Albaydha University Journal – Vol.(5)-Issue(4) 2023(special Issue of the Fourth Scientific Conference of Albaydha University)

Application of Adomian Decomposition Method for Solving Helmholtz Equation

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DOI: https://doi.org/10.56807/buj.v5i4.473

Abstract
The Adomian decomposition method (ADM) is effective for solving linear and nonlinear ordinary and partial differential equations, integral equations, algebraic equations and integro-differential equations and it leads to an analytical solution in the form of an infinite power series. We will solve the Helmholtz equation in two dimensional by the Adomian decomposition method.
Key words: *Application of Adomian Decomposition Method, Helmholtz Equation, Adomian Decomposition Method.*

I. Introduction

Various phenomenon of mathematical biology, engineering and physical problems can be described by linear or nonlinear parabolic equations. These linear or nonlinear models, as well as their analytic solutions, are of fundamental importance for applied sciences [2].

The Helmholtz equation is an important equation to be solved in many numerical problems such as waveguide related problems and appears in a variety of engineering applications and physical phenomena such as heat conduction, acoustic radiation, and water wave propagation [5]. In scattering applications and direct acoustic propagation the Helmholtz equation is a model, in the frequency domain, of the sound propagation and backscattering caused, usually, by a point source which emits a continuous time-harmonic signal [4]. The Adomain decomposition method is very actual and can easily be used to manipulate numerous linear or nonlinear, ordinary or partial differential equations and linear and nonlinear integrals equations [1].

This method was introduced and developed from the 1970s to the 1990s by George Adomian [1]. In recent years, the Adomain decomposition method has been receiving a lot of attention because of its many applications in the area of applied mathematics, biology, engineering and physics.

Definition 1.1.

The two-dimensional Helmholtz equation for smooth function Ψ in a 2-dimensinal real Euclidean space has the form

$$\Delta\Psi(\xi,\eta)+\lambda^2\,\Psi(\xi,\eta)=\Phi(\xi,\eta), \qquad (1)$$

where λ is a constant and $\Phi(\xi,\eta)$ is unknown function and Δ is the Laplace operator [3],[7].

Remark 1.2 .

The general form of (1) includes, as specializations, the following cases:

1) Laplace equation, with $\lambda=0$ and $\Phi=0$.

2) Poisson equation, with $\lambda=0$ and $\Phi\neq 0$.

2. Basic ideas of the Adomian decomposition method

The Adomian decomposition method consists of decomposing the unknown function $\Psi(\xi,\eta)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$\Psi(\xi,\eta)=\sum_{n=0}^{\infty}\Psi_n(\xi,\eta), \qquad (2)$$

where the components $\Psi_n(\xi,\eta),\ n\geq 0$ are to be determined in a recursive method.

The decomposition method concerns itself with finding the components $\Psi_0,\Psi_1,\Psi_2,\ldots$ individually [6].

For the goal of applications, illustration of the methodology of the suggested method using Adomian decomposition method, we consider the Helmholtz equation in two dimensional ξ ,η

$$\Delta\Psi(\xi,\eta)+\lambda^2\,\Psi(\xi,\eta)=\Phi(\xi,\eta), \qquad (3)$$

with the conditions

$$\Psi(\xi,0)=P(\xi), \qquad (4)$$
$$\Psi_\eta(\xi,a)=Q(\xi),$$

where $\Psi(\xi,\eta)$ is a solution of the Helmholtz equation.

We first write (3) an operator form by

$$L_\eta \Psi(\xi,\eta) = \Phi(\xi,\eta) - L_\xi \Psi(\xi,\eta) - \lambda^2 \Psi(\xi,\eta), \qquad (5)$$

where the differential operators L_ξ and L_η are defined by

$$L_\xi = \frac{\partial^2}{\partial \xi^2}, \qquad L_\eta = \frac{\partial^2}{\partial \eta^2}.$$

So that the inverse operators L_ξ^{-1} and L_η^{-1} are two fold integral operator defined by

$$L_\xi^{-1}(\cdot) = \int_0^\xi \int_0^\xi (\cdot)\, d\xi\, d\xi,$$

$$L_\eta^{-1}(\cdot) = \int_0^\eta \int_0^\eta (\cdot)\, d\eta\, d\eta.$$

Applying the inverse operator L_η^{-1} to both sides of (5) and using the boundary conditions, we get

$$L_\eta^{-1}[L_\eta \Psi(\xi,\eta)] = L_\eta^{-1}[\Phi(\xi,\eta) - L_\xi \Psi(\xi,\eta) - \lambda^2 \Psi(\xi,\eta)]. \qquad (6)$$

Since λ^2 is a constant, it can be factored out and the above equation can be written as:

$$\Psi(\xi,\eta) = P(\xi) + Q(\xi)\eta + L_\eta^{-1}[\Phi(\xi,\eta)] - L_\eta^{-1}[L_\xi \Psi(\xi,\eta)] - L_\eta^{-1}\left[\lambda^2 \Psi(\xi,\eta)\right]$$
$$= P(\xi) + Q(\xi)\eta + L_\eta^{-1}[\Phi(\xi,\eta)] - L_\eta^{-1}[L_\xi \Psi(\xi,\eta)] - \lambda^2 L_\eta^{-1}[\Psi(\xi,\eta)], \qquad (7)$$

where unknown function Ψ is decomposed into a sum of components defined by the decomposition series:

$$\Psi(\xi,\eta) = \sum_{n=0}^{\infty} \Psi_n(\xi,\eta),$$

with Ψ_0 identified as $\Psi(\xi,0)$ and the components $\Psi_n(\xi,\eta)$ obtained from the recurrently formula:

$$\Psi_0(\xi,\eta) = P(\xi) + Q(\xi)\eta + L_\eta^{-1}[\Phi(\xi,\eta)], \qquad (8)$$
$$\Psi_{n+1}(\xi,\eta) = -L_\eta^{-1}[L_\xi \Psi_n(\xi,\eta)] - L_\eta^{-1}[\lambda^2 \Psi_n(\xi,\eta)], \qquad n \geq 0,$$

i.e.,

$$\Psi_0(\xi,\eta) = P(\xi) + Q(\xi)\eta + L_\eta^{-1}[\Phi(\xi,\eta)],$$
$$\Psi_1(\xi,\eta) = -L_\eta^{-1}[L_\xi(\Psi_0(\xi,\eta))] - L_\eta^{-1}[\lambda^2 \Psi_0(\xi,\eta)], \qquad (9)$$
$$\Psi_2(\xi,\eta) = -L_\eta^{-1}[L_\xi(\Psi_1(\xi,\eta))] - L_\eta^{-1}[\lambda^2 \Psi_1(\xi,\eta)],$$
$$\vdots$$
$$\Psi_{n+1}(\xi,\eta) = -L_\eta^{-1}[L_\xi \Psi_n(\xi,\eta)] - L_\eta^{-1}[\lambda^2 \Psi_n(\xi,\eta)].$$

Therefore, the Adomian decomposition method is concerned with finding the components $\Psi_0, \Psi_1, \ldots$, individually and summing them up to obtain the solution $\Psi(\xi,\eta)$.

3. Illustrative examples

In this section, we solve Helmholtz equation with boundary conditions by Adomian decomposition method

Example 3.1.

Consider the Helmholtz equation

$$\Delta\Psi(\xi,\eta) + \Psi(\xi,\eta) = \xi\eta, \qquad (10)$$

with the conditions

$$\Psi(\xi,0) = 0, \qquad \Psi_\eta(\xi,0) = \xi \qquad (11)$$

Solution.

We first rewrite (10) in an operator form by

$$L_\eta \Psi(\xi,\eta) = \xi\eta - L_\xi \Psi(\xi,\eta) - \Psi(\xi,\eta), \qquad (12)$$

where the differential operators L_ξ and L_η are defined by

$$L_\xi = \frac{\partial^2}{\partial \xi^2}, \qquad L_\eta = \frac{\partial^2}{\partial \eta^2},$$

The inverse operator L_ξ^{-1} and L_η^{-1} are two-fold integral operators defined by

$$L_\xi^{-1}(\cdot) = \int_0^\xi \int_0^\xi (\cdot)d\xi d\xi,$$

$$L_\eta^{-1}(\cdot) = \int_0^\eta \int_0^\eta (\cdot)d\eta d\eta.$$

Applying the inverse operator L_η^{-1} to both sides of (12) and using the boundary conditions we get

$$L_\eta^{-1} L_\eta \Psi(\xi,\eta) = \xi\eta + \frac{\xi}{3!}\eta^3 - L_\eta^{-1}\left[L_\xi\big(\Psi(\xi,\eta)\big)\right] - L_\eta^{-1}[\Psi(\xi,\eta)]. \qquad (13)$$

Using both decomposition series

$$\Psi(\xi,\eta) = \sum_{n=0}^{\infty} \Psi_n(\xi,\eta), \qquad (14)$$

into both sides of (7) gets

$$\sum_{n=0}^{\infty} \Psi_n(\xi,\eta) = \xi\eta + \frac{\xi}{3!}\eta^3 - L_\eta^{-1}\left[L_\xi\left(\sum_{n=0}^{\infty} \Psi_n(\xi,\eta)\right)\right] - L_\eta^{-1}\left[\sum_{n=0}^{\infty} \Psi_n(\xi,\eta)\right]. \qquad (15)$$

Adomian's analysis admits the use of the recursive relation

$$\Psi_0(\xi,\eta) = \xi\eta + \frac{\xi}{3!}\eta^3,$$

$$\Psi_{n+1}(\xi,\eta) = -L_\eta^{-1}\left[L_\xi\big(\Psi_n(\xi,\eta)\big)\right] - L_\eta^{-1}[\Psi_n(\xi,\eta)]\,. \qquad (16)$$

Hence, we find

$$\Psi_0(\xi,\eta) = \xi\eta + \frac{\xi}{3!}\eta^3,$$

$$\Psi_1(\xi,\eta) = -L_\eta^{-1}\left[L_\xi\big(\Psi_0(\xi,\eta)\big)\right] - L_\eta^{-1}[\Psi_0(\xi,\eta)] = -\frac{1}{3!}\xi\eta^3 - \frac{1}{5!}\xi\eta^5,. \qquad (17)$$

$$\Psi_2(\xi,\eta) = -L_\eta^{-1}\left[L_\xi\big(\Psi_1(\xi,\eta)\big)\right] - L_\eta^{-1}[\Psi_1(\xi,\eta)] = \frac{1}{5!}\xi\eta^5 + \frac{1}{7!}\xi\eta^7,$$

and so on. The solution $\Psi(\xi,\eta)$ in a series form is given by

$$\Psi(\xi,\eta) = \Psi_0(\xi,\eta) + \Psi_1(\xi,\eta) + \Psi_2(\xi,\eta) + \cdots$$

$$= \xi\eta + \frac{1}{3!}\xi\eta^3 - \frac{1}{3!}\xi\eta^3 - \frac{1}{5!}\xi\eta^5 + \frac{1}{5!}\xi\eta^5 + \frac{1}{7!}\xi\eta^7 + \cdots \qquad (18)$$

Hence

$$\Psi(\xi,\eta) = \xi\eta. \qquad (19)$$

Example 3.2

Consider the Helmholtz equation in two dimensional ξ , η

$$\Psi_{\xi\xi}(\xi,\eta) + \Psi_{\eta\eta}(\xi,\eta) + \Psi(\xi,\eta) = 0, \qquad (20)$$

with the conditions

$$\Psi(\xi,0)=0, \qquad \Psi_\eta(\xi,0)=\xi \tag{21}$$

Solution.

We first rewrite (20) in an operator form by

$$L_\eta\Psi(\xi,\eta)=-L_\xi\Psi(\xi,\eta)-\Psi(\xi,\eta), \tag{22}$$

Applying the inverse operator L_η^{-1} to both sides of (22) and using the boundary conditions we obtain

$$L_\eta^{-1}L_\eta\Psi(\xi,\eta)=\xi\eta-L_\eta^{-1}\left[L_\xi\left(\Psi(\xi,\eta)\right)\right]-L_\eta^{-1}[\Psi(\xi,\eta)]. \tag{23}$$

Using both decomposition series

$$\Psi(\xi,\eta)=\sum_{n=0}^{\infty}\Psi_n(\xi,\eta), \tag{24}$$

into both sides of (23) gets

$$\sum_{n=0}^{\infty}\Psi_n(\xi,\eta)=\xi\eta-L_\eta^{-1}\left[L_\xi\left(\sum_{n=0}^{\infty}\Psi_n(\xi,\eta)\right)\right]-L_\eta^{-1}\left[\sum_{n=0}^{\infty}\Psi_n(\xi,\eta)\right]. \tag{25}$$

Adomian's analysis admits the use of the recursive relation

$$\Psi_0(\xi,\eta)=\xi\eta,$$
$$\Psi_{n+1}(\xi,\eta)=-L_\eta^{-1}\left[L_\xi\left(\Psi_n(\xi,\eta)\right)\right]-L_\eta^{-1}[\Psi_n(\xi,\eta)]. \tag{26}$$

Hence, we find

$$\begin{aligned}\Psi_0(\xi,\eta)&=\xi\eta,\\ \Psi_1(\xi,\eta)&=-L_\eta^{-1}\left[L_\xi\left(\Psi_0(\xi,\eta)\right)\right]-L_\eta^{-1}[\Psi_0(\xi,\eta)]=-\frac{1}{3!}\xi\eta^3,\\ \Psi_2(\xi,\eta)&=-L_\eta^{-1}\left[L_\xi\left(\Psi_1(\xi,\eta)\right)\right]-L_\eta^{-1}[\Psi_1(\xi,\eta)]=\frac{1}{5!}\xi\eta^5,\\ \Psi_3(\xi,\eta)&=-L_\eta^{-1}\left[L_\xi\left(\Psi_2(\xi,\eta)\right)\right]-L_\eta^{-1}[\Psi_2(\xi,\eta)]=-\frac{1}{7!}\xi\eta^7,\end{aligned} \tag{27}$$

and so on. The solution $\Psi(\xi,\eta)$ in a series form is given by

$$\begin{aligned}\Psi(\xi,\eta)&=\Psi_0(\xi,\eta)+\Psi_1(\xi,\eta)+\Psi_2(\xi,\eta)+\Psi_3(\xi,\eta)+\cdots\\ \Psi(\xi,\eta)&=\xi\eta-\frac{1}{3!}\xi\eta^3+\frac{1}{5!}\xi\eta^5-\frac{1}{7!}\xi\eta^7+\cdots\\ &=\xi\left(\eta-\frac{1}{3!}\eta^3+\frac{1}{5!}\eta^5-\frac{1}{7!}\eta^7+\cdots\right),\end{aligned} \tag{28}$$

where noise term vanishes in the limit. The solution in a closed form

$$\Psi(\xi,\eta)=\xi sin\eta. \tag{29}$$

Conclusion

In Conclusion, the Adomain decomposition method suggested in this paper has been applied for calculating the solutions of Helmholtz equations successfully. The Adomain decomposition method is actual and can easily be used to solve linear and nonlinear, ordinary or partial differential equations. For further work, we suggest that researches use applications of this method to solve the generalized Helmholtz equation.

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