalently,

$$u(x,t) = K_x^{-1}S_t^{-1}[F(\gamma,\delta,\mu)] = \frac{1}{(2\pi i)^2}\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\frac{x}{\gamma}}d\gamma\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{\mu}e^{\frac{\delta}{\mu}t}F(\frac{1}{\gamma},\delta,\mu)d\delta, \tag{1.6}$$

where κ and ω are real constants.

2. Existence and Uniqueness of the Double Kamal-Shehu Transform

Definition 2.1. *([8]) A function $u(x,t)$ is said to be of exponential orders $\alpha>0$, $\beta>0$, on $0\le x<\infty$, $0\le t<\infty$, if there exists positive constants L, X and T such that*

$$|u(x,t)| \le Le^{\alpha x+\beta t},\ \forall\, x>X,\ t>T,$$

and we write

$$u(x,t) = o(e^{\alpha x+\beta t}),\ as\ x\to\infty,\ t\to\infty.$$

Or equivalently,

$$\lim_{x\to\infty,t\to\infty} e^{-(\alpha x+\beta t)}|u(x,t)| \le L\lim_{x\to\infty,t\to\infty} e^{-(\frac{1}{\gamma}-\alpha)x}e^{-(\frac{\delta}{\mu}-\beta)t} = 0,\ \ \frac{1}{\gamma}>\alpha,\ \ \frac{\delta}{\mu}>\beta.$$

Theorem 2.2. *([7]) Let $u(x,t)$ be a continuous function in every finite intervals $(0,X)$ and $(0,T)$ and of exponential order $e^{(\alpha x+\beta t)}$, then the double Kamal-Shehu transform of $u(x,t)$ exists for all $\frac{1}{\gamma}>\alpha$ and $\frac{\delta}{\mu}>\beta$.*

Proof. Let $u(x,t)$ be of exponential order $e^{(\alpha x+\beta t)}$ such that

$$\Big|u(x,t)\Big| \le Le^{(\alpha x+\beta t)},\ \ \forall\, x>X,\ t>T.$$

Then, we have

$$\begin{aligned}
\Big|F(\gamma,\delta,\mu)\Big| &= \Big|\int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}u(x,t)dxdt\Big| \\
&\leqq \int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}|u(x,t)|dxdt \\
&\leqq L\int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}e^{(\alpha x+\beta t)}dxdt \\
&= L\int_0^\infty e^{-(\frac{1}{\gamma}-\alpha)x}dx\int_0^\infty e^{-(\frac{\delta}{\mu}-\beta)t}dt \\
&= \frac{L\gamma\mu}{(1-\gamma\alpha)(\delta-\beta\mu)}.
\end{aligned}$$

Thus, the proof is complete. □

Theorem 2.3. *Let $F_1(\gamma,\delta,\mu)$ and $F_2(\gamma,\delta,\mu)$ be the double Kamal-Shehu transform of the continuous functions $u_1(x,t)$ and $u_2(x,t)$ defined for $x,t\geq 0$ respectively. If $F_1(\gamma,\delta,\mu)=F_2(\gamma,\delta,\mu)$, then $u_1(x,t)=u_2(x,t)$.*

Proof. Assume that κ and ω are sufficiently large, since

$$u(x,t)=K_x^{-1}S_t^{-1}[F(\gamma,\delta,\mu)]=\Big(\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}e^{\frac{x}{\gamma}}d\gamma\Big)\Big(\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{\mu}e^{\frac{\delta}{\mu}t}F(\frac{1}{\gamma},\delta,\mu)d\delta\Big),$$

we deduce that

$$\begin{aligned}
u_1(x,t) &= \Big(\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}e^{\frac{x}{\gamma}}d\gamma\Big)\Big(\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{\mu}e^{\frac{\delta}{\mu}t}F_1(\frac{1}{\gamma},\delta,\mu)d\delta\Big) \\
&= \Big(\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}e^{\frac{x}{\gamma}}d\gamma\Big)\Big(\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{\mu}e^{\frac{\delta}{\mu}t}F_2(\frac{1}{\gamma},\delta,\mu)d\delta\Big) \\
&= u_2(x,t).
\end{aligned}$$

This ends the proof of the theorem. □

3. Some Useful Properties of the Double Kamal-Shehu Transform

3.1 Linearity property

If the double Kamal-Shehu transform of functions $u_1(x,t)$ and $u_2(x,t)$ are $F_1(\gamma,\delta,\mu)$ and $F_2(\gamma,\delta,\mu)$ respectively, then the double Kamal-Shehu transform of $\alpha u_1(x,t)+\beta u_2(x,t)$ is given by $\alpha F_1(\gamma,\delta,\mu)+\beta F_2(\gamma,\delta,\mu)$, where α and β are arbitrary constants.

Proof.

$$
\begin{aligned}
K_xS_t[\alpha u_1(x,t)+\beta u_2(x,t)] &= \int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}\Big(\alpha u_1(x,t)+\beta u_2(x,t)\Big)dxdt \\
&= \alpha\int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}u_1(x,t)dxdt \\
&+ \beta\int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}u_2(x,t)dxdt \\
&= \alpha F_1(\gamma,\delta,\mu)+\beta F_2(\gamma,\delta,\mu). \qquad (3.7)
\end{aligned}
$$

□

3.2 Shifting property

If the double Kamal-Shehu transform of $u(x,t)$ is $F(\gamma,\delta,\mu)$, then double Kamal-Shehu transform of the function $e^{(\alpha x+\beta t)}u(x,t)$ is given by
$F(\frac{\gamma}{1-\gamma\alpha},\delta-\beta\mu,\mu)$.

Proof.

$$
\begin{aligned}
K_xS_t[e^{(\alpha x+\beta t)}u(x,t)] &= \int_0^\infty\int_0^\infty e^{(\alpha x+\beta t)}e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}u(x,t)dxdt \\
&= \int_0^\infty\int_0^\infty e^{-\big((\frac{1-\gamma\alpha}{\gamma})x+(\frac{\delta-\beta\mu}{\mu})t\big)}u(x,t)dxdt \\
&= F(\frac{\gamma}{1-\gamma\alpha},\delta-\beta\mu,\mu). \qquad (3.8)
\end{aligned}
$$

□

3.3 Change of scale property

If the double Kamal-Shehu transform of the function $u(x,t)$ is $F(\gamma,\delta,\mu)$ then the double Kamal-Shehu transform of $u(\alpha x,\beta t)$ is given by $\frac{1}{\alpha\beta}F(\alpha\gamma,\frac{\delta}{\beta},\mu)$.

Proof. Using the definition of double Kamal-Shehu transform, we get

$$K_xS_t[u(\alpha x,\beta t)] = \int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}u(\alpha x,\beta t)dxdt.$$

Let $\upsilon=\alpha x,\quad \tau=\beta t$, then

$$
\begin{aligned}
K_xS_t[u(\alpha x,\beta t)] &= \frac{1}{\alpha\beta}\int_0^\infty\int_0^\infty e^{-\frac{\upsilon}{\alpha\gamma}}e^{-\frac{\delta\tau}{\beta\mu}}u(\upsilon,\tau)d\upsilon d\tau \\
&= \frac{1}{\alpha\beta}F(\alpha\gamma,\frac{\delta}{\beta},\mu). \qquad (3.9)
\end{aligned}
$$

□

3.4 Derivatives properties

If $K_xS_t[u(x,t)] = F(\gamma,\delta,\mu)$, then

$$(1). \qquad K_xS_t\Big[\frac{\partial u(x,t)}{\partial x}\Big] = \frac{1}{\gamma}F(\gamma,\delta,\mu) - S[u(0,t)]. \tag{3.10}$$

Proof.

$$\begin{aligned} K_xS_t\Big[\frac{\partial u(x,t)}{\partial x}\Big] &= \int_0^\infty \int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}\frac{\partial u(x,t)}{\partial x}dxdt \\ &= \int_0^\infty e^{-\frac{\delta}{\mu}t}dt\Big(\int_0^\infty e^{-\frac{x}{\gamma}}u_x(x,t)dx\Big). \end{aligned}$$

Using integration by parts, let u $= e^{-\frac{x}{\gamma}}$, $\;dv = u_x(x,t)dx$, then we obtain

$$\begin{aligned} K_xS_t\Big[\frac{\partial u(x,t)}{\partial x}\Big] &= \int_0^\infty e^{-\frac{\delta}{\mu}t}dt\Big(-u(0,t) + \frac{1}{\gamma}\int_0^\infty e^{-\frac{x}{\gamma}}u(x,t)dx\Big) \\ &= \frac{1}{\gamma}F(\gamma,\delta,\mu) - S[u(0,t)]. \end{aligned}$$

□

$$(2). \qquad K_xS_t\Big[\frac{\partial u(x,t)}{\partial t}\Big] = \frac{\delta}{\mu}F(\gamma,\delta,\mu) - K[u(x,0)]. \tag{3.11}$$

Proof.

$$\begin{aligned} K_xS_t\Big[\frac{\partial u(x,t)}{\partial t}\Big] &= \int_0^\infty \int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}\frac{\partial u(x,t)}{\partial t}dxdt \\ &= \int_0^\infty e^{-\frac{x}{\gamma}}dx\Big(\int_0^\infty e^{-\frac{\delta}{\mu}t}u_t(x,t)dt\Big). \end{aligned}$$

Using integration by parts, let u $= e^{-\frac{\delta}{\mu}t}$, $\;dv = u_t(x,t)dt$, then we obtain

$$\begin{aligned} K_xS_t\Big[\frac{\partial u(x,t)}{\partial t}\Big] &= \int_0^\infty e^{-\frac{x}{\gamma}}dx\Big(-u(x,0) + \frac{\delta}{\mu}\int_0^\infty e^{-\frac{\delta}{\mu}t}u(x,t)dt\Big) \\ &= \frac{\delta}{\mu}F(\gamma,\delta,\mu) - K[u(x,0)]. \end{aligned}$$

□

$$(3). \qquad K_xS_t\Big[\frac{\partial^2 u(x,t)}{\partial x^2}\Big] = \frac{1}{\gamma^2}F(\gamma,\delta,\mu) - \frac{1}{\gamma}S[u(0,t)] - S[u_x(0,t)]. \tag{3.12}$$

Proof.

$$\begin{aligned} K_xS_t\Big[\frac{\partial^2 u(x,t)}{\partial x^2}\Big] &= \int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}\frac{\partial^2 u(x,t)}{\partial x^2}dxdt \\ &= \int_0^\infty e^{-\frac{\delta}{\mu}t}dt\Big(\int_0^\infty e^{-\frac{x}{\gamma}}u_{xx}(x,t)dx\Big). \end{aligned}$$

Integration by parts twice, we obtain

$$\begin{aligned} K_xS_t\Big[\frac{\partial^2 u(x,t)}{\partial x^2}\Big] &= \int_0^\infty e^{-\frac{\delta}{\mu}t}dt\Big(-u_x(0,t)-\frac{1}{\gamma}u(0,t)+\frac{1}{\gamma^2}\int_0^\infty e^{-\frac{x}{\gamma}}u(x,t)dx\Big) \\ &= \frac{1}{\gamma^2}F(\gamma,\delta,\mu)-\frac{1}{\gamma}S[u(0,t)]-S[u_x(0,t)]. \end{aligned}$$

□

$$(4). \qquad K_xS_t\Big[\frac{\partial^2 u(x,t)}{\partial t^2}\Big]=\frac{\delta^2}{\mu^2}F(\gamma,\delta,\mu)-\frac{\delta}{\mu}K[u(x,0)]-K[u_t(x,0)]. \tag{3.13}$$

Proof.

$$\begin{aligned} K_xS_t\Big[\frac{\partial^2 u(x,t)}{\partial t^2}\Big] &= \int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}\frac{\partial^2 u(x,t)}{\partial t^2}dxdt \\ &= \int_0^\infty e^{-\frac{x}{\gamma}}dx\Big(\int_0^\infty e^{-\frac{\delta}{\mu}t}u_{tt}(x,t)dt\Big). \end{aligned}$$

Integration by parts twice, we obtain

$$\begin{aligned} K_xS_t\Big[\frac{\partial^2 u(x,t)}{\partial t^2}\Big] &= \int_0^\infty e^{-\frac{x}{\gamma}}dx\Big(-u_t(x,0)-\frac{\delta}{\mu}u(x,0)+\frac{\delta^2}{\mu^2}\int_0^\infty e^{-\frac{\delta}{\mu}t}u(x,t)dt\Big) \\ &= \frac{\delta^2}{\mu^2}F(\gamma,\delta,\mu)-\frac{\delta}{\mu}K[u(x,0)]-K[u_t(x,0)]. \end{aligned}$$

□

$$(5). \qquad K_xS_t\Big[\frac{\partial^2 u(x,t)}{\partial x\partial t}\Big]=\frac{\delta}{\gamma\mu}F(\gamma,\delta,\mu)-\frac{1}{\gamma}K[u(x,0)]-S[u_t(0,t)]. \tag{3.14}$$

Proof.

$$\begin{aligned} K_xS_t\Big[\frac{\partial^2 u(x,t)}{\partial x\partial t}\Big] &= \int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}\frac{\partial^2 u(x,t)}{\partial x\partial t}dxdt \\ &= \int_0^\infty e^{-\frac{\delta}{\mu}t}dt\Big(\int_0^\infty e^{-\frac{x}{\gamma}}u_{xt}(x,t)dx\Big). \end{aligned}$$

Integration by parts twice, we obtain

$$\begin{aligned}
K_xS_t\Big[\frac{\partial^2 u(x,t)}{\partial x\partial t}\Big] &= \int_0^\infty e^{-\frac{\delta}{\mu}t}dt\Big(-u_t(0,t)+\frac{1}{\gamma}\int_0^\infty e^{-\frac{x}{\gamma}}u_t(x,t)dx\Big)\\
&= -S[u_t(0,t)]+\frac{1}{\gamma}\int_0^\infty e^{-\frac{x}{\gamma}}dx\int_0^\infty e^{-\frac{\delta}{\mu}t}u_t(x,t)dt\\
&= -S[u_t(0,t)]+\frac{1}{\gamma}\int_0^\infty e^{-\frac{x}{\gamma}}dx\Big(-u(x,0)+\frac{\delta}{\mu}\int_0^\infty e^{-\frac{\delta}{\mu}t}u(x,t)dt\Big)\\
&= \frac{\delta}{\gamma\mu}F(\gamma,\delta,\mu)-\frac{1}{\gamma}K[u(x,0)]-S[u_t(0,t)].
\end{aligned}$$

□

4. The Double Kamal-Shehu Transform of Some Elementary Functions

(1). If the function $u(x,t)=1$, then

$$K_xS_t[u(x,t)]=\int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}dxdt=\frac{\gamma\mu}{\delta}. \tag{4.15}$$

(2). If the function $u(x,t)=xt$, then

$$K_xS_t[u(x,t)]=\int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}xtdxdt=\frac{\gamma^2\mu^2}{\delta^2}. \tag{4.16}$$

(3). If the function $u(x,t)=x^nt^m,\ \ n,m=0,1,2,\ \ ...$, then

$$K_xS_t[u(x,t)] = \int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}x^nt^mdxdt=n!m!\gamma^{n+1}\Big(\frac{\mu}{\delta}\Big)^{m+1}. \tag{4.17}$$

(4). If the function $u(x,t)=x^\sigma t^\nu,\ \ \sigma\geq -1,\ \ \nu\geq -1$, then

$$K_xS_t[u(x,t)]=\int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}x^\sigma t^\nu dxdt \ = \int_0^\infty e^{-\frac{x}{\gamma}}x^\sigma dx\int_0^\infty e^{-\frac{\delta}{\mu}t}t^\nu dt.$$

Let $\xi=\frac{x}{\gamma}$ and $\eta=\frac{\delta}{\mu}t$, then we have

$$\begin{aligned}
K_xS_t[u(x,t)] &= \gamma^{\sigma+1}\int_0^\infty e^{-\xi}\xi^\sigma d\xi(\frac{\mu}{\delta})^{\nu+1}\int_0^\infty e^{-\eta}\eta^\nu d\eta\\
&= \Gamma(\sigma+1)\gamma^{\sigma+1}\Gamma(\nu+1)\Big(\frac{\mu}{\delta}\Big)^{\nu+1}.
\end{aligned} \tag{4.18}$$

Where $\Gamma(.)$ is the Euler gamma function.

(5). If the function $u(x,t)=e^{nx+mt},\ \ n,m=0,1,2,\ \ ...$, then

$$K_xS_t[u(x,t)]=\int_0^\infty\int_0^\infty e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)}e^{nx+mt}dxdt=\frac{\gamma\mu}{(1-n\gamma)(\delta-m\mu)}. \tag{4.19}$$

(6). If the function $u(x,t) = \sin(nx+mt), \ \ n,m = 0,1,2, \ \ ...$, then

$$\begin{aligned} K_x S_t[u(x,t))] &= \int_0^{\infty}\int_0^{\infty} e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)} \sin(nx+mt)dxdt \\ &= \frac{\gamma\mu(n\gamma\delta+m\mu)}{(1+n^2\gamma^2)(\delta^2+m^2\mu^2)}. \end{aligned} \tag{4.20}$$

(7). If the function $u(x,t) = \cos(nx+mt), \ \ n,m = 0,1,2, \ \ ...$, then

$$\begin{aligned} K_x S_t[u(x,t)] &= \int_0^{\infty}\int_0^{\infty} e^{-(\frac{x}{\gamma}+\frac{\delta}{\mu}t)} \cos(nx+mt)dxdt \\ &= \frac{\gamma\mu(\delta-nm\gamma\mu)}{(1+n^2\gamma^2)(\delta^2+m^2\mu^2)}. \end{aligned} \tag{4.21}$$

Consequently,

$$K_x S_t[\sinh(nx+mt)] = \frac{\gamma\mu(n\gamma\delta+m\mu)}{(1-n^2\gamma^2)(\delta^2-m^2\mu^2)}, \tag{4.22}$$

$$K_x S_t[\cosh(nx+mt)] = \frac{\gamma\mu(\delta+nm\gamma\mu)}{(1-n^2\gamma^2)(\delta^2-m^2\mu^2)}. \tag{4.23}$$

5. Applications

In this section, we applied the double Kamal-Shehu transform method to linear partial differential equations. Let the second-order nonhomogeneous partial differential equation in two independent variables be in the form:

$$Au_{xx}(x,t) + Bu_{tt}(x,t) + Cu_x(x,t) + Du_t(x,t) + Eu(x,t) = h(x,t), (x,t) \in \mathbb{R}^2_+, \tag{5.24}$$

with the initial conditions:

$$u(x,0) = \hbar_1(x), \qquad u_t(x,0) = \hbar_2(x), \tag{5.25}$$

and the boundary conditions:

$$u(0,t) = \hbar_3(t), \qquad u_x(0,t) = \hbar_4(t), \tag{5.26}$$

where A, B, C, D and E are constants and $h(x,t)$ is the source term.
Using the property of partial derivative of the double Kamal-Shehu transform for equation (5.24), single Kamal transform for equation (5.25) and single Shehu transform for equation (5.26) and simplifying, we obtain that:

$$F(\gamma,\delta,\mu) = \Big[\frac{(B\frac{\delta}{\mu}+D)\hbar_1(\gamma) + B\hbar_2(\gamma) + (\frac{A}{\gamma}+C)\hbar_3(\delta,\mu) + A\hbar_4(\delta,\mu) + H(\gamma,\delta,\mu)}{(\frac{A}{\gamma^2}+B\frac{\delta^2}{\mu^2}+\frac{C}{\gamma}+D\frac{\delta}{\mu}+E)}\Big], \tag{5.27}$$

where $H(\gamma,\delta,\mu) = K_x S_t[h(x,t)]$.
Finally, solving this algebraic equation in $F(\gamma,\delta,\mu)$ and taking the inverse double Kamal-Shehu transform on both sides of equation (5.27), yields:

$$u(x,t) = K_x^{-1} S_t^{-1}\Big[\frac{(B\frac{\delta}{\mu}+D)\hbar_1(\gamma)+B\hbar_2(\gamma)+(\frac{A}{\gamma}+C)\hbar_3(\delta,\mu)+A\hbar_4(\delta,\mu)+H(\gamma,\delta,\mu)}{(\frac{A}{\gamma^2}+B\frac{\delta^2}{\mu^2}+\frac{C}{\gamma}+D\frac{\delta}{\mu}+E)}\Big], \tag{5.28}$$

which represent the general formula for the solution of equation (5.24) by the double Kamal-Shehu transform method.

Example 5.1. *Use double Kamal-Shehu transform method to solve the following first order inhomogeneous partial differential equation*

$$u_x(x,t) + u_t(x,t) = (1+x)e^t, \quad u(x,0) = x, \quad u(0,t) = 0. \tag{5.29}$$

Solution.
Applying the double Kamal-Shehu transform on both sides of equation (5.29), we have

$$K_x S_t[u_x(x,t) + u_t(x,t)] = K_x S_t[(1+x)e^t].$$

By linearity property and partial derivative properties of double Kamal-Shehu transform, we get

$$\frac{1}{\gamma}F(\gamma,\delta,\mu) - S[u(0,t)] + \frac{\delta}{\mu}F(\gamma,\delta,\mu) - K[u(x,0)] = \frac{\gamma\mu}{\delta-\mu} + \frac{\gamma^2\mu}{\delta-\mu}, \tag{5.30}$$

where

$$K_x S_t[(1+x)e^t] = \frac{\gamma\mu}{\delta-\mu} + \frac{\gamma^2\mu}{\delta-\mu}.$$

Substituting

$$\hbar_1(\gamma) = \gamma^2, \qquad \hbar_3(\delta,\mu) = 0,$$

into equation (5.30) and simplify to obtain

$$\Big(\frac{\mu+\gamma\delta}{\gamma\mu}\Big)F(\gamma,\delta,\mu) = \frac{\gamma(\gamma\delta+\mu)}{\delta-\mu},$$

or equivalently

$$F(\gamma,\delta,\mu) = \Big(\frac{\gamma\mu}{\gamma\delta+\mu}\Big)\Big(\frac{\gamma(\gamma\delta+\mu)}{\delta-\mu}\Big) = \frac{\gamma^2\mu}{\delta-\mu}. \tag{5.31}$$

Taking the inverse double Kamal-Shehu transform of equation (5.31), we get

$$u(x,t) = K_x^{-1} S_t^{-1}\Big[\frac{\gamma^2\mu}{\delta-\mu}\Big] = xe^t.$$

Which is the required solution of (5.29).

Example 5.2. *Consider the following linear problem*

$$u_{xt}(x,t) = -x + u(x,t), \tag{5.32}$$

subject to the conditions

$$u(x,0) = x + e^{x}, \qquad u(0,t) = e^{t}, \qquad u_t(0,t) = e^{t}.$$

Solution.
Applying the double Kamal-Shehu transform on both sides of equation (5.32), we get

$$K_x S_t[u_{xt}(x,t)] = K_x S_t[-x + u(x,t)].$$

By linearity property and partial derivative properties of double Kamal-Shehu transform, we get

$$\frac{\delta}{\gamma\mu}F(\gamma,\delta,\mu) - \frac{1}{\gamma}K[u(x,0)] - S[u_t(0,t)] = -\frac{\gamma^2\mu}{\delta} + F(\gamma,\delta,\mu). \tag{5.33}$$

Substituting

$$K[u(x,0)] = \gamma^2 + \frac{\gamma}{1-\gamma}, \qquad S[u_t(0,t)] = \frac{\mu}{\delta-\mu},$$

in (5.33) and simplifying, we get

$$\left(\frac{\delta-\gamma\mu}{\gamma\mu}\right)F(\gamma,\delta,\mu) = \gamma + \frac{1}{1-\gamma} + \frac{\mu}{\delta-\mu} - \frac{\gamma^2\mu}{\delta},$$

or equivalently

$$F(\gamma,\delta,\mu) = \frac{\gamma^2\mu}{\delta} + \frac{\gamma}{(1-\gamma)}\frac{\mu}{(\delta-\mu)}. \tag{5.34}$$

Taking the inverse double Kamal-Shehu transform of equation (5.34), we get a solution of (5.32)

$$\begin{aligned} u(x,t) &= K_x^{-1}S_t^{-1}\left[\frac{\gamma^2\mu}{\delta} + \frac{\gamma}{(1-\gamma)}\frac{\mu}{(\delta-\mu)}\right] \\ &= x + e^{x+t}. \end{aligned}$$

Which is same as solution obtained by decomposition method [14].

Example 5.3. *Consider the following boundary Poisson equation*

$$u_{xx}(x,t) + u_{tt}(x,t) = t\sin x, \tag{5.35}$$

subject to the conditions

$$u(x,0) = 0 = \hbar_1(x),\ u_t(x,0) = -\sin x = \hbar_2(x),\ u(0,t) = 0 = \hbar_3(t),\ u_x(0,t) = -t = \hbar_4(t).$$

Solution.
Applying the double Kamal-Shehu transform on both sides of equation (5.35), we get

$$\begin{aligned}\frac{1}{\gamma^2}F(\gamma,\delta,\mu) &- \frac{1}{\gamma}S[u(0,t)] - S[u_x(0,t)] + \frac{\delta^2}{\mu^2}F(\gamma,\delta,\mu) - \frac{\delta}{\mu}K[u(x,0)] - K[u_t(x,0)]\\ &= \frac{\mu^2}{\delta^2}\frac{\gamma^2}{(1+\gamma^2)}.\end{aligned} \tag{5.36}$$

Substituting

$$K[\hbar_1(x)] = 0, \qquad K[\hbar_2(x)] = -\frac{\gamma^2}{1+\gamma^2}, \qquad S[\hbar_3(t)] = 0, \qquad S[\hbar_4(t)] = -\frac{\mu^2}{\delta^2},$$

in (5.36) and simplifying, we get

$$F(\gamma,\delta,\mu) = -\frac{\mu^2}{\delta^2}\frac{\gamma^2}{(1+\gamma^2)}. \tag{5.37}$$

Taking the inverse double Kamal-Shehu transform of equation (5.37), we get a solution of (5.35)

$$\begin{aligned}u(x,t) &= K_x^{-1}S_t^{-1}\Big[-\frac{\mu^2}{\delta^2}\frac{\gamma^2}{(1+\gamma^2)}\Big]\\ &= -t\sin x.\end{aligned}$$

Example 5.4. *Consider the following nonhomogeneous Wave equation*

$$u_{tt}(x,t) - u_{xx}(x,t) + 3u(x,t) = 3, \tag{5.38}$$

with the conditions

$$\begin{aligned}u(x,0) &= 1 = \hbar_1(x), & u_t(x,0) &= 2\sin x = \hbar_2(x),\\ u(0,t) &= 1 = \hbar_3(t), & u_x(0,t) &= \sin 2t = \hbar_4(t).\end{aligned}$$

Solution.
Substituting

$$\hbar_1(\gamma) = \gamma,\ \hbar_2(\gamma) = \frac{2\gamma^2}{1+\gamma^2},\ \hbar_3(\delta,\mu) = \frac{\mu}{\delta},\ \hbar_4(\delta,\mu) = \frac{2\mu^2}{\delta^2+4\mu^2},\ H(\gamma,\delta,\mu) = \frac{3\gamma\mu}{\delta},$$

in (5.28) and simplifying, we get a solution of (5.38)

$$\begin{aligned}u(x,t) &= K_x^{-1}S_t^{-1}\Big[\frac{\gamma\mu}{\delta} + \frac{\gamma^2}{(1+\gamma^2)}\frac{2\mu^2}{(\delta^2+4\mu^2)}\Big]\\ &= 1 + \sin x\sin 2t.\end{aligned}$$

Example 5.5. *Use the double Kamal-Shehu transform method to solve the inhomogeneous partial differential equation*

$$u_t(x,t) - u_{xx}(x,t) = -6x, \tag{5.39}$$

subject to the conditions

$$\begin{aligned}u(x,0) = x^3 + \sin x = \hbar_1(x), \qquad u_t(x,0) &= -\sin x = \hbar_2(x),\\ u(0,t) = 0 = \hbar_3(t), \qquad u_x(0,t) &= e^{-t} = \hbar_4(t).\end{aligned}$$

Solution.
Substituting

$$\hbar_1(\gamma) = 6\gamma^4 + \frac{\gamma^2}{1+\gamma^2}, \quad \hbar_3(\delta,\mu) = 0, \quad \hbar_4(\delta,\mu) = \frac{\mu}{\delta+\mu}, \quad H(\gamma,\delta,\mu) = -\frac{6\gamma^2\mu}{\delta},$$

in (5.28) and simplifying, we get a solution of (5.39)

$$\begin{aligned} u(x,t) &= K_x^{-1} S_t^{-1}\Big[\frac{6\gamma^4\mu}{\delta} + \frac{\mu}{(\delta+\mu)}\frac{\gamma^2}{(1+\gamma^2)}\Big] \\ &= x^3 + e^{-t}\sin x. \end{aligned}$$

6. Conclusion

We introduced a new method for solving linear partial differential equations subject to the initial and boundary conditions called double Kamal-Shehu transform. We also discussed some theorems and properties about the double Kamal-Shehu transform and gave the double Kamal-Shehu transform of some elementary functions. The efficiency of the considered method was illustrated by some examples. The results reveal that the proposed method is very effective and can be applied for other linear problems in mathematical physics. We suggest that researchers continue to research and develop new methods for solving linear and nonlinear partial differential equations due to the importance of these equations and their multiple applications in various aspects of science.

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