n-Dual Square Free R-Module

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DOI: https://doi.org/10.56807/buj.v6i1.560

Abstract

In this Article we introduce and study a new module which is called n-dual-Square free modules (n- DSF for a short) and is de_ned as: For a ring R and a positive integer n, an R-module M is called n dual square free module, if M has no proper submodules $M_1, M_2, \ldots, M_n$ *with* $M = M_1 + M_2 + \ldots + M_n$ *and* $M/M_i \cong M/M_j$, *I , j = 1, 2, ..., n with i ≠ j and n ≥ 2. In the second part we study some algebra operations on n-DSF R- module. In the third part we introduce and study some results of a finitely generated modules with n-DSF R-module. And give some properties, theorems, corollaries and examples about them.*

Keywords: *A Proper R-submodule, a dual square free R-module, n-Dual square free R-module, a distributive R-module, a finitely generated R-module and a maximal R-module.*

الملخص

في هذه المقالة سنقدم وندرس مفهوم جديد من الحلقيات يدعى $n-$ ثنائي الحلقية الحرة المربعة ويعرف كالتالي : لتكن R حلقة وليكن n عدد صحيح موجب ولتكن M حلقية معرفة فوق الحلقة R تسمى الحلقية M ب n– ثنائي الحلقية الحرة المربعة اذا وجدت n من الحلقيات الجزئية غير الفعلية

$M_1, M_2 \ldots \ldots \ldots M_n$ من الحلقية M بحيث $M = M_1 + M_2 + \cdots + M_n$ وبحيث يكون $M/M_i \cong M/M_j$ و $i, j = 1, 2, \ldots \ldots, n$ و $i \neq j$ و $n \geq 2$. وفي القسم الثالث من المقالة درسنا بعض العمليات الجبرية على هذا المفهوم وفي القسم الرابع درسنا بعض النتائج المهمة مع الحلقية منتهية التوليد ثم قدمنا بعض النظريات....المبرهنات...النتائج ...والامثلة على كل واحدة منهم.

الكلمات المفتاحية: وحدة R فرعية مناسبة، وحدة R حرة مربعة مزدوجة، وحدة R خالية من المربع المزدوج، وحدة R توزيعية، وحدة R مولدة بشكل محدود ووحدة R قصوى.

1 Introduction

Throughout this Article all rings are noncommutative with unit unless otherwise noted and all modules are assumed to be left unitary R-modules. In [15] the concept of a square free module was introduced and studied and was defined as : A module M was called square-free if it contains no nonzero isomorphic submodule A and B with A ∩ B = 0. The notion of square-freeness is very important in group, ring and module theories and is used as a tool to establish many useful and significant results in the research most notably.

A dual notion to square-free modules was introduced most recently in [26] and was defined as : A module M was called dual square free (DSF- module for a short), if M has no proper submodules A and B with M = A + B and $M/A = M/B$. This consist of three parts, in second part we introduce and study the concept of n-DSF R-module as the generalized of the concept of a dual square free R- module and is defined as : For a ring R and a positive integer n, An R-module M is called n-dual-square free module, (n-DSF for a short), if M has no proper submodules M_1, $M_2, \ldots, M_n$ with M = $M_1 + M_2 + \ldots + M_n$ and $M/M_i \cong M/M_j$, i ,j = 1, 2, ..., n with i ≠ j and n ≥ 2.

In 3.2, we give some important remarks as the following: Let R be a ring and M be an R-module. M is a DSF R-module this equivalent M is 2-DSF module. M is called distributive $n-$ DSF R-module if $A \cap (B_1 + B_2 + ... + B_n) = A \cap B_1 + A \cap B_2 + ... + A \cap B_n$ for all $A, B_1, B_2, ..., B_n \in M$. M is distributive $n-$ DSF R- module if and only if every submodule of M is n-DSF-module. An $R-$ submodule N of an R-module M is called a proper if $N \subsetneqq M$. Finally, it is obviously that every n-DSF R module is k-DSF R module for every positive integer $k \leq n$.

In 3.3, we give the characterization of n-DSF R-module as: M is n-DSF module if and only if M satisfies the following conditions: M has no proper submodules $M_1, M_2, \ldots, M_n$ with $M = M_1 + M_2 + \ldots + M_n$ and $M \diagup M_i \cong M \diagup M_j, i, j = 1, 2, \ldots, n, i \neq j$ and n is a positive integer and $n \geq 2$. In 3.4, we explain that if M is a distributive, M is n-DSF module if and only if M is n^2 - DSF module. In 3.5, M is a DSF-R module if and only if M is $n-$ DSF module. In 3.6, very important definition is given where we can decide whether the module is n-DSF-R module or not and is defined as: Let M, N be R-modules, M, N are called factor orthogonal if, no nonzero a factor module of M is isomorphic to a factor module of N. We give an example about it 3.7. In 3.8, M is a DSF-R module if and only if M is ∞-DSF R module. In 3.9, we give the important characterization of n-DSF-module R-module as: M is n-DSF module if and only if M is finitely generated R-module. In 3.10, if M is a $n-$ DSF module, then every submodule of M is n-DSF module and 3.11, M is n-DSF module, then $M \diagup N$ is $n-$ DSF module where N is a submodule of M. In 3.12, let R be a ring and M, N be R-modules, let $f : M \longrightarrow N$ be a homomorphism R-module then, if f is surjective and M is n-DSF module, then N, Imf and $Cokerf$ are n-DSF R-modules. And if f is injective and N is n-DSF module, then M, $Kerf$ and $coimf$ are n-DSF R-module. In 3.13, we study the behavior of n-DSF-module on the short exact sequence of R- homomorphism modules such that $0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} K \longrightarrow 0$, then we have the following cases: M is n-DSF if and only if N and K are n-DSF, M and N are n-DSF, then K is n-DSF, M and K are n-DSF, then N is n-DSF. Finally, M is n-DSF if and only if $N \bigoplus K$ is n-DSF.

In the fourth part we introduce and study some algebra operations on n DSF R-module such that in 4.1, let R be a ring and M, N be n-DSF R-modules, then $M + N$ is n-DSF R-module and $M \bigcap N$ is n-DSF R-module. In 4.2, we generalized 4.1, such that $\Sigma_{i=1}^{n} M_i$ is n-DSF R-module and $\bigcap_{i=1}^{n} M_i$ is n-DSF R-module. In 4.3, is generalized 4.2, and 4.4, are generalized 4.3, and in 4.5, let R be a ring and M_i be R_i-modules, $i = 1, 2$, then M_i are n-DSF R_i-modules, $i = 1, 2$ if and only if $M = M_1 \times M_2$ is n-DSF $R = R_1 \times R_2$-module. In 4.6, we generalized 4.5, such that M_i are n-DSF R_i-module, $i = 1, 2, ..., n$ if and only if $M = \prod_{i=1}^{n} M_i$ is n-DSF $R = \prod_{i=1}^{n} R_i$-module. In 4.7, and 4.9, we study the intersection of n-DSF R module such that let R_i be rings and M_i be R_i-modules, $i = 1, 2$, then M_i are n-DSF R_i-modules, $i = 1, 2$ if and only if $M = M_1 \cap M_2$ is n-DSF $R = R_1 \cap R_2$-module. In 4.9, we generalized 4.7, as: Let R_i be rings and M_i be R_i-modules, $i = 1, 2, ..., n$, then M_i are n-DSF R_i-modules, $i = 1, 2, ..., n$ if and only if $M = \cap_{i=1}^{n} M_i$ is n-DSF $R = \cap_{i=1}^{n} R_i$-module.

In the fifth part we introduce and study some results with finitely generated and n-Dual Square Free R-Module such that in 5.5, we give a characterization of n DSF R module as: Let R be a ring and M be an R-modules. The following are equivalents: M is n-DSF R-module if and only if M is finitely generated R-module if and only if M is Noetherian R-module if and only if M is finitely presented R-module if and only if M is a coherent R-module. In 5.6, Let R be a ring and M be an R-module. The following are holds: Every submodule of finitely generated R-module is n-DSF module, every submodule of Neotherian R-module is n-DSF module, every submodule of finitely presented R-module is n-DSF module and every submodule of a coherent R-module is n-DSF module. In 5.7, and 5.10, Let R be a ring and M be finitely generated or a local R-modules(An R-module M is called a local if it has a unique maximal submodule), then M is is n-DSF R-module. In 5.9, Let R be a ring and M be R-module and N be R-submodule of M. If N and M/N are $n-$ DSF R-modules, then M is also n-DSF R-module. In 5.11, we give an example and proved $\mathbb{Z}$ is a n-DSF as $\mathbb{Z}$-module.

2 Preliminaries

Definition 2.1. *[7] An R-submodule N of an R-module M is called a maximal if* $N \neq M$ *and* N *is such that if* L *is a submodule with* $N \subseteq L \subseteq M$, *then* $N = L$ *or* $L = M$.

Theorem 2.2. *[3] Let R be a ring and M be an R-modules, the following are equivalent :*

1. *M is finitely generated R-module,*
2. *M is Noetherian R-module.*

Definition 2.3. *In [26], A dual square-free module (DSF-module for a short) is defined as A module M is called dual square-free if M has no proper submodules A and B with* $M = A + B$ *and* $M/A \cong M/B$.

Definition 2.4. *[9] For a ring R, an R-module M is called a finitely generated (f. g for short), for every infinite family* $\{M_i\}_{i\in I}$ *of submodules of M where (I is an infinite index set) with* $M = \Sigma_{i\in I} M_i$, *there is a finite subset* $J \subset I$ *such that* $M = \Sigma_{i\in J} M_i$.

3 n-Dual Square Free R-Module

In this part we will introduce and study n-dual square free R-module ($n-$ DSF module for a short) and also study some examples, lemmas, theorems, properties and corollaries about them.

Definition 3.1. *For a ring R and a positive integer n. An R-module M is called n-dual-square free R-module n-DSF R-module, if M has no proper submodules* $M_1, M_2, \ldots, M_n$ *with* $M = M_1 + M_2 + \ldots + M_n$ *and* $M/M_i \cong M/M_j$ *and* $i, j = 1, 2, \ldots, n$, $i \neq j$ *and* $n \geq 2$.

Remarks 3.2. *Let R be a ring and M be an R-module.*

1. *Let M be a distributive module, then M is a DSF R-module this equivalent M is* 2*-DSF module.*
2. *Let M be a DSF-R module, M is called distributive R-module if* $A\cap(B+C) = A\cap B + A\cap C$ *for all* A, B *and* $C \in M$. *it easily to prove that M is distributive module if and only if every submodule of M is DSF-module.[15]*
3. *Let M be n-DSF-R module, M is called distributive n-DSF R-module if* $A \cap (B_1 + B_2 + ... + B_n) = A \cap B_1 + A \cap B_2 + ... + A \cap B_n$ *for all* $A, B_1, B_2, ..., B_n \in M$. *it easily to prove that M is distributive module if and only if every submodule of M is n-DSF-module.*
4. *An* $R-$ *submodule N of a n-DSF R-module M is called a proper if* $N \subsetneqq M$.
5. *Obviously, every n-DSF R-module is k-DSF R-module for every positive integer* $k \leq n$.

The following result gives the characterization of n-DSF R-module.

Lemma 3.3. *Let R be a ring and M be an R-module, M is n-DSF R module if and only if M satisfies the following conditions:*

1. *M has no proper submodules* $M_1, M_2, \ldots, M_n$ *with* $M = M_1 + M_2 + \ldots + M_n$
2. $M/M_i \cong M/M_j, i, j = 1, 2, \ldots, n, i \neq j$ *and n is a positive integer and* $n \geq 2$.

Proof. Suppose that M is n-DSF module, then from 3.1 M has no proper submodules $M_1, M_2, \ldots, M_n$ with $M = M_1 + M_2 + \ldots + M_n$. And there is a map $f : M/M_i \longrightarrow M/M_j$, $i, j = 1, 2, \ldots, n$ and $i \neq j$. It easily to prove this map is an isomorphism.

Conversely, suppose that the two conditions in 3.3 are satisfied, hence by 3.1 M is n-DSF module. ∎

Proposition 3.4. *Let R be a ring and M be an R-module, if M is a distributive. M is n-DSF module if and only if M is n^2 - DSF module.*

Proof. By mathematical induction. Suppose that $n = 2$ and $2 - DSF$ module, then by 3.3 M has no proper M_1, M_2 submodules with $M = M_1 + M_2$ and $M/M_1 \cong M/M_2$. Since M is distributive, then every submodule of M is a DSF-module, then by 2.3 M_1 has no proper submodules N_1, N_2 and M_2 has no proper N_3 and N_4 with $M_1 = N_1 + N_2$ and $M_2 = N_3 + N_4$ and this implies that $M = N_1 + N_2 + N_3 + N_4$ and $M/N_i \cong M/N_j, j = 1, 2, 3, 4$ and $i \neq j$, hence M is $4 - DSF$ module.

Suppose that the assertion is true for n, i.e n-DSF module if and only if M is n^2-DSF module. ....(*)

We will prove that the assertion is true for $n + 1$. Suppose that M is $n + 1$-DSF R-module, then M has no proper submodules $M_1, M_2, \ldots, M_n, M_{n+1}$ with $M = M_1 + M_2 + \ldots + M_n + M_{n+1}$ and $M/M_i \cong M/M_j, i, j = 1, 2, \ldots, n, n + 1$ and $i \neq j$, then M is $(n + 1) - DSF$ module and by $(*)$ M is $(n + 1)^2$-DSF module, hence it is true for every n. ■

Proposition 3.5. *Let R be a ring and M be R-module, if M is a distributive, M is DSF-R module if and only if M is $n-$ DSF module.*

Proof. Suppose that M is a DSF R-module this is equivalent that M has no proper submodules A and B such that $M = A + B$ with $M/A \cong M/B$ and M is a distributive R-module, then every submodule of a DSF R-module M is a DSF R-R-module that is equivalent that M is 4-DSF $R-$module from 3.4, and we repeat this process and then generalize this result to n, i.e M has no proper submodules $M_1, M_2, \ldots, M_n$ with $M = M_1 + M_2 + \ldots + M_n$ with $M/M_i \cong M/M_j, i, j = 1, 2, \ldots, n, i \neq j$, hence from 3.3, M is n-DSF R-module.

Conversely, suppose that M is n-DSF $R-$ module and by 3.3, this equivalent that $M = M_1 + M_2 + \ldots + M_n$ with $M/M_i \cong M/M_j, i, j = 1, 2, \ldots, n, i \neq j$. And we know that the sum of any number of submodules of R-module M is also submodule of M, hence we can write $M = M_1 + B$ and $M/M_1 \cong M/B$ and from 3.3, hence M is a DSF-R module. ■

In following very important definition where we can decide whether the module is n-DSF-R module or not.

Definition 3.6. *Let M, N be R-modules, M, N are called factor orthogonal if, no nonzero factor module of M is isomorphic to a factor module of N.*

Example 3.7.

1. *For example in [26] if $M = A + B$ and $A, B \subseteq M$, then $M/A \cong M/B$ are factor orthogonal and hence in this case is a DSF R module this equivalent that by 3.5 M is n-DSF- R module.*

2. *If $M = \mathbb{Z} \oplus \mathbb{Z}_p$, then $\mathbb{Z}$ and $\mathbb{Z}_p$ are not factors orthogonal and by 3.6 the $\mathbb{Z} \oplus \mathbb{Z}_p$ is not a n-DSF- $\mathbb{Z}$ module.*

Proposition 3.8. *Let R be a ring and M be an R-module, if M is a distributive R-module, M is a DSF-R module if and only if M is ∞-DSF R module.*

Proof. Suppose that M is a DSF module that is equivalent that M is n-DSF module and M is a distributive R-module that is equivalent that M is n^2-DSF module from 3.5, and so forth this implies that the numbers of submodules of M are continuous increasing and we get the infinitely family of submodules $(M_i)_i \in I$, where I is infinite index set such that $M = \Sigma_{i \in I} M_i$ with $M/M_i \cong M/M_j, i, j \in I$ and $i \neq j$. That is means M is ∞-DSF $-R$ module.

Conversely, suppose that M is ∞-DSF module and we know that the sum of any number of submodules of M is also a submodule of M, hence we can get the a finite family of submodules of M such that $M = M_1 + B$ with $M/M_1 \cong M/B$ and $B = \Sigma_{i \in I, i \geq 2} M_i$ and from 2.3, hence M is a DSF-R module. ■

Theorem 3.9. *Let R be a ring and M be an R-module. If M is a distributive R-module, M is a n-DSF module if and only if M is a finitely generated R-module.*

Proof. Suppose that M is n-DSF R-module, then by 3.8, M is ∞-DSF R-module that is means there exist infinite index set I, i.e there is infinitely family of submodules $(M_i)_{i\in I}$ of M such that $M = \Sigma_{i\in I} M_i$.

We can extract a finite index set $J \subset I$, i.e there is a finite family of submodules of M with $M = \Sigma_{j\in J} M_j$ and by 2.4, hence M is a finitely generated R-module.

Conversely, suppose that M is a finitely generated R-module, then by 2.4, there is an infinite index set I such that $M = \Sigma_{i\in I} M_i$ and we can extract the finite family of no proper submodules $(M_j)_{j\in J}$ where $J \subset I$ as : suppose that the set $\{A_1, A_2, ..., A_n\}$ are maximal R-submodules of M. Let $x \in M$ but it does not belong to $A_i, i = 1, 2, ..., n$, then we get the no proper submodules $M_1 = Rx + A_1, M_2 = Rx + A_2, ..., M_n = Rx + A_n$ in M with $M = \Sigma_{i=n}^{n} M_i = M_1 + M_2 + ... + M_n$ and it easily to prove $M/M_i \cong M/M_j, i, j = 1, 2, ..., n$ and $i \neq j$, hence by 3.3, M is $n-$DSF R-module. ■

Proposition 3.10. *Let R be a ring and M be an R-module, If M is distributive $n-$DSF module, then every submodule of M is n-DSF module.*

Proof. Suppose that N is a submodule of M and Let M be a $n-$ DSF module this equivalent that M is a finitely generated R-module this implies that N is a finitely generated and hence by 3.9, N is n-DSF $R-$module. ■

Proposition 3.11. *Let R be a ring and M be an R-module. If M is distributive n-DSF module, then $M \diagup N$ is $n-$ DSF module, where N is a submodule of M.*

Proof. Let N be $R-$ submodule of M, then M/N is the quotient R-module and let M be a $n-$ DSF $R-$ module this is equivalents that M is a finitely generated R-module this implies that $M \diagup N$ is also a finitely generated and hence $M \diagup N$ is n-DSF $R-$ module. ■

Proposition 3.12. *Let R be a ring and M, N be R-modules. Let $f : M \longrightarrow N$ be a homomorphism R- module then,*

1. *If f is surjective and M is n-DSF module, then N, Imf and $Cokerf$ are n-DSF R-modules.*
2. *If f is injective and N is n-DSF module, then M, $Kerff$ and $coimf$ are n-DSF R-module.*

Proof.

1. Let M be n-DSF module and f be a surjective, then $Imf = N$ and from the first isomorphism theorem $M/Kerf \cong Imf = N$ and by 3.11, $M/Kerf$ is n-DSF where $kerf$ is a submodule of M that is implies that and hence N is n-DSF R-module and also Imf is n-DSF R module. we know that $cokerf = codomf/Imf = N/Imf$ is a factor of N, then from3.11, $cokerf$ is n-DSF R module.

2. Let N be n-DSF module and f be an injective, then $Kerff = 0$ and from the first isomorphism theorem $M/Kerf \cong Imf$ and N is n-DSF that implies that Imf is n-DSF as R-submodule of N, hence M is n-DSF R-module and also $kerf$ is n- DSF R module. we know that $coimf = domf/kerf = M/kerf$ is a factor of M, then from 3.11, $coimf$ is n-DSF R module.

■

In the following theorem we will study the behavior of n-DSF-module on the short exact sequence.

Theorem 3.13. *Let R be a ring and N, M and K be R-modules and $0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} K \longrightarrow 0$ be a short exact sequence of R- homomorphism modules, then we have the following cases :*

1. *M is n-DSF if and only if N and K are n-DSF.*
2. *M and N are n-DSF, then K is n-DSF.*
3. *M and K are n-DSF, then N is n- DSF.*
4. *M is n-DSF if and only if $N \bigoplus K$ is n-DSF.*

Proof.

1. Suppose that N and K are n-DSF R-modules, then they are finitely generated by 3.9, and hence by [3], M is a finitely generated and by 3.9, M is n-DSF module by.

   Conversely, suppose that M is n-DSF R-module, then M is finitely generated R-module this implies that N and K must are finitely generated because if N or K is not finitely generated this implies that M is not finitely generated and this is contradiction and hence N and K are finitely generated and by 3.9, then they are n-DSF R-modules.

2. Suppose that N and M are n-DSF R-modules, then from the first isomorphism theorem $M/kerg \cong Img = K$ and by 3.11, $M/kerg$ is $n-$ DSF R-module, hence K is $n-$ DSF R-module.

3. Suppose that K and M are n-DSF R-modules, then from the first isomorphism theorem $N/kerf \cong Imf$ and $kerf = 0$ and by 3.10, Imf is $n-$ DSF R-submodule of M, then $N \cong Imf$ and hence N is $n-$ DSF R-module.

4. Let $0 \longrightarrow N \longrightarrow M = N \bigoplus M \longrightarrow K \longrightarrow 0$ be the short exact sequence of R-homomorphism modules: we Apply (1)N and K are n-DSF R-modules, then $M = N \bigoplus K$ is n-DSF R-module.

   The conversely, since N and K are R-submodules of $M = N \bigoplus K$, then they by 3.10, they are n-DSF R-modules.

■

4 Some algebra operations on n-Dual Square Free R-Module

In this part we study some algebra operations as: the intersection, addition and multiplication of n-dual square free R-module and generalized them.

Proposition 4.1. *Let R be a ring and M, N be n-DSF R-modules. Then*

1. *$M + N$ is n-DSF R-module;*
2. *$M \bigcap N$ is n-DSF R-module.*

Proof.

1. Let M and N are n-DSF R-modules. then by 3.3, M has no proper submodules $M_1, M_2, ..., M_n$ such that $M = \Sigma_{i=1}^{n} M_i$ and $M/M_i \cong M/M_j$ and $i, j = 1.2, ..., n$, $i \neq j$. And N by 3.3, has no proper $N_1, N_2, ..., N_n$ such that $N = \Sigma_{i=1}^{n} N_i$ and $N/N_i \cong N/N_j$ and $i, j = 1.2, ..., n$, $i \neq j$.

   Now we collect $M + N = (M_1 + N_1) + (M_2 + N_2) + ... + (M_n + N_n)$ and we can write $A = M + N = A_1 + A_2 + ... + A_n$ and we know that the sum of two no proper R-submodules of R-module is also stay no proper R-submodule, then the submodules $A_1, A_2, ..., A_n$ are no proper R-submodules of A and one can easily to prove that $A/A_i \cong A/A_j$ and $i, j = 1.2, ..., n$, $i \neq j$. And from 3.3, $A = M + N$ is n-DSF R-module.

2. Let M and N are n-DSF R-modules then they are finitely generated R-modules and $M \bigcap N$ is an R-submodule of M and an R-submodule of N this implies that $M \bigcap N$ is finitely generated R-module and hence from 3.9, $M \bigcap N$ is $n-$ DSF R-module. ■

Proposition 4.2. *Let R be a ring and M be n-DSF R-module. Then*

1. *$\Sigma_{i=1}^{n} M_i$ is n-DSF R-module;*
2. *$\bigcap_{i=1}^{n} M_i$ is n-DSF R-module.*

Proof.

1. We prove it by induction mathematical. For $n = 2$, then from 4.1, $M_1 + M_2$ is n-DSF R-module.

   Suppose that the assertion is true for n, i.e $\Sigma_{i=1}^{n} M_i$ is n-DSF R-module..........(*)

   We will prove it is true for $n + 1$. Let $M_1, M_2, ..., M_n, M_{n+1}$ be n-DSF R-modules, then $\Sigma_{i=1}^{n+1} M_i = \Sigma_{i=1}^{n} M_i + M_{n+1}$ and from 4.1, $\Sigma_{i=1}^{n} M_i$, n-DSF R-module and by ....(*) and M_{n+1} is n-DSF R-module. Hence, $\Sigma_{i=1}^{n} M_i$ is n-DSF R-module for every n. ■

2. We prove it by induction mathematical. For $n = 2$, then from 3.7 $M_1 \bigcap M_2$ is n-DSF R-module.

   Suppose that the statement is true for n, i.e $\bigcap_{i=1}^{n} M_i$ is n-DSF R-module.............(*)

   We prove it is true for $n + 1$. Let $M_1, M_2, ..., M_n, M_{n+1}$ be n-DSF R-modules, then $\bigcap_{i=1}^{n+1} M_i = \bigcap_{i=1}^{n} M_i \bigcap M_{n+1}$ and from 3.7, $\bigcap_{i=1}^{n} M_i$, n-DSF R-module from ...(*) and M_{n+1} is n-DSF R-module. Hence, $\bigcap_{i=1}^{n+1} M_i$ is n-DSF R-module for every n. ■

Corollary 4.3. *Let R be a ring and M be n-DSF R-distributive module. If $N, \grave{N}$ are R-submodules of M,then*

1. *$N + \grave{N}$ is n-DSF module;*
2. *$N \cap \grave{N}$ is n-DSF module.*

Proof.

1. Let N and $\grave{N}$ are R-submodules of M and M is distributive, then they are n-DSF R-submodules and this implies that they are finitely generated R-submodules and $N + \grave{N}$ is an R-submodule of M and this implies that $N + \grave{N}$ is a finitely generated R-module and by 3.9, $N + \grave{N}$ is $n-$ DSF R-module.

2. Let N and $\grave{N}$ are n-DSF R-submodules of M, then they are finitely generated R-modules and $N \bigcap N$ is an R-submodule of N and also an R-submodule of $\grave{N}$, this implies that $N \bigcap \grave{N}$ is a finitely generated R-submodule of M and hence by 3.9, $N \bigcap \grave{N}$ is $n-$ DSF R-module. ■

Corollary 4.4. *Let R be a ring and M be n-DSF R-module. If M is distributive and $N_1, N_2, ..., N_n$ are R submodules of M, then*

1. *$\Sigma_{i=1}^{n} N_i$ is n-DSF module;*
2. *$\bigcap_{i=1}^{n} N_i$ is n-DSF module.*

Proof.

1. Let $N_1, N_2, ..., N_n$ be R-submodules of M and M is distributive, then from 3.10 they are n-DSF R-submodules and this equivalents that they are finitely generated R-submodules and $\Sigma_{i=1}^{n} N_i$ is also an R-submodule of M and this equivalents that $\Sigma_{i=1}^{n} N_i$ is a finitely generated R-module and hence by 3.9, $\Sigma_{i=1}^{n} N_i$ is $n-$ DSF R-module.
2. Let $N_1, N_2, ..., N_n$ be n-DSF R-submodules of M and M is distributive, then they are finitely generated R-modules and $\bigcap_{i=1}^{n} N_i$ is an R-submodule of M and this equivalents that $\bigcap_{i=1}^{n} N_i$ is a finitely generated R-submodule of M and hence by 3.9, $\bigcap_{i=1}^{n} N_i$ is $n-$ DSF R-module. ■

Proposition 4.5. *Let R be a ring and M_i be R_i-modules, $i = 1, 2$. Then M_i are n-DSF R_i-module , $i = 1, 2$ if and only if $M = M_1 \times M_2$ is n-DSF $R = R_1 \times R_2$-module.*

Proof. Suppose that M_i are n-DSF R_i-module , $i = 1, 2$ then by 3.3, M_1 has no proper submodules $A_1, A_2, ..., A_n$ with $M_1 = \Sigma_{i=1}^{n} A_i$ and $M_1/A_i \cong M_1/A_j$ and $i, j = 1.2, ..., n$, $i \neq j$. And M_2 by 3.3, has no proper $B_1, B_2, ..., B_n$ with $M_2 = \Sigma_{i=1}^{n} B_i$ and $M_1/B_i \cong M_2/B_j$ and $i, j = 1.2, ..., n$, $i \neq j$.

Now we multiply $M_1 \times M_2 = (A_1 \times B_1) + (A_2 \times B_2) + ... + (A_n \times B_n)$ and we can write $M = M_1 \times M_2 = D_1 + D_2 + ... + D_n$ where $D_i = A_i \times B_i, i = 1, 2, ..., n$ and we know that the multiple of two no proper R-submodules of R-module is also stay no proper R-submodule, then the submodules $D_1, D_2, ..., D_n$ are no proper R-submodules of M and one can easily to prove that $M/D_i \cong M/D_j$ and $i, j = 1.2, ..., n$, $i \neq j$. And from 3.3, $D = M_1 \times M_2$ is n-DSF R-module.

Conversely, suppose that $M = M_1 \times M_2$ is n-DSF $R = R_1 \times R_2$-module and M_i be R_i-submodules , $i = 1, 2$ of M, hence M_i are n-DSF R_i-modules , $i = 1, 2$. ■

Proposition 4.6. *Let R_i be rings and M_i be R_i-modules , $i = 1, 2, ..., n$. Then M_i are n-DSF R_i-modules , $i = 1, 2, ..., n$ if and only if $M = \prod_{i=1}^{n} M_i$ is n-DSF $R = \prod_{i=1}^{n} R_i$-module.*

Proof. We will prove it by induction mathematical. For $n = 2$, then from 4.5 $M_1 \times M_2$ is n-DSF $R = R_1 \times R_2$-module.

Suppose that the assertion is true for n, i.e $M = \prod_{i=1}^{n} M_i$ is n-DSF $R = \prod_{i=1}^{n} R_i$-module if and only if M_i are n-DSF R_i-module , $i = 1, 2, ..., n$. ....(*)

We prove it is true for $n+1$. Let $M_1, M_2, ..., M_n, M_{n+1}$ be n-DSF R_i-modules, $i = 1, 2, ..., n$, then $\prod_{i=1}^{n+1} M_i = \prod_{i=1}^{n} M_i \times M_{n+1}$ and from 4.5, $\prod_{i=1}^{n} M_i$, n-DSF $R = \prod_{i=1}^{n} R_i$-module and by ......(*) and M_{n+1} is n-DSF R_{n+1} is -module. Hence, $\prod_{i=1}^{n} M_i$ is n-DSF $R = \prod_{i=1}^{n} R_i$-module for every n.

Conversely, suppose that $M = \prod_{i=1}^{n} M_i$ is n-DSF $R = \prod_{i=1}^{n} R_i$-module. And M_i are R_i-submodules of M, $i = 1, 2, ..., n$, hence M_i are n-DSF R_i-modules, $i = 1, 2, ..., n$. ■

Proposition 4.7. *Let R_i be rings and M_i be R_i-modules , $i = 1, 2$, then M_i are n-DSF R_i-modules, $i = 1, 2$ if and only if $M = M_1 \cap M_2$ is n-DSF $R = R_1 \cap R_2$-module.*

Proof. Suppose that M_i are n-DSF R_i-module , $i = 1, 2$, then by 3.3 M_1 has no proper submodules $S_1, S_2, ..., S_n$ with $M_1 = \Sigma_{i=1}^{n} S_i$ and $M_1/S_i \cong M_1/S_j$ and $i, j = 1.2, ..., n$, $i \neq j$. And M_2 by 3.3 has no proper $T_1, T_2, ..., T_n$ with $M_2 = \Sigma_{i=1}^{n} T_i$ and $M_1/T_i \cong M_2/T_j$ and $i, j = 1.2, ..., n$, $i \neq j$.

Now we intersect $M_1 \cap M_2 = (S_1 \cap T_1) + (S_2 \cap T_2) + ... + (S_n \cap T_n)$ and we can write $M = M_1 \cap M_2 = K_1 + K_2 + ... + K_n$ where $K_i = S_i \cap B_i, i = 1, 2, ..., n$ and we know that the intersection of two no proper R-submodules of R-module is also stay no proper R-submodule, then the submodules $K_1, K_2, ..., K_n$ are no proper R-submodules of M and one can easily to prove that $M/K_i \cong M/K_j$ and $i, j = 1.2, ..., n,\ i \neq j$. And from 3.3 $K = M_1 \bigcap M_2$ is n-DSF R-module.

Conversely, suppose that $M = M_1 \cap M_2$ is n-DSF $R = R_1 \cap R_2$-module, then we have the following short exact sequence $0 \longrightarrow \cap_{i=1}^{2} M_i \longrightarrow M_i \longrightarrow \cap_{i=1}^{3} M_i \longrightarrow 0$ and we have and $\cap_{i=1}^{3} M_i$ is R_i-submodules of M_i, then it is n-DSF R-module and hence by 3.13, M_i is n-DSF R_i-modules, $i = 1, 2$. ■

Theorem 4.8. *Let R_i be rings and M_i be R_i-modules, $i = 1, 2, ..., n$, then M_i are n-DSF R_i-modules, $i = 1, 2, ..., n$ if and only if $M = \bigcap_{i=1}^{n} M_i$ is n-DSF $R = \cap_{i=1}^{n} R_i$-module.*

Proof. We prove it by induction mathematical. For $n = 2$, then from 4.7 $M_1 \cap M_2$ is n-DSF $R = R_1 \cap R_2$-module.

Suppose that the assertion is true for n, i.e $M = \bigcap_{i=1}^{n} M_i$ is n-DSF $R = \bigcap_{i=1}^{n} R_i$-module if and only if M_i are n-DSF R_i-module , $i = 1, 2, ..., n$. ....(*)

We prove it is true for $n+1$. Let $M_1, M_2, ..., M_n, M_{n+1}$ be n-DSF R_i-modules, $i = 1, 2, ..., n$, then $\bigcap_{i=1}^{n+1} M_i = \bigcap_{i=1}^{n} M_i \bigcap M_{n+1}$ and from 4.7, $\bigcap_{i=1}^{n} M_i$, is n-DSF $R = \bigcap_{i=1}^{n} R_i$-module and by ....(*) and M_{n+1} is n-DSF R_{n+1}-module. Hence, $\bigcap_{i=1}^{n} M_i$ is n-DSF $R = \bigcap_{i=1}^{n} R_i$-module for every n.

Conversely, suppose that $M = \bigcap M_i$ is n-DSF $R = \bigcap R_i$-module, then we have the following short exact sequence $0 \longrightarrow \bigcap_{i=1}^{n-2} M_i \longrightarrow M_i \longrightarrow \bigcap_{i=1}^{n} M_i \longrightarrow 0$ and we have and $\bigcap_{i=1}^{n-2} M_i$ and $\bigcap_{i=1}^{n-2} M_i$ are n-DSF R-submodules by hypothesis, hence by 3.13, M_i are n-DSF R_i-modules, $i = 1, 2, ..., n$. ■

Theorem 4.9. *Let R_i be rings and M_i be R_i-modules, $i = 1, 2, ..., n$, then M_i are n-DSF R_i-modules, $i = 1, 2, ..., n$ if and only if $M = \bigoplus_{i=1}^{n} M_i$ is n-DSF $R = \prod_{i=1}^{n} R_i$-module.*

Proof. Suppose that M_i are n-DSF R_i-modules, $i = 1, 2, ..., n$, then we have the following short exact sequence $0 \longrightarrow M_{n-1} \longrightarrow M = \bigoplus_{i=1}^{n} M_i \longrightarrow M_n \longrightarrow 0$ and we have and M_{n-1} and M_n are n-DSF R-submodules by hypothesis, hence by 3.13, $M = \bigoplus_{i=1}^{n} M_i$ is n-DSF R_i-modules.

Conversely, suppose that $M = \bigoplus_{i=1}^{n} M_i$ is n-DSF $R = \prod_{i=1}^{n} R_i$-module. And M_i are R_i-submodules of M, $i = 1, 2, ..., n$, hence by 3.10, M_i are n-DSF R_i-modules, $i = 1, 2, ..., n$. ■

5 Some results with a finitely generated and n-Dual Square Free R-Module

In this part we introduce and study some important characterizations of n-dual square free R-module, important results and example.

Definition 5.1. *[11] An R-module N is called a finitely presented (for short f.p.) if,*

1. *N is finitely generated and*
2. *In every exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow N \longrightarrow 0$, with L finitely generated, K is also finitely generated.*

Theorem 5.2. *Let R be a ring and M be an R-modules, the following are equivalent :*

1. *M is n-DSF R-module.*
2. *M is a finitely presented R-module.*

Proof. (1) $\Rightarrow$ (2). Suppose that M is n-DSF R-module this implies that M is a finitely generated R-module and let $0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} M \longrightarrow 0$ be a short exact sequence with L finitely generated and from the first isomorphism theorem $K/kerf \cong Imf$ and Imf is a R-submodule of L, then it is a finitely generated R-submodule and f is injective, then $kerf = 0$ this implies that $K \cong Imf$ and hence K is also finitely generated and we conclude by 5.1 M is finitely presented R-module.
(2) $\Rightarrow$ (1). Suppose that M is a finitely presented R-module, then M is a finitely generated R-module and by 3.9, M is n-DSF R-module. ■

Definition 5.3. *[18] Let R be a ring and M be an R-module. A module N is called a coherent if*

1. *M is a finitely generated and*
2. *Any finitely generated submodule of M is finitely presented.*

Theorem 5.4. *Let R be a ring and M be an R-modules, the following are equivalent :*

1. *M is n-DSF R-module.*
2. *M is a coherent R-module.*

Proof. (1) $\Rightarrow$ (2). Suppose that M is n-DSF R-module this is equivalent that M is finitely generated R-module and let N be an R-submodule of M, then N is also a finitely generated R-module. Now let $0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} N \longrightarrow 0$ be a short exact sequence with L is finitely generated and from the first isomorphism theorem $K/kerf \cong Imf$ and Imf is a R-submodule of L, then it is a finitely generated R-submodule and f is injective, then $kerf = 0$ this implies that $K \cong Imf$ and hence K is also a finitely generated and we conclude by 5.3, N is a finitely presented R-module.

(2) $\Rightarrow$ (1). Suppose that M is a finitely presented R-module, then M is a finitely generated R-module and by 3.9 M is n-DSF R-module. ■

Theorem 5.5. *Let R be a ring and M be an R-modules, the following are equivalent :*

1. *M is n-DSF R-module.*
2. *M is finitely generated R-module.*
3. *M is Noetherian R-module.*
4. *M is finitely presented R-module.*
5. *M is a coherent R-module.*

Proof. (1) $\Leftrightarrow$ (2). Suppose that M is n-DSF module that is equivalent that M is by 3.9, finitely generated module.

(2) $\Leftrightarrow$ (3). Suppose that M is finitely generated R-module that is equivalent that by [3], M is Noetherian R-module.

(3) $\Leftrightarrow$ (4). Suppose that M is Noetherian R-module that is equivalent that by [3], M is a finitely generated R-module and hence M is n-DSF module by 5.2, M is finitely presented R-module.

(4) $\Leftrightarrow$ (5). Suppose that M is finitely presented R-module that is equivalent that by [3], M is a finitely generated R-module and by **??** M is n-DSF module and hence by 5.4, M is a

coherent R-module.

(5) $\Leftrightarrow$ (1). Suppose that M is a coherent R-module that is equivalent that by 5.3, M is finitely generated R-module that is equivalent that by 3.9, M is n-DSF module. ∎

Corollary 5.6. *Let R be a ring and M be an R-module, the following are holds :*

1. *Every submodule of a finitely generated R-module is n-DSF module.*
2. *Every submodule of a Neotherian R-module is n-DSF module.*
3. *Every submodule of a finitely presented R-module is n-DSF module.*
4. *Every submodule of a coherent R-module is n-DSF module.*

Proof.

1. Let M be finitely generated R-module and let N be an R submodule of M this implies that N finitely generated and hence by 3.9, N is n-DSF module.
2. Let M be Neotherian R-module and let N be an R submodule of M this implies that by [3], N Neotherian R submodule of M and then N is finitely generated and by 3.9, hence N is n-DSF module.
3. Let M be finitely presented R-module and let N be an R submodule of M this implies that [17] N a finitely presented, then N is finitely generated and by 3.9, hence N is n-DSF module.
4. Let M be coherent R-module and let N be an R submodule of M this implies that [17] N coherent R submodule of M, then N is finitely generated and by 3.9, hence N is n-DSF module.

Proposition 5.7. *Let R be a ring and M be finitely generated R-module. Then M is n-DSF R-module.*

Proof. Let M be a finitely generated R-module. Suppose that A is a maximal R-submodule of M [16]. Let $x_i, \in M, i = 1, 2, ..., n$ but they do not belong to A, then we get the no proper submodules $M_1 = Rx_1 + A, M_2 = Rx_2 + A, ..., M_n = Rx_n + A$ in M with $M = M_1 + M_2 + ... + M_n$ and it easy to prove $M/M_i \cong M/M_j, i, j = 1, 2, ..., n$ and $i \neq j$, and by 3.3, hence M is $n-$DSF R-module. ∎

Proposition 5.8. *Let R be a ring and M be R-module. If M is a local R-module, then M is n-DSF R-module.*

Proof. Let M be a local R-module, then M has a unique maximal submodule A. Let $x_1, x_2, ..., x_n \in M$ but they do not belong to A, then we get the no proper submodule $M_1 = Rx_1 + A, M_2 = Rx_2 + A, ..., M_n = Rx_n + A$ in M with $M = M_1 + M_2 + ... + M_n$ and it easy to prove $M/M_i \cong M/M_j, i, j = 1, 2, ..., n$ and $i \neq j$, by 3.3, hence M is $n-$DSF R-module. ∎

Proposition 5.9. *Let R be a ring and M be R-module, let N be R-submodule of M, if N and M/N are $n-$ DSF R-modules, then M is n-DSF R-module.*

Proof. Let M be R-module and N be R-submodule of M, then we have $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ be a short exact sequence of homomorphism R-modules. N and M/N are $n-$DSF R-modules, then they are finitely generated R-modules this equivalent that M is finitely generated R-module and hence 3.9, M is n-DSF R-module. ∎

Proposition 5.10. *Let R be a ring and M be R-module, if M is Noetherian R-module, then M is n-DSF R-module.*

Proof. Let M be M Noetherian R-module and from the definition of Noetherian R-module [3], M has no proper R-submodule $M_1, M_2, ..., M_n$ and $n \geq k$ with $M = \Sigma_{i=1}^{n} M_i$ and it easily to prove $M/M_i \cong M/M_j, i, j = 1, 2, ..., n$ and $i \neq j$, hence by 3.3, M is $n-$ DSF R-module. ■

Example 5.11. *Let $\mathbb{Z}$ be the ring of the integer set, let $x \in \mathbb{Z}$ and x does not belong to any maximal ideal of a ring $\mathbb{Z}$, then $\mathbb{Z}$ is a n-DSF as $\mathbb{Z}$-module.*

Proof. It is clear that $\mathbb{Z}$ is $\mathbb{Z}$-module and $\mathbb{Z}$ is a PID, every ideal of $\mathbb{Z}$ is a finitely generated $\mathbb{Z}$-submodule of $\mathbb{Z}$ as $\mathbb{Z}$- module [4]. Suppose that $\mathbb{Z}$ has maximal $\mathbb{Z}$-submodules $p_i\mathbb{Z}$ and $i = 1, 2, ..., n$. Because p_i are prime numbers and $p_i\mathbb{Z} \subsetneqq \mathbb{Z}$, hence $\mathbb{Z}$ has no proper $\mathbb{Z}$-submodules $J_1 = x\mathbb{Z} + p_1\mathbb{Z}, J_2 = x\mathbb{Z} + p_2\mathbb{Z}, ..., J_n = x\mathbb{Z} + p_n\mathbb{Z}$. and $J_i \subseteq \mathbb{Z}$ with $\mathbb{Z} = J_1 + J_2 +, ..., +J_n$. And it is easily prove that $\mathbb{Z}/J_i \cong \mathbb{Z}/J_j$ and $i, j = 1, 2, ..., n$, and $i \neq j$. and hence by 3.3, $\mathbb{Z}$ is n-DSF $\mathbb{Z}$-module. ■

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