Hub-integrity graph of graphs

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***ABSTRACT*.**
The hub-integrity graph $H_I(G)$ of a graph G is a graph with $V(H_I(G)) = V' \cup S$, where S is the set of all nonempty HI-sets of G and V' is the set of all vertices belonging to S, and with two vertices $u, v \in V(H_I(G))$ adjacent if $u \in V'$ and v isHI-set of G containing u. In this paper, we initiate the study of this new graph valued function, and characterizations are given for graphs whose hub-integrity graphs are trees. Also some properties of this graph are established.

الملخص
مخطط كفاءة المحور للمخطط G هو المخطط ذات الرؤوس $V(H_I(G)) = V' \cup S$ حيث S مجموعة كل المجموعات غير الخالية HI-sets و V' مجموعة كل الرؤوس المنتمية الى S, ولأي رأسين $u, v \in V(H_I(G))$ متجاورة اذا كان $u \in V'$ و v هومجموعة كل HI-set التي تحوي الرأس u . في هذا البحث انشأنا دراسة لمخطط جديد لدوال القيمة, وعممنا المخططات التي مخطط كفاءة المحور لها عباره عن مخطط أشجار, أيضا ناقشنا بعض الخصائص لهذا المخطط.

Keywords: *Integrity, Hub set, Hub-integrity, HI-set of a graph G.*

1 Introduction

Throughout this paper we consider a finite, undirected graph with neither loops nor multiple edges. For a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. We use p to denote the number of vertices and q to denote the number of edges of a graph G. The distance between the vertices v_i and v_j is the length of the shortest path joining v_i and v_j. The shortest $v_i v_j$ path is often called a geodesic. The diameter of a connected graph G is the length of any longest geodesic, denoted by $diam(G)$. The reader follow Harary (1969), for graph-theoretical terminology and notation not defined here. The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex set, two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The line graph $L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. The girth of a graph is the length of a shortest cycle contained in the graph [Harary (1969)]. The symbols $\alpha(G)$, and $\beta(G)$ denote the vertex cover number, and the independence number of G, respectively. A gear graph G_p is obtained from the wheel $W_{1,p-1}$ by adding a vertex between every pair of adjacent vertices of the cycle C_{p-1}. A firefly graph $F_{s,t,p-2s-2t-1} (s \geq 0, t \geq 0 \; and \; p - 2s - 2t - 1 \geq 0)$ is a graph of order p that consists of s triangles, t pendant paths of length 2 and $p - 2s -$

$2t-1$ pendant edges sharing a common vertex [Li et al. (2013)]. The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph G obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2 [Frucht et al. (1970)].

Networks appear in many different applications and settings. The most common networks are telecommunication networks, computer networks, the internet, road and rail networks and other logistic networks. In all applications, vulnerability and reliability are crucial and have important features. Network designers often build a network configuration around specific processing, performance and cost requirements. But there is little consideration given to the stability of the networks communication structure when under the pressure of link or node loses. This lack of consideration makes the networks have low survivability. Therefore network design process must identify the critical points of failure and be able to modify the design to eliminate them [Newport et al. (1991)].

The stability of a communication network composed of processing nodes and communication links is of prime importance to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. In an analysis of the stability of a communication network to disruption, two questions that come to mind are: (i) How many vertices can still communicate? (ii) How difficult is it to reconnect the graph? The concept of integrity was introduced as a measure of graph stability by Barefoot et al. (1987). Formally, the integrity is $I(G) = min_{S\subset V}\{|S| + m(G-S)\}$, where $m(G-S)$ denotes the order of a largest component of $G-S$. The integrity is a measure which deals with the first question stated above, namely how many vertices can still communicate? If the set S achieves the integrity, then it is called an I-set of G. That is, if $|S| + m(G-S) = I(G)$ for any set S, then S is called an I-set. For more details on the integrity see [Bagga et al. (1992), Barefoot et al. (1987), Goddard et al. (1990), Mathad et al. (2015), Mahde et al. (2017), Mahde et al. (2020)].

Suppose that $H \subseteq V(G)$ and let $x, y \in V(G)$. An H-path between x and y is a path where all intermediate vertices are from H. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$, call such an H-path trivial). A set $H \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) - H$, there is an H-path in G between x and y. The smallest size of a hub set in G is called a hub number of G, and is denoted by $h(G)$ [Walsh (2006)]. For more details on the hub parameters see [Khalaf et al. (2018), Khalaf et al. (2018), Khalaf et al. (2019), Khalaf et al. (2020)]. A set $S \subseteq V(G)$ is called a dominating set of G if each vertex of $V-S$ is adjacent to at least one vertex of S. The domination number of a graph G, denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G [Haynes et al. (1998)].

Mahde et al. (2015) have introduced the concept of hub-integrity of a graph as a new measure of vulnerability which is defined as follows.

Definition 1.1 *[*Mahde et al. (2015)*]*
The hub-integrity of a graph G denoted by $HI(G)$ is defined by, $HI(G) = min\{|S| + m(G-S), S \text{ is a hub set of } G\}$, where $m(G-S)$ is the order of a maximum component of $G-S$.

For more details on hub-integrity of graphs see [Mahde et al. (2015), Mahde et al. (2016), Mahde et al. (2017), Mahde et al. (2018)].

Definition 1.2 *A subset S of $V(G)$ is said to be a HI-set, if $HI(G) = |S| + m(G-S)$.*

We use the following result for our later results.

Theorem 1.1 [Harary *(1969)] A graph G is bipartite if and only if all its cycles are even.*

By using the concepts of hub set and hub-integrity with HI-sets, we have integrated these concepts to get a new concept. Motivated by this, we introduce hub-integrity graph of a graph as a new graph valued function in the field of hub set in graphs defined as follows.

2. Hub-integrity graph of graphs

Definition 2.1 *The hub-integrity graph $H_I(G)$ of a graph G is a graph with $V(H_I(G)) = V' \cup S$, where S is the set of all nonempty HI-sets of G and V' is the set of all vertices belonging to S, and with two vertices $u, v \in V(H_I(G))$ adjacent if $u \in V'$ and v isHI-set of G containing u.*

In Figure 1, we show a graph G and its hub-integrity graph $H_I(G)$. We have $S_1 = \{1,3\}, S_2 = \{1,4\}, S_3 = \{2,4\}, S_4 = \{2,5\}, S_5 = \{3,5\}, S_6 = \{2,3,5\}, S_7 = \{1,2,4\}, S_8 = \{1,3,4\}, S_9 = \{2,4,5\}$ and $S_{10} = \{1,3,5\}$ are HI-sets of G.

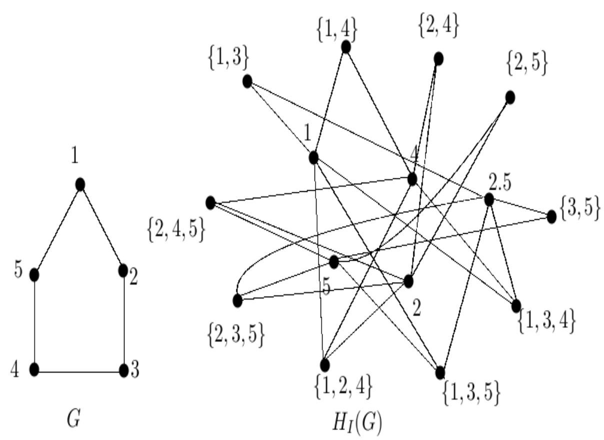

Figure 1: G and $H_I(G)$

The following theorem determiners the number of vertices and edges in hub-integrity graph of a graph.

Theorem 2.1 *For any (p, q) graph G,*
$|V(H_I(G))| = |S| + |\cup S_i|$, where $S_i \in S$ and

$$|E(H_I(G))| = \sum_{S_i \in S} |S_i|.$$

Remark 2.1 *For any graph G, $H_I(G)$ is bipartite graph, and no two vertices of V' and S are adjacent vertices in $H_I(G)$.*

In the following theorem we establish a necessary and sufficient conditions for the graphs whose hub-integrity graphs are trees.

Theorem 2.2 *For any graph $G \neq K_{n,n}$, the hub integrity graph $H_I(G)$ of a graph G is a tree if and only if for every two vertices u and v, the following two conditions hold:*

(1) If there is a HI-set of G containing u and v, then this set is unique.

(2) If the vertices u and v are in different HI-sets, then there is no HI-set containing both u and v.

Proof. Suppose the conditions (1) and (2) hold. We must to prove that any two vertices u and v of $H_I(G)$ are connected by exactly one path. For contrary, suppose there are two vertices u and v joined by at least two paths, thus there exist three HI-sets H_1, H_2 and H_3 such that $u \in H_1, v \in H_2$ and $u, v \in H_3$ or there are two HI-sets at least containing u and v, this is contrary to the conditions (1) and (2). For this reason, every pair of vertices in $H_I(G)$ is uniquely connected. Then $H_I(G)$ is a tree.
Conversely, suppose $H_I(G)$ is a tree, and condition (1) does not hold. Then there are at least two HI-sets containing u and v and hence there is a cycle of length four in $H_I(G)$, a contradiction. Now, assume that for vertices u and v, for contrary, suppose condition (2) does not hold. Then there are two vertices u' and v', also there are three HI-sets H_1, H_2 and H_3 such that $u, u' \in H_1$, $v, v' \in H_2$ and $u, v \in H_3$. Then u' and v' are joined by at least two paths in $H_I(G)$, a contradiction. This completes the proof.

We now present a characterization of graphs whose hub-integrity graphs are K_p.

Theorem 2.3 *The hub-integrity graph $H_I(G)$ of a graph G is a complete graph K_2 if and only if $G \cong F_{s,0,p-2s-1}, s \geq 2$.*

Proof. Suppose $H_I(G)$ is a complete graph K_2. Since no two vertices in $H_I(G)$ corresponding to HI-sets of G are adjacent, the HI-sets contain one vertex as hub set of $H_I(G)$, so the graph that has minimum hub set of one vertex only is $F_{s,0,p-2s-1}, s \geq 2$. Converse is clear because $h(F_{s,0,p-2s-1}) = 1$ and HI-set of $F_{s,0,p-2s-1}$ contains only one vertex, hence the result.

The following theorem characterizes graphs whose $H_I(G)$ graphs are connected.

Theorem 2.4 *For any graph $G \neq K_{n,n}$, $H_I(G)$ is a connected.*

Proof. Since G has either one or at least two HI-sets such that there exists at least one vertex belonging to two sets, $H_I(G)$ is connected. But the graph $K_{n,n}$ contains two disjoint HI-sets. Therefore $H_I(K_{n,n})$ is disconnected.

By Theorem 2.4 we have the following lemma.

Lemma 2.1 *For any graph G, $H_I(G)$ is disconnected if and only if $G \cong K_{n,n}$.*

Depending upon the cardinality of the set S, we characterize the graphs G for which $H_I(G)$ is a tree.

Proposition 2.1 *For any graph G,*

1. $|S| = 1$ if and only if $H_I(G) \cong K_{1,p-1}$.
2. If $|S| = 2$, then

$$H_I(G) \cong \begin{cases} disconnected, & if\ S_1 \cap S_2 = \phi\ ; \\ tree, & if\ |S_1 \cap S_2| = 1. \end{cases}$$

Proof. Suppose that $|S| = 1$ and $S = \{S_1\}$, depending on the number of elements in set S_1, we have the following cases:

Case 1: If S_1 has one element, then by definition of $HI(G)$, S_1 is adjacent to this element. So we get $H_I(G) \cong P_2 = K_{1,1}$.

Case 2: If S_1 has two elements, then S_1 is adjacent to these elements, thus the resulting graph is $K_{1,2}$. In general, if S_1 has $p-1$ elements, the resulting graph is $K_{1,p-1}$.

Conversely, $H_I(G) \cong K_{1,p-1}$ is considered, let $V(K_{1,p-1}) = \{v, v_1, v_2, v_3, \ldots, v_{p-1}\}$, with v as center vertex. By definition of $H_I(G)$, all HI-sets of G form an independent set in $H_I(G)$. Consider an HI-set S_i of G such that $S_i = v$ in $K_{1,p-1}$. Then $v_1, v_2, \ldots, v_{p-1} \in S_i$ and hence $|S| = 1$. Now, if $S_i \neq v$ in $K_{1,p-1}$ then $S_i = v_i$ for some $i, 1 \leq i \leq p-1$. Then $S_i = v_i$, $1 \leq i \leq p-1$ are HI-sets each contains on element v. So, $|S| = p-1$. Since $vv_i \in E(K_{1,p-1}), v \in S_i, 1 \leq i \leq p-1$, a contradiction to the fact that HI-sets form an independent sets in $H_I(G)$. So $|S| = 1$.

Observation 2.1 *$H_I(G) \cong K_{1,p-1}$ if $G \cong F_{s,0,p-2s-1}, s \geq 2, \overline{K_p}, S_{n,m}, C_4 + e, (C_4 + e) \cup \overline{K_p}, K_2 \circ nK_2, n \geq 1, \overline{K_p} \cup pK_{1,p-1}, \overline{K_p} \cup F_{s,0,p-2s-1}, s \geq 2, G \cup K_{1,p-1}: |V(G)| < |V(K_{1,p-1})|, C_p \circ K_1, L(C_p \circ K_1)$, and the following graphs as in the figure 2.*

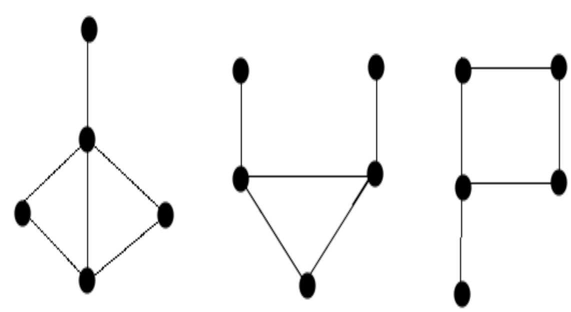

Figure 2

Theorem 2.5 *For any graph G, $\gamma(H_I(G)) \leq |S|$, where $|S|$ is the number of HI-sets of G.*

Proof. The following cases are considered.

Case 1: $|S| = 1$, the vertex in $H_I(G)$ corresponding to the element of S is adjacent to all remaining vertices, so $\gamma(H_I(G)) = 1$.

Case 2: $|S| \geq 2$, from definition of $H_I(G)$, no two vertices in $H_I(G)$ corresponding to elements of $V(G)$ are adjacent and no two vertices corresponding to HI-sets are adjacent. Since all vertices in $H_I(G)$ corresponding to HI-sets are adjacent to all vertices which belong to HI-sets, the HI-sets are dominating sets of $H_I(G)$. Then $\gamma(H_I(G)) \leq |S|$.

In the next proposition, we characterize the graphs G for which $\overline{H_I(G)}$is totally disconnected and $diam(H_I(G)) = 1$.

Proposition 2.2

1. For any graph G, $\overline{H_I(G)}$is totally disconnected graph if and only if $G \cong F_{s,0,p-2s-1}, s \geq 2$.
2. For any graph G, $diam(H_I(G)) = 1$ if and only if $G \cong F_{s,0,p-2s-1}, s \geq 2$.

Proof. (1) Since $\overline{H_I(G)}$is totally disconnected, $H_I(G)$ is complete graph and hence $H_I(G) \cong K_2$, by Theorem 2.3, $G \cong F_{s,0,p-2s-1}, s \geq 2$. Converse is obvious.

(2) Since $diam(H_I(G)) = 1$, $H_I(G)$ is complete graph, so $H_I(G) \cong K_2$, and by Theorem 2.3 G is $F_{s,0,p-2s-1}, s \geq 2$. Converse is clear.

Theorem 2.6 *$H_I(G)$ triangle-free for any graph G.*

Proof. By Remark 2.1, $H_I(G)$ is bipartite and Theorem 1.1 completes the proof.

In the next result, we obtain bounds on the domination number of $H_I(G)$.

Proposition 2.3 *For any graph G, $1 \leq \gamma(H_I(G)) \leq p$.*

Proof. If $|S| = 1$, then clearly $\gamma(H_I(G)) = 1$, and the upper bound is satisfied if $G \cong K_p$, because we have p HI-sets consisting of one vertex, so we must choose all vertices in $H_I(G)$ that are adjacent to vertices corresponding to these sets.

Proposition 2.4 *For any graph G,*

$$\beta(H_I(G)) = \begin{cases} |S|, & if \ |S| > |V'|; \\ |V'|, & if \ |S| \leq |V'|. \end{cases}$$

Proof. Let M denote the set of vertices of $H_I(G)$, which correspond to the HI-sets of G and let N denote the set of all vertices which belong to element of M, which correspond to vertices of G. It is clear from definition of $H_I(G)$ that the set M is independent, also the set N is independent in $H_I(G)$. Now depending on the number of vertices in $H_I(G)$, we have two cases.

Case 1: $|M| > |N|$, we have to show that the set M is maximal independent set of $H_I(G)$. Considering the existence of maximal independent set M' containing M, M' contains other vertices of N. Then there exists at least one vertex of M adjacent to one vertex of $H_I(G)$ from the set N, so M' is not maximal independent. In addition to that $|M| > |N|$. Thus, $\beta(H_I(G)) = |S|$.

Case 2: $|N| > |M|$, the proof is similar to Case 1, hence $\beta(H_I(G)) = |V'|$.

Now we determine vertex cover number α of the hub-integrity graph.

Proposition 2.5 *For any graph G,*

$$\alpha(H_I(G)) = \begin{cases} |S|, & if \ |S| \leq |V'|; \\ |V'|, & if \ |S| > |V'|. \end{cases}$$

Proof. By Remark 2.1, S and V' are independent sets of $H_I(G)$ and every vertex of V' is adjacent with some vertices of S. Now, if $|S| > |V'|$, the vertices of V' cover all the vertices of $H_I(G)$. Otherwise if $|S| \leq |V'|$, the vertices of S cover all vertices of $H_I(G)$, this completes the proof.

We now present a characterization of graph whose $H_I(G)$ has domination number 1.

Theorem 2.7 *For any graph* G, $\gamma(H_I(G)) = 1$ *if and only if* $|S| = 1$.

Proof. Consider $|S| = 1$, then by Observation 2.1, $H_I(G) \cong K_{1,p-1}$, so $\gamma(H_I(G)) = 1$.
Now if $\gamma(H_I(G)) = 1$, then there exists at least one vertex v adjacent to all vertices of $H_I(G)$, so the following cases are considered. Suppose v *is* S and by definition $H_I(G)$, it follows that $|S| = 1$. if $v \in S$, then there exists at least one set S such that v is adjacent to it and if $|S| = 1$ this is required, and if there exist two sets S_1 and S_2 and $v \in S_1 \cup S_2$, there exists a path P_4 and this case $\gamma(H_I(G)) \neq 1$. Hence $|S| = 1$.

Observation 2.2 $H_I(G) \cong G_p$ *if* $G \cong W_{1,5}, W_{1,7}$.

Theorem 2.8 *For any graph* G, $\gamma(\overline{H_I(G)}) = 2$.

Proof. Choose a vertex $v \in V$ and S be HI-set of G such that $v \in S$. It is clear, both v and S form a minimal dominating set of $\overline{H_I(G)}$,since v and S are adjacent in $H_I(G)$. Then $\gamma\left(\overline{H_I(G)}\right) = 2$.

Finally, we calculate the girth of $H_I(G)$.

Theorem 2.9 *For any graph* $G \neq K_{n,n}$, *if* $H_I(G)$ *is not tree, then* $g(H_I(G)) = 4$.

Proof. Let $S_1, S_2 \in V(H_I(G))$ such that S_1, S_2 are vertices of $H_I(G)$ corresponding to HI-sets of G. Since $H_I(G)$ is not tree, there exist at least two vertices $u, v \in S_1 \cap S_2$ and since u and v are adjacent with both S_1 and S_2, it follows that the vertices S_1, S_2, u, v form a cycle C_4 in $H_I(G)$. Since, by Theorem 2.6, $H_I(G)$ is triangle-free we have, $g(H_I(G)) = 4$.

3. Conclusion

Hub concept and integrity are very valuable measures for network designers. Integrity gives information about the stability of graph model of network. Our new concept hub- integrity graph combines these two important concepts.
In this paper, we introduced the concept of the hub –integrity graph of graphs. Some properties of the $H_I(G)$ and $\overline{H_I(G)}$are established. For future work we plan to extend our results on other important classes of graphs under some operation and generalize the obtained results.

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