Comparison Between Two Inspiral Models

(Post-Newton and Eccentric)

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Abstract
Gravitational waves are related to black holes in the context of two-body motion. However, the two-body problems are not held in general relativity. Consequently, analytical models have been developed to describe the behavior of binary systems before they merge. During this process, gravitational waves emit as the separation between the components of the binary system decreases. These models can be categorized into two parts: models for the inspiral phase, which rely on the post-Newtonian approximation, and models for the merger phase, which are based on numerical solutions to the equations of general relativity. We implemented two distinct models for the inspiral phase to compare their agreement. We found a complete agreement in all aspects except for the timing of the detection of gravitational waves, even though both models were conducted under the same conditions.
***Keywords*: *Inspiral Models*- Post-Newton –Eccentric**

مقارنة بين نموذجي بوست نيوتن والتخالف المركزي

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المخلص
ترتبط موجات الجاذبية بالثقوب السوداء في سياق حركة الجسمين. ومع ذلك، فإن مسائل الجسمين لا تنطبق على النسبية العامة. وبالتالي، تم تطوير نماذج تحليلية لوصف سلوك الأنظمة الثنائية قبل اندماجها. خلال هذه العملية، تنبعث موجات الجاذبية مع انخفاض المسافة بين مكونات النظام الثنائي. يمكن تصنيف هذه النماذج إلى قسمين: نماذج للمرحلة الملهمة، والتي تعتمد على التقريب ما بعد النيوتوني، ونماذج لمرحلة الاندماج، والتي تعتمد على الحلول العددية لمعادلات النسبية العامة. قمنا بتنفيذ نموذجين متميزين للمرحلة الملهمة لمقارنة اتفاقهما. وقد وجدنا اتفاقاً كاملاً في جميع الجوانب باستثناء توقيت الكشف عن موجات الجاذبية، على الرغم من أن كلا النموذجين تم إجراؤهما في نفس الظروف
الكلمات المفتاحية: النموذج الحلزوني- بوسن نيوتن- التخالف المركزي.

1. Introduction

Celestial motion, a well-known phenomenon, has intrigued civilizations throughout history. Numerous attempts to explain this phenomenon have been made by various cultures. Johannes Kepler pioneered planetary motion laws, followed by Isaac Newton's theory that encompassed Kepler's laws. Newton's theory, focusing on mass, distance, time, kinetic energy, and potential energy, separated time and space [1]. It successfully explained solar planet and satellite motion. General relativity, an extension of Newton's theory, introduced space-time as a unified entity. High mass curving space-time led to gravity, forming the basis for Einstein's theory. The theory later led to the concept of gravitational waves, detected by ground-based observatories like LIGO [2]. Gravitational waves signal binary system mergers. Newton's two-body problem lacked a direct solution in general relativity, prompting approximate solutions for the inspiral, merger, and ring-down stages. Kepler's laws explained inspiral, while numerical solutions tackled the merger. Gravitational waves propagate transversely, with effects akin to light. Detected waves' effects involve space stretching and shrinking, particularly near a potential source like a binary black hole system in the Andromeda galaxy. Wave amplitude inversely correlates with source-detector distance, explaining their faintness. The wave-strain is constructed by the plus and cross polarizations, given by

$$h(t) = h(t)_+ + h(t)_\times; \qquad (1)$$

$$h_+ = -\frac{M\eta}{R}\left[p1\left[\left(-\dot{r}^2 + r^2\dot{\phi}^2 + \frac{M}{r}\right)\cos 2\phi + 2\dot{r}r\dot{\phi}\sin 2\phi\right] + p2\right] \qquad (2)$$

$$h_\times = -\frac{2M\eta}{R}\cos\theta\left(\left(-\dot{r}^2 + r^2\dot{\phi}^2 + \frac{M}{r}\right)\sin 2\phi + 2\dot{r}r\dot{\phi}\cos 2\phi\right) \qquad (3)$$

with $p1 = \cos^2\theta + 1$, $p2 = \left(-\dot{r}^2 + r^2\dot{\phi}^2 + \frac{M}{r}\right)\sin\theta$. The angle θ known as the inclination angle (Figure 1), is the angle between the plane that contains the binary system and the frame reference of detector known as fundamental frame [3, 4].

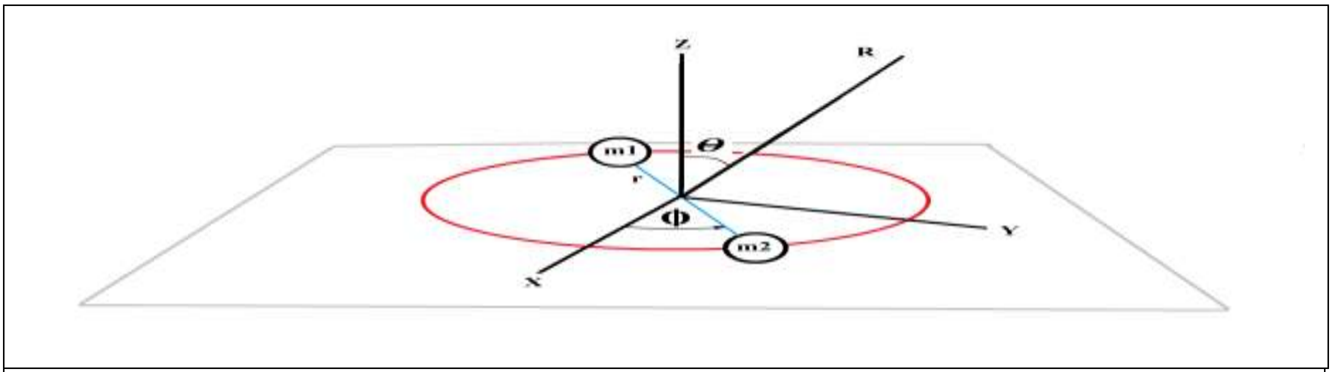

Figure 1: The illustration of the location of binary system with respect to an observer, system lies on the plane where *X*,*Y* lies, and separates its components by *r*, the observer locates at a large distance *R*, *R* >> *r*.

We set θ = 0 in order to simplify our calculation, hence the plus and cross polarizations read:

$$h_+ = -2\frac{M\eta}{R}\left[\left(-\dot{r}^2 + r^2\dot{\phi}^2 + \frac{M}{r}\right)\cos 2\phi + 2\dot{r}r\dot{\phi}\sin 2\phi\right] \qquad (4)$$

$$h_\times = -\frac{2M\eta}{R}\cos\theta\left(\left(-\dot{r}^2 + r^2\dot{\phi}^2 + \frac{M}{r}\right)\sin 2\phi + 2\dot{r}r\dot{\phi}\cos 2\phi\right) \qquad (5)$$

2. Geometrical Units and Stiff Time

The measurement in the field of black holes is performed by *Geometrical Units*, in this system, the light's speed, *c*, and gravitational constant *G* both are set to unity. In this mean $c = 2.998\times10^{8}\,\mathrm{m/s}) = 1$ lead to $1\mathrm{m} = 3.336\times10^{-9}\,\mathrm{s}$ and $\mathrm{G} = (6.667\times10^{-11}\,\mathrm{m}^3/1\mathrm{kg.s}^3) = 1$ lead to $1\mathrm{kg} = 7.42\times10^{-28}\,\mathrm{m}$. The mass of sun, Msused as a measure of black holes masses, the black hole of mass up to *M* (*M* be a dimensionless quantity) times sun mass, is simply written $\mathrm{MM_s}$, hence we see *black hole of mass M in solar units*. Regarded to equations above one can express *Ms* in terms of seconds or meters: $\mathrm{Ms} = 1.988\times10^{30}\,\mathrm{kg}$, $\mathrm{Ms} = 1476\mathrm{m}$ or $\mathrm{Ms} = 4.923\times10^{-6}\,\mathrm{s}$.

Stiffness: The stiffness in the context of differential equations is a property of the numerical solutions of differential equations (spatially by computers). A problem is stiff if it has solutions such that one varies slowly, but there exist another one which varies rapidly in the range of error tolerance. The final time of the inspiral is obtained by such property. The stiff time is chosen to be the total time $t_t = t_s$ of the evaluation of the underline binary black holes (BBHs) (inspiral, merger and ring-down). The evaluation of merger and ring-down is extend from the separation reaches innermost stable circular orbital (ISCO), to light ring (LR), hence the separation be in the range $(6M, 4M)$ this range corresponds to the time $t_l = 6M - 4M = 2M$. Thus the inspiral time is $t_i = t_s - t_l$.

3. Problem and models

The detected wave is supposed to be gravitational wave, the candidate source for wave considered to be a binary system of black holes located in Andromeda galaxy which is the nearest galaxy to our planets, where the distance is about 2.5 million light year. The models of inspiral phase that performed here are Post-Newoton approximations (PN) and eccentric anomaly model which based on PN approximations.

4. Inspiral Models

The phase of inspiral describes the evaluation of binary system by linearzed the equation of general relativity when the gravitational field being week, this is by series of approximations in power of parameter x, $x = v/c$, the rate of velocity of binary system to light speed. The validity of this phase accrue while $v << c$ [4], [5]. General relativity suggests that the orbital energy E of binary system lose gradually, which propagated as gravitational waves. Energy flux given by balance equation $F = -dE/dt$, used to represents x, that is $\dot{x} = F/dE/dx.$ the two body equation in the picture of PN approximations is read

$$\frac{dv}{dt} = -\frac{GM}{r^2}\left(-n + x^2 A1PN + x^3 A1.5PN + x^4 A2PN + x^5 A2PN + x^6 A2.5PN + ..\right)$$

where M is the total mass, r the separation, r =| x1 - x2 |, *n* = |*x*1 - *x*2|/*r*.

For qausi-circular orbit (PN approximation model), parameter x calculated by integrated the equation (see[6, 7])

$$M\frac{dx}{dt}\Big|^{6pn} = \frac{64}{5}\eta x^5\left(1 + \sum_{k=2}^{k=12} a_{\frac{k}{2}} x^{\frac{k}{2}}\right). \qquad (6)$$

Where η is the reduced mass and a_k are constants (see [3], [4], [5]).The relative phase ϕ calculated by

$$M\frac{d\phi_{orb}}{dt} = M\omega_{orb} = x^{3/2}\ (7)$$

The separation r is given by the equation

$$r(t) = M(r^{0pn}x^{-1} + r^{1pn} + r^{2pn}x + r^{3pn}x^{2})\ (8)$$

In eccentric anomaly model, the two body problem reduced to one body problem (we refer interest reader to see [8]) by Kepler's equations which describe the evolution of binary system in terms of eccentric anomaly E_a, mean anomaly M_a and mean motion n, $M_a = E_a - e\sin E_a = n(t_0 - t)$. where e is the orbit eccentricity and t the time. The model based on PN approximation, hence the parameter x has main role .The parameter x can be obtained by solving the system of differential equation below,

$$\dot{x} = M^{-1}\left(\dot{x}_{0pn}x^{5} + \dot{x}_{1pn}x^{6} + \dot{x}_{2pn}x^{7} + \dot{x}_{3pn}x^{8}\right),\ (9)$$

$$\dot{e}_t = M^{-1}\left(\dot{e}_{0pn}x^{4} + \dot{e}_{1pn}x^{5} + \dot{e}_{2pn}x^{6} + \dot{e}_{3pn}x^{7}\right),\ (10)$$

The PN corrections x_{ipn} are functions of *et* and η (see [6, 10]),and PN correction $\dot{e}_{ipn}$ of the temporal eccentricity are given in reference [9**].** Then one can implement the parameter x to calculate residual quantities (see Figure 2) in order to calculate the relative phase and separation

$$r = M(r^{0pn}x^{-1} + r^{1pn} + r^{2pn}x + r^{3pn}x^{3}),\quad (11)$$

$$\phi = M^{-1}(\phi^{0pn}x^{3/2} + \phi^{1pn}x^{5/2} + \phi^{2pn}x^{7/2} + \phi^{3pn}x^{9/2}),\ (12)$$

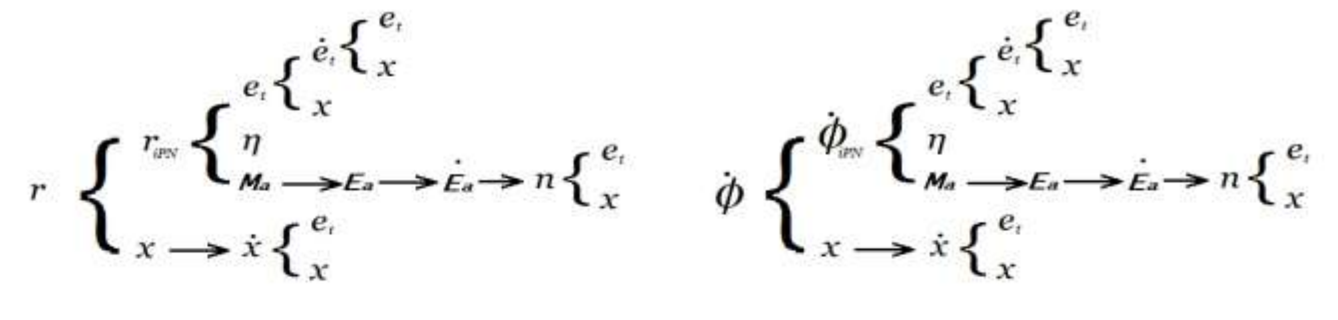

Figure 2:Diagram represents the eccentric model.

5. Results

The models below are performed under the consideration that the binary system of non-spinning black holes of M = 40 in solar unit as a total mass, time is in seconds, distance R to the binary is the distance between close galaxy to our planet, Andromeda, where R about 2.5 million light years.

The environment of implementation inspiral's models comes by their theoretical idea and initial data, we then at the stage of setting initial values, hence getting the results. Following the decencies in the context, the initial time is zero ($t_0 = 0$). For PN model, the differential equation (6) is the key of solution and needs lower and upper boundaries. In eccentric model, we start by solving the system consisting of two differential equations (9) , (10) hence they will need lower and upper boundaries. The common initial data for both is the value of x at t = 0, (i.e. $x_0 = x(0)$) which corresponds to the value of the frequency that detected by the ground-base LIGO, it is $f_0 = 10Hz$, therefore we can obtain x_0 in terms of ω_0 by Kepler's third law, the upper boundary for both models can be obtained after determining final time. For eccentric model, the another initial data for the system (9), (10) is the eccentricity, $e_0 = e(0)$, we set its value to ($e_0 = 10\times10^{-7}$) in order to investigates the evaluation of inspiral by these two models. Now we proceed to perform our model, eccentric model, at first the initial values of x and e that are $x_0 = 0.033699744$, $e_0 = 10\times10^{-7}$. We solve the system consisting of two differential equations (mention above), by Matlab. We employ the routine *ode15s* with relative error tolerance 10^{-8}, the method breakdown at the stiff time, we then can determine the final time tf obtaining the values of $x(t)$ and $e(t)$. We plot the evaluation of eccentricity in Figure (3),

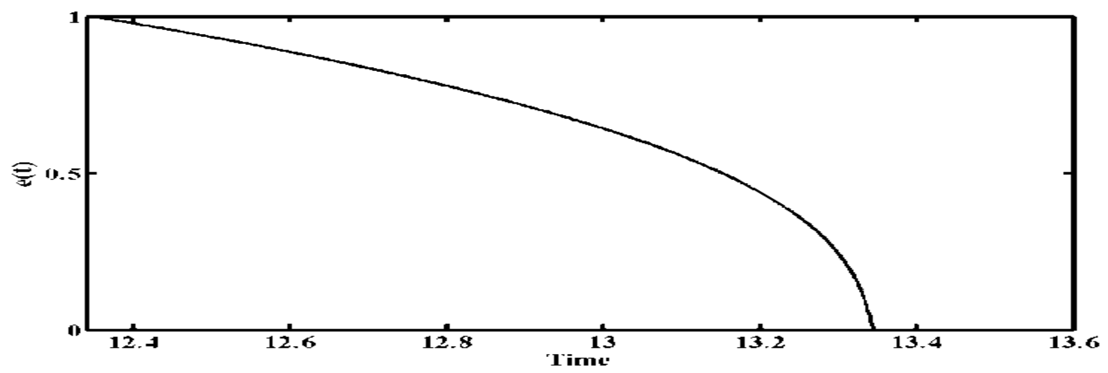

Figure 3: The evaluation of normalized eccentricity *e*, for a binary system of total mass 40*Ms*,the time is in second.

Figure 3 shows that the eccentricity tens to zero at late time of this stage, such behavior regarded to be a measurement for model performance in correct manner. After getting *e*, we were ready to calculate the mean anomaly, we begin by calculating the corrections *nipn*, then integrating equation $\dot{M}_a = n$. For eccentric anomaly E_a we follow the diagram represented in Figure (2) and relevant relation (see [3,5]). Then, we employ η and E_a for calculate r_{inp} and $\emptyset_{inp}$ in order to obtain the separation (equation 11) and its derivative in a numerical manner, and the derivative of phase (equation 12), hence, the phase by performing a numerical integration of equation. In Figure (4) we plot the evaluation of x and r. Now we have all what we want for calculating the strain of GWs by equations (4 and 5). Because our calculations performed in a second (in this case). In Figure (5), the plot of the real part of h and its amplitude, is about 5.4×10^{-19} , (both are in normal representation). We conclude that the velocity of binary BHs reaches 0.45 of light speed at the end of this stage, and the gravitational waves of this binary have a life expected to be 13.5 seconds. Further, we find that the maximum frequency is up to 190*Hz* token at 13.346455*sec* which corresponds to r = 3*M*.

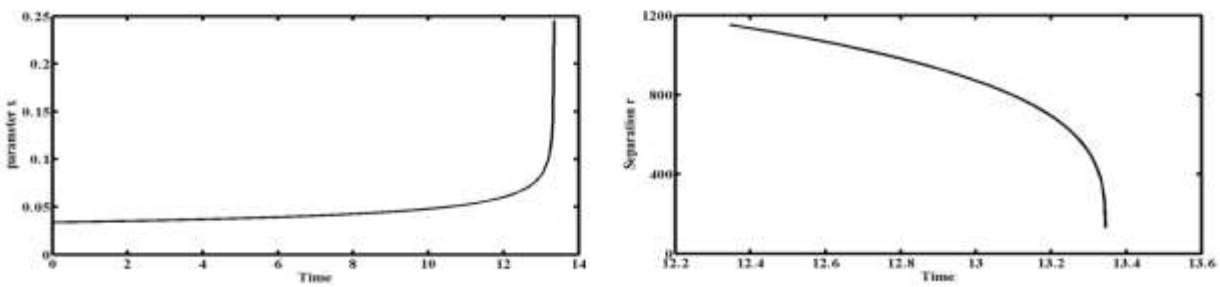

Figure 4: The evaluation of parameter *x* (left), and separation *r* (right), for a binary system of total mass 40*Ms*.. Time in seconds.

Now, we present a comparison of our method versus the PN model in order to investigate their rapprochement. We perform PN model under the same conditions and find that the final time in this model is 11.9244 seconds which defers from our result. Hence the figures below plotted under shifting in time. In Figure (6), the plot is of real strain of two models. The figure shows that the two models are in agreement.

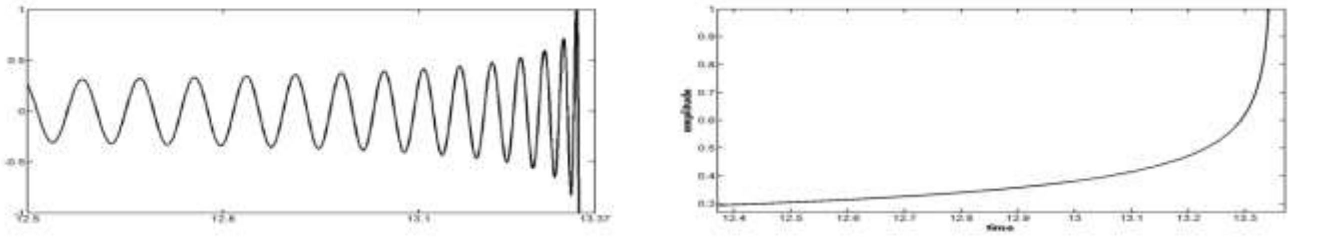

Figure 5: The real strain of GWs (left), and its amplitude (right), for binary system of total mass 40*Ms.* The amplitude is taken by both real and imaginary parts of strain, and plotted in a normal representation, it is proportional to the inverse of the distance between the wave source and detector location.

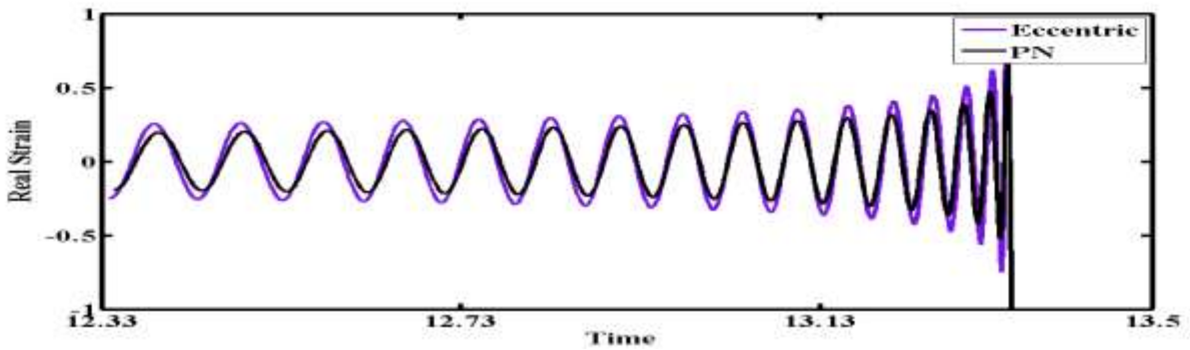

Figure 6: Real strain of GWs plotted by eccentric and PN models. Strainrepresented in last second by both models, a shifting in time have been madeto bring them at phase**.**

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