EXACT SOLUTION OF LINEAR AND NONLINEAR SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS
BY DOUBLE ABOODH-SHEHU AND ADOMIAN DECOMPOSITION METHOD

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Abstract
In this work, we developed a new method to obtain approximate solutions of linear and nonlinear coupled partial differential equations with the help of Double Aboodh-Shehu decomposition method (DASDM). The nonlinear term can easily be handled with the help of Adomian polynomials. The results of the present technique have closed agreement with approximate solutions obtained with the help of Adomian decomposition method (ADM).

1. Introduction
The topic of partial differential equations is one of the most important subjects in mathematics and other sciences. Therefore it is very important to know methods to solve such partial differential equations. In the literature, several different transforms are introduced and applied to find the solution of partial differential equations such as Laplace transform [8], Shehu transform [10], Aboodh transform [2], and so on. Two of the most popular methods for solving partial differential equations are the integral transforms method and Adomian decomposition method [13]. The decomposition method has been shown to solve efficiently, easily and accurately a large class of linear and nonlinear ordinary, partial, deterministic or stochastic differential equations [7, 16]. The method is very well suited to physical problems since it does not require unnecessary linearization, perturbation, discretization, or any unrealistic assumptions. The Adomian decomposition method is relatively easy to implement, and it can be used with other methods. It can also be used to solve both initial value problems and boundary value problems. In [9], the authors used Laplace transform with Adomian decomposition method to solve nonlinear coupled partial differential equations. The main objective of this paper is to obtained the exact solutions of coupled linear and nonlinear partial differential equations with initial value problems by using double Aboodh-Shehu transform algorithm based on decomposition method. First, we recall the definitions of Aboodh, Shehu and double Aboodh-Shehu transforms.

Definition 1.1. The single Aboodh transform of the real function \( f(x) \) of exponential order is defined over the set of functions
\[
M = \{ f(x) : \exists K, \tau_1, \tau_2 > 0, |f(x)| < Ke^{x|x|}, x \in (-1)^i \times [0, \infty), i = 1, 2 \}.
\]

Key words and phrases. Double Aboodh-Shehu decomposition method, Adomian decomposition method, linear and nonlinear system Partial differential equations.
EXACT SOLUTION OF LINEAR AND NONLINEAR SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS BY DOUBLE ABOODH-SHEHU AND ADOMIAN DECOMPOSITION METHOD

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ABSTRACT. In this work, we develop a new method to obtain approximate solutions of linear and nonlinear coupled partial differential equations with the help of Double Aboodh-Shehu decomposition method (DASDM). The nonlinear term can easily be handled with the help of Adomian polynomials. The results of the present technique have closed agreement with approximate solutions obtained with the help of Adomian decomposition method (ADM).

1. Introduction

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The main objective of this paper is to obtained the exact solutions of coupled linear and nonlinear partial differential equations with initial value problems by using double Aboodh-Shehu transform algorithm based on decomposition method. First, we recall the definitions of Aboodh, Shehu and double Aboodh-Shehu transforms.

Definition 1.1. The single Aboodh transform of the real function $f(x)$ of exponential order is defined over the set of functions

$$\mathcal{M} = \left\{ f(x) : \exists K, \tau_1, \tau_2 > 0, |f(x)| < Ke^{x|\tau_1|}, x \in (-1)^i \times [0, \infty), i = 1, 2 \right\}.$$
by the following integral

$$A[f(x)] = F(r) = \frac{1}{r} \int_{0}^{\infty} e^{-rx} f(x) dx, \quad \tau_1 \leq r \leq \tau_2.$$ 

And the inverse Aboodh transform is

$$A^{-1}[F(r)] = f(x) = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} re^{rx} F(r) dr, \quad \omega \geq 0.$$ 

Aboodh transform was introduced by K. Aboodh [2] in 2013 to facilitate the process of solving ordinary and partial differential equations in the time domain. For further details and properties of the Aboodh transform and its derivatives we refer to [1, 5].

**Definition 1.2.** The single Shehu transform of the function $f(t)$ of exponential order is defined over the set of functions

$$\mathcal{B} = \left\{ f(t) : \exists N, \rho_1, \rho_2 > 0, |f(t)| < Ne^{\rho j}, t \in (-1)^j \times [0, \infty), \ j = 1, 2 \right\},$$

by the following integral

$$\mathcal{S}[f(t)] = F(s, u) = \int_{0}^{\infty} e^{-\frac{ru}{s}} f(t) dt, \ s > 0, \ u > 0.$$ 

Moreover, the inverse Shehu transform is given by

$$f(t) = \mathcal{S}^{-1}[F(s, u)] = \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} \frac{1}{u} e^{\frac{s}{u}} F(s, u) ds,$$

where $s$ and $u$ are the Shehu transform variables, and $w$ is a real constant and the integral in Eq.(1.2) is taken along $s = w$ in the complex plane $s = x + iy$. For further details and properties of Shehu transform and its derivatives we refer to [3, 4, 10, 12].

**Definition 1.3.** The double Aboodh-Shehu transform of the continuous function $f(x,t)$, $x,t > 0$ is denoted by the operator $A_xS_t[f(x,t)] = F(r,s,u)$ and defined by

$$A_xS_t[f(x,t)] = F(r,s,u) = \frac{1}{r} \int_{0}^{\infty} \int_{0}^{\infty} e^{-(rx+\frac{ru}{s})} f(x,t) dx dt$$

$$= \frac{1}{r} \lim_{a \to \infty, b \to \infty} \int_{0}^{a} \int_{0}^{b} e^{-(rx+\frac{ru}{s})} f(x,t) dx dt.$$ 

It converges if the limit of the integral exists, and diverges if not. The inverse of double Aboodh-Shehu transform is defined by

$$f(x,t) = A_x^{-1}S_t^{-1} [F(r,s,u)] = \frac{1}{(2\pi i)^2} \int_{\rho_1-i\infty}^{\rho_1+i\infty} \int_{\rho_2-i\infty}^{\rho_2+i\infty} re^{rx} \left\{ \frac{1}{u} e^{\frac{s}{u}} F(r,s,u) ds \right\} dr,$$

where $\rho_1$ and $\rho_2$ are real constants.
SOLUTION OF LINEAR AND NONLINEAR SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

Double Aboodh-Shehu transform for second partial derivatives property

\[
A_xS_t\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = r^2F(r, s, u) - S[f(0, t)] - \frac{1}{r}S[f_x(0, t)],
\]
\[
A_xS_t\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{s^2}{u^2}F(r, s, u) - \frac{s}{u}A[f(x, 0)] - A[f_t(x, 0)],
\]
\[
A_xS_t\left[\frac{\partial^2 f(x,t)}{\partial x\partial t}\right] = \frac{sr}{u}F(r, s, u) - rA[f(x, 0)] - \frac{1}{r}S[f_t(0, t)].
\]

where \(A[.]\) and \(S[.]\) denote to single Aboodh transform and single Shehu transform respectively.

In [11], some theorems and properties of the double Aboodh-Shehu transform and its derivatives were established. Moreover, double Aboodh-Shehu transform for some functions are showed.

We consider the general inhomogeneous nonlinear partial differential equation with initial conditions given below:

\[
Lu(x, t) + Ru(x, t) + Nu(x, t) = f(x, t),
\]
\[
u(x, 0) = f_1(x), \quad u_t(x, 0) = f_2(x),
\]

where \(L = \frac{\partial^2}{\partial t^2}\) is the second order derivative which is assumed to be easily invertible, \(R\) is the remaining linear differential operator, \(Nu\) represents the nonlinear terms and \(f(x, t), f_1(x)\) and \(f_2(x)\) are known functions.

The methodology consists of applying double Aboodh-Shehu transform first on both sides of Eq. (1.1)

\[
A_xS_t[Lu(x, t)] + A_xS_t[Ru(x, t)] + A_xS_t[Nu(x, t)] = A_xS_t[f(x, t)].
\]

Using the differentiation property of double Aboodh-Shehu transform, we have

\[
\frac{s^2}{u^2}U(r, s, u) - \frac{s}{u}A[u(x, 0)] - A[u_t(x, 0)] + A_xS_t[Ru(x, t)] \\
+ A_xS_t[Nu(x, t)] = A_xS_t[f(x, t)].
\]

Using given initial conditions and arrangement, Eq. (1.4) becomes

\[
U(r, s, u) = \frac{u}{s}A[f_1(x)] + \frac{u^2}{s^2}A[f_2(x)] + \frac{u^2}{s^2}A_xS_t[f(x, t)] \\
- \frac{u^2}{s^2}A_xS_t[Ru(x, t)] - \frac{u^2}{s^2}A_xS_t[Nu(x, t)].
\]

Application of inverse double Aboodh-Shehu transform to (1.5) leads to

\[
u(x, t) = A_x^{-1}S_t^{-1}\left[\frac{u}{s}A[f_1(x)] + \frac{u^2}{s^2}A[f_2(x)] + \frac{u^2}{s^2}A_xS_t[f(x, t)]\right] \\
- A_x^{-1}S_t^{-1}\left[\frac{u^2}{s^2}A_xS_t[Ru(x, t)] + \frac{u^2}{s^2}A_xS_t[Nu(x, t)]\right].
\]

The second step in double Aboodh-Shehu decomposition method is that we represent solution as an infinite series:

\[
u(x, t) = \sum_{i=0}^{\infty} u_i(x, t),
\]
and the nonlinear term can be decomposed as

\[ Nu(x, t) = \sum_{i=0}^{\infty} A_i \]  

(1.8)

where \( A_i \) are Adomian polynomials \([15]\) of \( u_0, u_1, u_2, ..., u_n \) and it can be calculated by formula

\[ A_i = \frac{1}{i!} \frac{d^i}{d\lambda^i} \left[ \sum_{i=0}^{\infty} \lambda^i u_i \right]_{\lambda=0} \]  

(1.9)

Substituting Eq. (1.7) and Eq. (1.8) in Eq. (1.6), we get

\[ \sum_{i=0}^{\infty} u_i(x, t) = A^{-1}_x S^{-1}_t \left[ \frac{u}{s} A[f_1(x)] + \frac{u^2}{s^2} A[f_2(x)] + \frac{u^2}{s^2} A_x S_t[f(x, t)] \right] \]

(1.10)

On comparing both sides of the Eq. (1.10) and by using standard Adomian decomposition method (ADM), we then define the recurrence relations as

\[ u_0(x, t) = A^{-1}_x S^{-1}_t \left[ \frac{u}{s} A[f_1(x)] + \frac{u^2}{s^2} A[f_2(x)] + \frac{u^2}{s^2} A_x S_t[f(x, t)] \right], \]  

(1.11)

\[ u_1(x, t) = -A^{-1}_x S^{-1}_t \left[ \frac{u^2}{s^2} A_x S_t[R u_0(x, t)] + \frac{u^2}{s^2} A_x S_t[A_0] \right], \]  

(1.12)

\[ u_2(x, t) = -A^{-1}_x S^{-1}_t \left[ \frac{u^2}{s^2} A_x S_t[R u_1(x, t)] + \frac{u^2}{s^2} A_x S_t[A_1] \right]. \]  

(1.13)

In more general, the recursive relation is given by

\[ u_{i+1}(x, t) = -A^{-1}_x S^{-1}_t \left[ \frac{u^2}{s^2} A_x S_t[R u_i(x, t)] + \frac{u^2}{s^2} A_x S_t[A_i] \right], \quad i \geq 0. \]  

(1.14)

The recurrence relation generates the solution of (1.1) in series form given by

\[ u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + ... + u_i(x, t) + .... \]  

(1.15)

### 2. Applications

In order to illustrate the applicability and efficiency of the double Aboodh-Shehu decomposition method, we apply this method to solve some examples.

**Example 2.1.** Consider the following nonlinear partial differential equation

\[ u_{xx}(x, t) + \frac{1}{4} u_t^2(x, t) = u(x, t), \]  

(2.1)

subject to the initial conditions

\[ u(0, t) = 1 + t^2, \quad u_x(0, t) = 1. \]  

(2.2)
SOLUTION OF LINEAR AND NONLINEAR SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

Applying double Aboodh-Shehu transform first on both sides of Eq. (2.1)

\[ A_xS_t[u_{xx}(x, t)] + A_xS_t\left[\frac{1}{4}u_t^2(x, t)\right] = A_xS_t[u(x, t)]. \]  

(2.3)

Using the differentiation property of double Aboodh-Shehu transform, we have

\[ r^2U(r, s, u) - S[u(0, t)] - \frac{1}{r}S[u_x(0, t)] + A_xS_t\left[\frac{1}{4}u_t^2\right] = A_xS_t[u]. \]

Rearranging the terms and using given initial conditions, we have

\[ U(r, s, u) = \frac{1}{r^2}S[1 + t^2] + \frac{1}{r^3}S[1] + \frac{1}{r^2}A_xS_t[u] - \frac{1}{4r^2}A_xS_t[u_t^2]. \]  

(2.4)

Application of inverse double Aboodh-Shehu transform to (2.4) leads to

\[ u(x, t) = 1 + t^2 + x + A_x^{-1}S_t^{-1}\left[\frac{1}{r^2}A_xS_t[u] - \frac{1}{4r^2}A_xS_t[u_t^2]\right]. \]  

(2.5)

The double Aboodh-Shehu decomposition method assumes a series solution of the function \( u(x, t) \) is given by

\[ u(x, t) = \sum_{i=0}^{\infty} u_i(x, t). \]  

(2.6)

Using Eq. (2.6) into Eq. (2.5) we get

\[ \sum_{i=0}^{\infty} u_i(x, t) = 1 + t^2 + x + A_x^{-1}S_t^{-1}\left[\frac{1}{r^2}A_xS_t\left[\sum_{i=0}^{\infty} u_i\right] - \frac{1}{4r^2}A_xS_t\left[\sum_{i=0}^{\infty} A_i(u)\right]\right], \]  

(2.7)

where \( A_i \) are Adomian polynomials that represents nonlinear terms. So Adomian polynomials are given as follows:

\[ \sum_{i=0}^{\infty} A_i(u) = u_t^2(x, t). \]  

(2.8)

The few components of the Adomian polynomials are given as follow:

\[ A_0(u) = u_0^2, \quad A_1(u) = 2u_0u_1t, \quad ..., \quad A_i(u) = \sum_{r=0}^{i} u_r t u_{i-r}t. \]  

(2.9)

From Eqs. (2.7) and (2.8) we obtain

\[ u_0 = 1 + t^2 + x, \]  

(2.10)

\[ \sum_{i=0}^{\infty} u_{i+1}(x, t) = A_x^{-1}S_t^{-1}\left[\frac{1}{r^2}A_xS_t\left[\sum_{i=0}^{\infty} u_i - \frac{1}{4} \sum_{i=0}^{\infty} A_i(u)\right]\right], \quad i \geq 0. \]  

(2.11)
Then the first few components of $u_i(x, t)$ follows immediately upon setting
\[
\begin{align*}
  u_1(x, t) &= A_x^{-1}S_t^{-1}\left[\frac{1}{r^2}A_xS_t[u_0] - \frac{1}{4r^2}A_xS_t[A_0(u)]\right] \\
  &= A_x^{-1}S_t^{-1}\left[\frac{1}{r^2}A_xS_t[1 + t^2 + x] - \frac{1}{4r^2}A_xS_t[4t^2]\right] \\
  &= A_x^{-1}S_t^{-1}\left[\frac{u}{r^4} + \frac{u}{r^5}\right] \\
  &= \frac{1}{2!}x^2 + \frac{1}{3!}x^3,
\end{align*}
\]
\[
\begin{align*}
  u_2(x, t) &= A_x^{-1}S_t^{-1}\left[\frac{1}{r^2}A_xS_t[u_1] - \frac{1}{4r^2}A_xS_t[A_1(u)]\right] \\
  &= A_x^{-1}S_t^{-1}\left[\frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{1}{4r^2}A_xS_t[0]\right] \\
  &= A_x^{-1}S_t^{-1}\left[\frac{u}{r^6} + \frac{u}{r^7}\right] \\
  &= \frac{1}{4!}x^4 + \frac{1}{5!}x^5,
\end{align*}
\]
and so on for other components. Therefore the exact solution obtained by double Aboodh-Shehu decomposition method is given as follows:
\[
\begin{align*}
  u(x, t) &= \sum_{i=0}^{\infty} u_i(x, t) = t^2 + 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + ... \\
  &= t^2 + \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + ...\right) \\
  &= t^2 + e^x.
\end{align*}
\]
Which is same as solution obtained by variational iteration method (VIM) [14].

**Example 2.2.** Consider the following linear system of partial differential equations
\[
\begin{align*}
  u_t(x, t) - v_x(x, t) - (u - v) &= -2, \\
  v_t(x, t) + u_x(x, t) - (u - v) &= -2, \\
\end{align*}
\]
with initial conditions
\[
\begin{align*}
  u(x, 0) &= 1 + e^x, \\
  v(x, 0) &= -1 + e^x.
\end{align*}
\]
Taking the double Aboodh-Shehu transform on both sides of (2.12), then by using the differentiation property of double Aboodh-Shehu transform, we have
\[
\begin{align*}
  \frac{\partial}{\partial u}U(r, s, u) - A[u(x, 0)] &= A_xS_t[-2] + A_xS_t[v_x] + A_xS_t[u - v], \\
  \frac{\partial}{\partial u}V(r, s, u) - A[v(x, 0)] &= A_xS_t[-2] - A_xS_t[u_x] + A_xS_t[u - v].
\end{align*}
\]
Application of single Aboodh transform to (2.13) and substitute in (2.14), we have
\[
\begin{align*}
  U(r, s, u) &= \frac{u}{r^2} + \frac{u}{r(r-1)s} - \frac{2ru^2}{r^2s^2} + \frac{u}{s}A_xS_t[v_x + (u - v)], \\
  V(r, s, u) &= -\frac{u}{r^2} + \frac{u}{r(r-1)s} - \frac{2ru^2}{r^2s^2} - \frac{u}{s}A_xS_t[u_x - (u - v)].
\end{align*}
\]
Taking the inverse double Aboodh-Shehu transform in (2.15), our required recursive relation is given by

\[ u(x, t) = 1 + e^x - 2t + A_x^{-1}S_t^{-1}\left[ \frac{u}{s} A_x S_t \left[ u_x + (u - v) \right] \right], \]

\[ v(x, t) = -1 + e^x - 2t - A_x^{-1}S_t^{-1}\left[ \frac{u}{s} A_x S_t \left[ u_x - (u - v) \right] \right]. \]  

(2.16)

The double Aboodh-Shehu decomposition method assumes a series solution of the functions \( u(x, t) \) and \( v(x, t) \) are given by

\[ u(x, t) = \sum_{i=0}^{\infty} u_i(x, t), \quad v(x, t) = \sum_{i=0}^{\infty} v_i(x, t). \]  

(2.17)

Using Eq. (2.17) into Eq. (2.16) we obtain

\[ \sum_{i=0}^{\infty} u_i(x, t) = 1 + e^x - 2t + A_x^{-1}S_t^{-1}\left[ \frac{u}{s} A_x S_t \left[ \sum_{i=0}^{\infty} u_{ix} + \sum_{i=0}^{\infty} (u_i - v_i) \right] \right], \]  

(2.18)

\[ \sum_{i=0}^{\infty} v_i(x, t) = -1 + e^x - 2t - A_x^{-1}S_t^{-1}\left[ \frac{u}{s} A_x S_t \left[ \sum_{i=0}^{\infty} u_{ix} - \sum_{i=0}^{\infty} (u_i - v_i) \right] \right]. \]  

(2.19)

From (2.18) and (2.19) the recursive relations are

\[ u_0(x, t) = 1 + e^x - 2t, \]

\[ u_{i+1}(x, t) = A_x^{-1}S_t^{-1}\left[ \frac{u}{s} A_x S_t \left[ \sum_{i=0}^{\infty} u_{ix} + \sum_{i=0}^{\infty} (u_i - v_i) \right] \right], \quad i \geq 0, \]  

(2.20)

\[ v_0(x, t) = -1 + e^x - 2t, \]

\[ v_{i+1}(x, t) = -A_x^{-1}S_t^{-1}\left[ \frac{u}{s} A_x S_t \left[ \sum_{i=0}^{\infty} u_{ix} - \sum_{i=0}^{\infty} (u_i - v_i) \right] \right], \quad i \geq 0. \]
In view of the recursive relations (2.20) we obtained other components as follows:

\[
\begin{align*}
\quad u_1(x,t) & = A_x^{-1}S_t^{-1}\left[\frac{u}{s} A_x S_t\left[ u_{0x} + (u_0 - v_0) \right]\right] = A_x^{-1}S_t^{-1}\left[\frac{u^2}{r(r-1)s^2} + \frac{2u^2}{r^2s^2}\right] \\
& = te^x + 2t, \\
v_1(x,t) & = A_x^{-1}S_t^{-1}\left[\frac{u}{s} A_x S_t\left[ u_{0x} - (u_0 - v_0) \right]\right] = -A_x^{-1}S_t^{-1}\left[\frac{u}{s} A_x S_t[e^x - 2]\right] \\
& = -te^x + 2t, \\
u_2(x,t) & = A_x^{-1}S_t^{-1}\left[\frac{u}{s} A_x S_t\left[ u_{0x} + (u_1 - v_1) \right]\right] = A_x^{-1}S_t^{-1}\left[\frac{u^3}{r(r-1)s^3}\right] \\
& = \frac{t^2}{2!}e^x, \\
v_2(x,t) & = A_x^{-1}S_t^{-1}\left[\frac{u}{s} A_x S_t\left[ u_{0x} - (u_1 - v_1) \right]\right] = -A_x^{-1}S_t^{-1}\left[\frac{u}{s} A_x S_t[-te^x]\right] \\
& = -\frac{t^2}{2!}e^x, \\
\end{align*}
\]

and so on for other components. The series solutions are given by

\[
\begin{align*}
\quad u(x,t) & = \sum_{i=0}^{\infty} u_i(x,t) = 1 + e^x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right), \\
v(x,t) & = \sum_{i=0}^{\infty} v_i(x,t) = -1 + e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right).
\end{align*}
\]

Then the solutions obtained by double Aboodh-Shehu decomposition method are given as follows:

\[
\begin{align*}
\quad u(x,t) & = 1 + e^{x+t}, \\
v(x,t) & = -1 + e^{x-t}.
\end{align*}
\]

Which is same as solution obtained by Sumudu decomposition method [6].

**Example 2.3.** Consider the system of nonlinear partial differential equations

\[
\begin{align*}
\quad u_t + vu_y + u &= 1, \\
v_t - uv_y - v &= 1, \\
\end{align*}
\]

(2.21)

with initial conditions

\[
\begin{align*}
\quad u(y,0) &= e^y, \\
v(y,0) &= e^{-y}.
\end{align*}
\]

(2.22)
Applying the double Aboodh-Shehu transform to both sides of equations (2.21), we have

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} U(r, s, u) - A[u(y, 0)] &= A_y S_y[1] - A_y S_y[uv_y + u], \\
\frac{\partial^2}{\partial t^2} V(r, s, u) - A[v(y, 0)] &= A_y S_y[1] + A_y S_y[uv_y + v].
\end{align*}
\] (2.23)

Application of single Aboodh transform to (2.22) and substitute in (2.23), we have

\[
\begin{align*}
U(r, s, u) &= \frac{u}{r(r-1)s} + \frac{u^3}{s^2 r^2} - \frac{u}{s} A_y S_y[uv_y + u], \\
V(r, s, u) &= \frac{u}{r(r+1)s} + \frac{u^3}{s^2 r^2} + \frac{u}{s} A_y S_y[uv_y + v].
\end{align*}
\] (2.24)

By taking the inverse double Aboodh-Shehu transform in (2.24), we get

\[
\begin{align*}
u(y, t) &= e^y + t - A_y^{-1} S_t^{-1} \left[ \frac{u}{s} A_y S_y[uv_y + u] \right], \\
v(y, t) &= e^{-y} + t + A_y^{-1} S_t^{-1} \left[ \frac{u}{s} A_y S_y[uv_y + v] \right].
\end{align*}
\] (2.25)

The recursive relations are

\[
\begin{align*}
u_0(y, t) &= e^y, \\
u_{i+1}(y, t) &= t - A_y^{-1} S_t^{-1} \left[ \frac{u}{s} A_y S_y \left[ \sum_{i=0}^{\infty} C_i(v, u) + \sum_{i=0}^{\infty} u_i \right] \right], \quad i \geq 0, \\
v_0(y, t) &= e^{-y}, \\
v_{i+1}(y, t) &= t + A_y^{-1} S_t^{-1} \left[ \frac{u}{s} A_y S_y \left[ \sum_{i=0}^{\infty} D_i(u, v) + \sum_{i=0}^{\infty} v_i \right] \right], \quad i \geq 0,
\end{align*}
\] (2.26)

where \( u(y, t) \) and \( v(y, t) \) are linear terms represented by the decomposition series and \( C_i(v, u) \) and \( D_i(u, v) \) are Adomian polynomials representing the nonlinear terms [15]. The few components of Adomian polynomials are given as follow

\[
\begin{align*}
C_0(v, u) &= v_0 u_0, \\
C_1(v, u) &= v_0 u_1 + v_1 u_0, \\
C_2(v, u) &= v_0 u_2 + v_1 u_1 + v_2 u_0, \\
C_3(v, u) &= v_0 u_3 + v_1 u_2 + v_2 u_1 + v_3 u_0, \\
&\vdots \\
C_i(v, u) &= \sum_{n=0}^{i} v_n u(i-n)y, \\
D_0(u, v) &= u_0 v_0, \\
D_1(u, v) &= u_0 v_1 + u_1 v_0, \\
D_2(u, v) &= u_0 v_2 + u_1 v_1 + u_2 v_0, \\
D_3(u, v) &= u_0 v_3 + u_1 v_2 + u_2 v_1 + u_3 v_0, \\
&\vdots \\
D_i(u, v) &= \sum_{n=0}^{i} u_n v(i-n)y.
\end{align*}
\]
Using the derived Adomian polynomials into (2.26), we obtain

\[ u_0(y,t) = e^y, \]
\[ v_0(y,t) = e^{-y}, \]
\[ u_1(y,t) = t - A_y^{-1}S_t^{-1}\left[\frac{u}{s}A_yS_t[C_0(v,u) + u_0]\right] = t - A_y^{-1}S_t^{-1}\left[\frac{u}{s}A_yS_t[v_0u_0y + u_0]\right] \]
\[ = t - A_y^{-1}S_t^{-1}\left[\frac{u}{s}A_yS_t[1 + e^y]\right] = t - A_y^{-1}S_t^{-1}\left[\frac{u^2}{r^2s^2} + \frac{u^2}{r(r-1)s^2}\right] \]
\[ = -te^y, \]
\[ v_1(y,t) = t + A_y^{-1}S_t^{-1}\left[\frac{u}{s}A_yS_t[D_0(u,v) + v_0]\right] = t + A_y^{-1}S_t^{-1}\left[\frac{u}{s}A_yS_t[u_0v_0y + v_0]\right] \]
\[ = t + A_y^{-1}S_t^{-1}\left[\frac{u}{s}A_yS_t[-1 + e^{-y}]\right] = t + A_y^{-1}S_t^{-1}\left[-\frac{u^3}{r^2s^2} + \frac{u^3}{r(r+1)s^2}\right] \]
\[ = te^{-y}, \]
\[ u_2(y,t) = -A_y^{-1}S_t^{-1}\left[\frac{u}{s}A_yS_t[C_1(v,u) + u_1]\right] = -A_y^{-1}S_t^{-1}\left[\frac{u}{s}A_yS_t[v_0u_1y + v_1u_0y + u_1]\right] \]
\[ = -A_y^{-1}S_t^{-1}\left[\frac{u}{s}A_yS_t[-te^y]\right] = -A_y^{-1}S_t^{-1}\left[-\frac{u^3}{r(r-1)s^3}\right] \]
\[ = \frac{t^2}{2!}e^y, \]
\[ v_2(y,t) = A_y^{-1}S_t^{-1}\left[\frac{u}{s}A_yS_t[D_1(u,v) + v_0]\right] = A_y^{-1}S_t^{-1}\left[\frac{u}{s}A_yS_t[u_0v_1y + v_1u_0y + v_1]\right] \]
\[ = A_y^{-1}S_t^{-1}\left[\frac{u}{s}A_yS_t[te^{-y}]\right] = A_y^{-1}S_t^{-1}\left[\frac{u^3}{r(r+1)s^3}\right] \]
\[ = \frac{t^2}{2!}e^{-y}. \]

In the same way we can get

\[ u_3(y,t) = -\frac{t^3}{3!}e^y, \]
\[ v_3(y,t) = \frac{t^3}{3!}e^{-y}, \]

and so on for other components. Therefore the solutions obtained by double Aboodh-Shelu decomposition method are given by

\[ u(y,t) = \sum_{i=0}^{\infty} u_i(y,t) = e^y \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right) = e^{y-t}, \quad (2.27) \]
\[ v(y,t) = \sum_{i=0}^{\infty} v_i(y,t) = e^{-y} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right) = e^{-y+t}. \quad (2.28) \]

**Example 2.4.** Consider the system of nonlinear partial differential equations

\[ u_x(x,y,t) - v_xw_t = -1, \]
\[ v_y(x,y,t) - w_xu_t = 1, \]
\[ w_y(x,y,t) - u_xv_t = -5, \quad (2.29) \]
SOLUTION OF LINEAR AND NONLINEAR SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

with initial conditions

\[ u(x, 0, t) = x + 3t, \]
\[ v(x, 0, t) = x + 3t, \]
\[ w(x, 0, t) = -x + 3t, \]  

(2.30)

Taking the double Aboodh-Shehu transform to both sides of equations (2.29), we have

\[ ru(x, r, s, u) - \frac{1}{r} S[u(x, 0, t)] = -\frac{1}{r^2} + A_y S_t[v_x w_t], \]
\[ rv(x, r, s, u) - \frac{1}{r} S[v(x, 0, t)] = \frac{1}{r^2} + A_y S_t[w_x u_t], \]
\[ rw(x, r, s, u) - \frac{1}{r} S[w(x, 0, t)] = -\frac{5}{r^2} + A_y S_t[u_x v_t]. \]  

(2.31)

Application of single Shehu transform to (2.30) then Substitute in (2.31) and rearranging the terms, we have

\[ u(x, r, s, u) = \frac{xu}{r^2 s} + \frac{3u^2}{r^2 s^2} - \frac{1}{r^2} + \frac{1}{r} A_y S_t[v_x w_t], \]
\[ v(x, r, s, u) = \frac{xu}{r^2 s} + \frac{3u^2}{r^2 s^2} + \frac{1}{r^3} + \frac{1}{r} A_y S_t[w_x u_t], \]
\[ w(x, r, s, u) = -\frac{xu}{r^2 s} + \frac{3u^2}{r^2 s^2} - \frac{5}{r^2} + \frac{1}{r} A_y S_t[u_x v_t]. \]  

(2.32)

By taking the inverse double Aboodh-Shehu transform in (2.32), we get

\[ u(x, y, t) = x + 3t - y + A_y^{-1} S_t^{-1} \left[ \frac{1}{r} A_y S_t[v_x w_t] \right], \]
\[ v(x, y, t) = x + 3t + y + A_y^{-1} S_t^{-1} \left[ \frac{1}{r} A_y S_t[w_x u_t] \right], \]
\[ w(x, y, t) = -x + 3t - 5y + A_y^{-1} S_t^{-1} \left[ \frac{1}{r} A_y S_t[u_x v_t] \right]. \]  

(2.33)

The recursive relations are

\[ u_0(x, y, t) = x - y + 3t, \]
\[ u_{i+1}(x, y, t) = A_y^{-1} S_t^{-1} \left[ \frac{1}{r} A_y S_t \left( \sum_{i=0}^{\infty} E_i(v, w) \right) \right], \quad i \geq 0, \]
\[ v_0(x, y, t) = x + y + 3t, \]
\[ v_{i+1}(x, y, t) = A_y^{-1} S_t^{-1} \left[ \frac{1}{r} A_y S_t \left( \sum_{i=0}^{\infty} F_i(w, u) \right) \right], \quad i \geq 0, \]
\[ w_0(x, y, t) = -x - 5y + 3t, \]
\[ w_{i+1}(x, y, t) = A_y^{-1} S_t^{-1} \left[ \frac{1}{r} A_y S_t \left( \sum_{i=0}^{\infty} G_i(u, v) \right) \right], \quad i \geq 0, \]  

(2.34)
where $E_i(u,v)$, $F_i(w,u)$, and $G_i(u,v)$ are Adomian polynomials representing the nonlinear terms [15] in above equations. The few components of Adomian polynomials are given as follow

\[
E_0(u,v) = v_{0x}w_{0t}, \\
E_1(u,v) = v_{1x}w_{0t} + v_{0x}w_{1t}, \\
\vdots \\
E_i(u,v) = \sum_{n=0}^{i} v_{nx}w_{(i-n)t}, \\
F_0(w,u) = w_{0x}u_{0t}, \\
F_1(w,u) = w_{1x}u_{0t} + w_{0x}u_{1t}, \\
\vdots \\
F_i(w,u) = \sum_{n=0}^{i} w_{nx}u_{(i-n)t}, \\
G_0(u,v) = u_{0x}v_{0t}, \\
G_1(u,v) = u_{1x}v_{0t} + u_{0x}v_{1t}, \\
\vdots \\
G_i(u,v) = \sum_{n=0}^{i} u_{nx}v_{(i-n)t}.
\]

In view of this recursive relations we obtained other components of the solution as follows

\[
u_1(x, y, t) = A_y^{-1}S_t^{-1} \left[ \frac{1}{r} A_y S_t [E_0(u,v)] \right] = A_y^{-1}S_t^{-1} \left[ \frac{1}{r} A_y S_t [v_{0x}w_{0t}] \right] = A_y^{-1}S_t^{-1} \left[ \frac{3}{r^3} \right] = 3y, \\
v_1(x, y, t) = A_y^{-1}S_t^{-1} \left[ \frac{1}{r} A_y S_t [F_0(w,u)] \right] = A_y^{-1}S_t^{-1} \left[ \frac{1}{r} A_y S_t [w_{0x}u_{0t}] \right] = A_y^{-1}S_t^{-1} \left[ \frac{-3}{r^3} \right] = -3y, \\
w_1(x, y, t) = A_y^{-1}S_t^{-1} \left[ \frac{1}{r} A_y S_t [G_0(u,v)] \right] = A_y^{-1}S_t^{-1} \left[ \frac{1}{r} A_y S_t [u_{0x}v_{0t}] \right] = A_y^{-1}S_t^{-1} \left[ \frac{3}{r^3} \right] = 3y, \\
u_2(x, y, t) = A_y^{-1}S_t^{-1} \left[ \frac{1}{r} A_y S_t [E_1(v,w)] \right] = A_y^{-1}S_t^{-1} \left[ \frac{1}{r} A_y S_t [v_{1x}w_{0t} + v_{0x}w_{1t}] \right] = 0, \\
v_2(x, y, t) = A_y^{-1}S_t^{-1} \left[ \frac{1}{r} A_y S_t [F_1(w,u)] \right] = A_y^{-1}S_t^{-1} \left[ \frac{1}{r} A_y S_t [w_{1x}u_{0t} + w_{0x}u_{1t}] \right] = 0, \\
w_2(x, y, t) = A_y^{-1}S_t^{-1} \left[ \frac{1}{r} A_y S_t [G_1(u,v)] \right] = A_y^{-1}S_t^{-1} \left[ \frac{1}{r} A_y S_t [u_{1x}v_{0t} + u_{0x}v_{1t}] \right] = 0.
\]

Similarly, $u_3(x, y, t) = v_3(x, y, t) = w_3(x, y, t) = 0$ and so on for rest terms.

Therefore, the solution of system (2.29) of nonlinear partial differential equations
is given below
\[ u(x, y, t) = \sum_{i=0}^{\infty} u_i(x, y, t) = x + 2y + 3t, \]
\[ v(x, y, t) = \sum_{i=0}^{\infty} v_i(x, y, t) = x - 2y + 3t, \]
\[ w(x, y, t) = \sum_{i=0}^{\infty} w_i(x, y, t) = -x - 2y + 3t. \]

3. Conclusion

In the present paper, double Aboodh-Shehu transform method combined with Adomian decomposition method which so-called the double Aboodh-Shehu decomposition method (DASDM) is applied to solve linear and nonlinear coupled partial differential equations with initial conditions. Four examples have been presented. The results of these examples tell us that both methods can be used alternatively for the solution of high-order initial value problems.

References


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