Albaydha University Journal – Vol.(5)-Issue(4) 2023(special Issue of the Fourth Scientific Conference of Albaydha University)

ON TRIPLE LAPLACE-ABOODH-SHEHU TRANSFORM AND ITS PROPERTIES WITH APPLICATIONS

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DOI: https://doi.org/10.56807/buj.v5i4.471

ABSTRACT

In this paper, we combine the Laplace transform, Aboodh transform and Shehu transform to give another transform which is called the triple Laplace-Aboodh-Shehu transform. This interesting transform reduce a linear partial equation of second order with unknown function of three variables to an algebraic equation, which can then be solved by applying the inverse triple Laplace-Aboodh-Shehu transform.

1. INTRODUCTIN

The topic of partial differential equations is one of the most important subjects in mathematics and other sciences. Therefore, it is very important to know methods to solve such partial differential equations. One of most common and rather method for solving partial differential equations is the integral transform method. In the literature, several different transforms are introduced and applied to find the solution of partial differential equations such as Laplace transform [9, 10], Shehu transform [13], Aboodh transform [2, 3], and so on. Aboodh transform has deeper connection with Laplace and Shehu transforms, and the Shehu transform is a simple variant of the Laplace transform.

In recent years, great attention has been given to deal with double and triple integral transforms, see for example [1, 2, 6, 7, 8]. Aboodh [3] in 2013 introduced a new integral transform called Aboodh transform, which is derived from the Fourier integral and similar to Laplace transform, and applied it to solve ordinary differential equations, after that he introduced the double Aboodh transform and used it to solve integral differential equation and partial differential equation [2]. Recently, in 2020, the authors in [4] introduced a new double integral transform called Laplace-Shehu transform and applied it to solve partial differential equations. In [1], the concept of triple Laplace transform was used to solve third order partial differential equations and the properties have been determined and studied also. For further detail and theories about triple integrals transform and their characteristics, see [7, 8, 14, 15].

The aim of this work is to study a new operator integral transform called triple Laplace-Aboodh-Shehu transform with its fundamental properties and gave the triple Laplace-Aboodh-Shehu transform of some elementary functions.

In order to illustrate the applicability and efficiency of the triple Laplace-AboodhShehu transform, we apply this interesting transform to solve some kinds of partial differential equations.

Firs, we recall the definitions of Laplace, Aboodh, Shehu, double Laplace-Aboodh, double Laplace-Shehu and double Aboodh-Shehu transforms.

Definition 1.1. [5] The Laplace transform of the continuous function $g(x)$ is defined by

$$L[f(x)] = F(p) = \int_0^\infty e^{-px} g(x)dx. \tag{1.1}$$

The inverse Laplace transform is defined by

$$L^{-1}[F(p)] = f(x) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} e^{px} F(p)dp,$$

where α is a real constant.

Definition 1.2. [3] The Aboodh transform of the real function $f(y)$ of exponential order is defined over the set of functions,

$$\mathcal{A} = \left\{ f(y) : \exists K, \tau_1, \tau_2 > 0, |f(y)| < Ke^{|y|\tau_i}, y \in (-1)^i \times [0, \infty), i = 1, 2 \right\},$$

by the following integral

$$A[f(y)] = F(r) = \frac{1}{r}\int_0^\infty e^{-ry} f(y)dy,\ \tau_1 \leq r \leq \tau_2. \tag{1.2}$$

And the inverse Aboodh transform is

$$A^{-1}[F(y)] = f(y) = \frac{1}{2\pi i}\int_{w - i\infty}^{w + i\infty} re^{ry} F(r)dr,\ w \geq 0.$$

Definition 1.3. [13] The Shehu transform of the function $f(t)$ of exponential order is defined over the set of functions,

$$\mathcal{B} = \left\{ f(t) : \exists N, \rho_1, \rho_2 > 0, |f(t)| < Ne^{\frac{|t|}{\rho_i}}, t \in (-1)^i \times [0, \infty) \right\},$$

by the following integral

$$\mathbb{S}[f(t)] = F(s, u) = \int_0^\infty e^{-\frac{s}{u}t} f(t)dt, \quad s > 0, \quad u > 0. \tag{1.3}$$

The inverse Shehu transform is given by

$$f(t) = \mathbb{S}^{-1}[F(s, u)] = \frac{1}{2\pi i}\int_{\lambda - i\infty}^{\lambda + i\infty} \frac{1}{u} e^{\frac{s}{u}t} F(s, u)ds,$$

where s and u are the Shehu transform variables, and λ is a real constant and the integral in $Eq.$ (1.3) is taken along $s = w$ in the complex plane $s = x + iy$.

Definition 1.4. [12] The double Laplace-Aboodh transform of the continuous function $g(x, y)$ and $x, y > 0$ is denoted by the operator $L_x A_y[g(x, y)] = G(p, r)$ and defined by

$$L_x A_y[f(x, y)] = F(p, r) = \frac{1}{r}\int_0^\infty \int_0^\infty e^{-(px + ry)} f(x, y)dxdy, \tag{1.4}$$

provided the integral exists.
The inverse double Laplace-Aboodh transform is defined by

$$f(x,y) = L_x^{-1}A_y^{-1}[F(p,r)] = \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}\left\{\int_{\omega-i\infty}^{\omega+i\infty} re^{ry}F(p,r)dr\right\}dp,$$

where α and ω are real constants.

Definition 1.5. [11] The double Laplace-Shehu transform of the function $f(x,t)$ of two variables $x > 0$ and $t > 0$ is denoted by $L_x\mathbb{S}_t[f(x,t)] = F(p,s,u)$ and defined as

$$L_x\mathbb{S}_t[f(x,t)] = F(p,s,u) = \int_0^\infty\int_0^\infty e^{-(px+\frac{s}{u}t)}f(x,t)dxdt \tag{1.5}$$

provided the integral exists.
The inverse double Laplace-Shehu transform is defined by

$$L_x^{-1}\mathbb{S}_t^{-1}[F(p,s,u)] = f(x,t) = \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}\left\{\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{1}{u}e^{\frac{s}{u}t}F(p,s,u)ds\right\}dp,$$

where α and ω are real constants.

Definition 1.6. [15] The double Aboodh-Shehu transform of the continuous function $f(y,t)$, $y,t > 0$ is denoted by the operator $A_y\mathbb{S}_t[f(y,t)] = F(r,s,u)$ and defined by

$$A_y\mathbb{S}_t[f(y,t)] = F(r,s,u) = \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(ry+\frac{s}{u}t)}f(y,t)dydt \tag{1.6}$$

$$= \frac{1}{r}\lim_{a\to\infty,b\to\infty}\int_0^a\int_0^b e^{-(ry+\frac{s}{u}t)}f(y,t)dydt. \tag{1.7}$$

It converges if the limit of the integral exists, and diverges if not.
The inverse of double Aboodh-Shehu transform is defined by

$$f(y,t) = A_y^{-1}\mathbb{S}_t^{-1}\left[F(r,s,u)\right] = \frac{1}{(2\pi i)^2}\int_{\omega-i\infty}^{\omega+i\infty} re^{ry}\left\{\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{1}{u}e^{\frac{s}{u}t}F(r,s,u)ds\right\}dr, \tag{1.8}$$

where ω and λ are real constants.

In the next, we introduce the definition of our main result in this work.

Definition 1.7. Let f be a continuous function of three variables say $x,y,t > 0$; then, the triple Laplace-Aboodh-Shehu transform of $f(x,y,t)$ is denoted by the operator $L_xA_y\mathbb{S}_t[f(x,y,t)] = F(p,r,s,u)$ and defined by

$$L_xA_y\mathbb{S}_t[f(x,y,t)] = F(p,r,s,u) = \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}f(x,y,t)dxdydt. \tag{1.9}$$

Provided the integral exists.
The inverse triple Laplace-Aboodh-Shehu transform is defined by

$$\begin{aligned} f(x,y,t) &= L_x^{-1}A_y^{-1}\mathbb{S}_t^{-1}[F(p,r,s,u)] \\ &= \frac{1}{(2\pi i)^3}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}\left\{\int_{\omega-i\infty}^{\omega+i\infty} re^{ry}\left\{\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{1}{u}e^{\frac{s}{u}t}F(p,r,s,u)ds\right\}dr\right\}dp, \end{aligned}$$

where $\alpha,\ \omega$ and λ are real constants.

2. Existence and Uniqueness of Triple Laplace-Aboodh-Shehu Transform

Definition 2.1. A function $f(x,y,t)$ is said to be of exponential order $e^{(ax+by+ct)}$, $a,\ b,\ c>0$ on $[0,\infty)$, if there are positive constants $M,\ X,\ Y$, and T such that

$$|f(x,y,t)|\le Me^{(ax+by+ct)}, \qquad \forall \quad x>X, \quad y>Y, \quad t>T,$$

and we write

$$f(x,y,t)=o\left(e^{(ax+by+ct)}\right) \qquad as \quad (x,y,t\to\infty).$$

Or, equivalently,

$$\sup_{x,y,t>0}\left(\frac{|f(x,y,t)|}{e^{(ax+by+ct)}}\right)<\infty.$$

Theorem 2.2. *Let $f(x,y,t)$ be a continuous function on the interval $[0,\infty)$ and of exponential order $e^{(ax+by+ct)}$. Then the triple Laplace-Aboodh-Shehu transform of $f(x,y,t)$ exists for all $p>a,\ r>b$ and $\frac{s}{u}>c$.*

Proof. Let $f(x,y,t)$ be of exponential order $e^{(ax+by+ct)}$ such that

$$|f(x,y,t)|\le Me^{(ax+by+ct)}, \quad \forall \quad x>X, \quad y>Y, \quad t>T.$$

Then, we have

$$\begin{aligned} |L_xA_y\mathbb{S}_t[f(x,y,t)]| &= \left|\frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}f(x,y,t)dxdydt\right| \\ &\le \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}\,|f(x,y,t)|\,dxdydt \\ &\le \frac{M}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}e^{(ax+by+ct)}dxdydt \\ &= M\int_0^\infty e^{-\left(\frac{s-cu}{u}\right)t}\left\{\frac{1}{r}\int_0^\infty e^{-(r-b)y}\left\{\int_0^\infty e^{-(p-a)x}dx\right\}dy\right\}dt \\ &= \frac{Mu}{r(p-a)(r-b)(s-cu)}. \end{aligned}$$

Thus, the proof is complete. □

Theorem 2.3. *Let $F_1(p,r,s,u)$ and $F_2(p,r,s,u)$ be the triple Laplace-Aboodh-Shehu transform of the continuous functions $f_1(x,y,t)$ and $f_2(x,y,t)$ defined for $x,\ y,\ t\ge 0$ respectively. If $F_1(p,r,s,u)=F_2(p,r,s,u)$, then $f_1(x,y,t)=f_2(x,y,t)$.*

Proof. Assume that $\alpha_1,\ \alpha_2$ and α_3 are sufficiently large, since

$$f(x,y,t)=\frac{1}{(2\pi i)^3}\int_{\alpha_1-i\infty}^{\alpha_1+i\infty}e^{px}\left\{\int_{\alpha_2-i\infty}^{\alpha_2+i\infty}re^{ry}\left\{\int_{\alpha_3-i\infty}^{\alpha_3+i\infty}\frac{1}{u}e^{\frac{s}{u}t}F(p,r,s,u)ds\right\}dr\right\}dp,$$

we deduce that

$$\begin{aligned}f_1(x,y,t)&=\frac{1}{(2\pi i)^3}\int_{\alpha_1-i\infty}^{\alpha_1+i\infty}e^{px}\left\{\int_{\alpha_2-i\infty}^{\alpha_2+i\infty}re^{ry}\left\{\int_{\alpha_3-i\infty}^{\alpha_3+i\infty}\frac{1}{u}e^{\frac{s}{u}t}F_1(p,r,s,u)ds\right\}dr\right\}dp\\&=\frac{1}{(2\pi i)^3}\int_{\alpha_1-i\infty}^{\alpha_1+i\infty}e^{px}\left\{\int_{\alpha_2-i\infty}^{\alpha_2+i\infty}re^{ry}\left\{\int_{\alpha_3-i\infty}^{\alpha_3+i\infty}\frac{1}{u}e^{\frac{s}{u}t}F_2(p,r,s,u)ds\right\}dr\right\}dp\\&=f_2(x,y,t).\end{aligned}$$

This proves the uniqueness of the triple Laplace-Aboodh-Shehu transform. □

3. Some Properties of Triple Laplace-Aboodh-Shehu Transform

4.1. Linearity property If $f_1(x,y,t)$ and $f_2(x,y,t)$ be two functions such that

$$L_xA_y\mathbb{S}_t[f_1(x,y,t)]=F_1(p,r,s,u),$$
$$L_xA_y\mathbb{S}_t[f_2(x,y,t)]=F_2(p,r,s,u).$$

Then for any constants a and b, we have

$$L_xA_y\mathbb{S}_t[ag_1(x,y,t)+bg_2(x,y,t)]=aL_xA_y\mathbb{S}_t[g_1(x,y,t)]+bL_xA_y\mathbb{S}_t[g_2(x,y,t)].$$

Proof. Using definition of triple Laplace-Aboodh-Shehu transform, we obtain

$$\begin{aligned}L_xA_y\mathbb{S}_t[af_1(x,y,t)+bf_2(x,y,t)]&=\frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}\left\{af_1(x,y,t)+bf_2(x,y,t)\right\}dxdydt\\&=\frac{a}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}f_1(x,y,t)dxdydt\\&+\frac{b}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}f_2(x,y,t)dxdydt\\&=aL_xA_y\mathbb{S}_t[f_1(x,y,t)]+bL_xA_y\mathbb{S}_t[f_2(x,y,t)].\end{aligned}$$

The proof is complete. □

4.2. Shifting property If the triple Laplace-Aboodh-Shehu transform of $f(x,y,t)$ is $F(p,r,s,u)$, then for real constants $a,\ b$ and c, we have

$$L_xA_y\mathbb{S}_t[e^{(ax+by+ct)}f(x,y,t)]=\frac{(r-b)}{r}F(p-a,r-b,s-uc,u). \tag{3.1}$$

Proof. Using the definition of triple Laplace-Aboodh-Shehu transform, we get

$$\begin{aligned}
L_xA_y\mathbb{S}_t[e^{(ax+by+ct)}f(x,y,t)] &= \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}e^{(ax+by+ct)}f(x,y,t)dxdydt\\
&= \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left((p-a)x+(r-b)y+(\frac{s}{u}-c)t\right)}f(x,y,t)dxdydt\\
&= \frac{(r-b)}{r(r-b)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left((p-a)x+(r-b)y+(\frac{s}{u}-c)t\right)}f(x,y,t)dxdydt\\
&= \frac{(r-b)}{r}F(p-a,r-b,s-uc,u).
\end{aligned}$$

The proof is complete. □

4.3. Changing of scale property Let $f(x,y,t)$ be a function such that

$$L_xA_y\mathbb{S}_t[f(x,y,t)] = F(p,r,s,u).$$

Then for $a,\ b$ and c are positive constants, we have

$$L_xA_y\mathbb{S}_t[f(ax,by,ct)] = \frac{1}{ab^2c}F(\frac{p}{a},\frac{r}{b},\frac{s}{c},u). \tag{3.2}$$

Proof. Using the definition of triple Laplace-Aboodh-Shehu transform, we deduce

$$L_xA_y\mathbb{S}_t[f(ax,by,ct)] = \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}f(ax,by,ct)dxdydt,$$

Let $\theta = ax,\ \phi = by$ and $\varphi = ct$, then

$$\begin{aligned}
L_xA_y\mathbb{S}_t[f(\theta,\phi,\varphi)] &= \frac{1}{rabc}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{p}{a}\theta+\frac{r}{b}\phi+\frac{s}{uc}\varphi)}f(\theta,\phi,\varphi)d\theta d\phi d\varphi\\
&= \frac{1}{ab^2c}\frac{b}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{p}{a}\theta+\frac{r}{b}\phi+\frac{s}{uc}\varphi)}f(\theta,\phi,\varphi)d\theta d\phi d\varphi\\
&= \frac{1}{ab^2c}F(\frac{p}{a},\frac{r}{b},\frac{s}{c},u).
\end{aligned}$$

The proof is complete. □

4.4. Derivatives properties If $L_xA_y\mathbb{S}_t[f(x,y,t)] = F(p,r,s,u)$, then

(1) $$L_xA_y\mathbb{S}_t\left[\frac{\partial f(x,y,t)}{\partial x}\right] = pF(p,r,s,u) - A_y\mathbb{S}_t[f(0,y,t)]. \tag{3.3}$$

Proof.

$$\begin{aligned}
L_xA_y\mathbb{S}_t\left[\frac{\partial f(x,y,t)}{\partial x}\right] &= \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}dt\frac{\partial f(x,y,t)}{\partial x}dxdydt\\
&= \frac{1}{r}\int_0^\infty e^{-ry}dy\int_0^\infty e^{-\frac{s}{u}t}dt\left\{\int_0^\infty e^{-px}f_x(x,y,t)dx\right\}.
\end{aligned}$$

Using integration by parts, let $\mathsf{u} = e^{-px}$, $d\mathsf{v} = \frac{\partial f(x,y,t)}{\partial x}$, then we obtain

$$\begin{aligned}
L_xA_y\mathbb{S}_t\left[\frac{\partial f(x,y,t)}{\partial x}\right] &= \frac{1}{r}\int_0^\infty e^{-ry}dy\int_0^\infty e^{-\frac{s}{u}t}dt\left\{-f(0,y,t)+p\int_0^\infty e^{-px}f(x,y,t)dx\right\}\\
&= pF(p,r,s,u) - A_y\mathbb{S}_t[f(0,y,t)].
\end{aligned}$$

This ends the proof. □

$$\textbf{(2)} \qquad L_xA_y\mathbb{S}_t\left[\frac{\partial f(x,y,t)}{\partial y}\right] = rF(p,r,s,u) - \frac{1}{r}L_x\mathbb{S}_t[f(x,0,t)]. \qquad (3.4)$$

Proof.

$$\begin{aligned} L_xA_y\mathbb{S}_t\left[\frac{\partial f(x,y,t)}{\partial y}\right] &= \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}dt\frac{\partial f(x,y,t)}{\partial y}dxdydt \\ &= \frac{1}{r}\int_0^\infty e^{-px}dx\int_0^\infty e^{-\frac{s}{u}t}dt\left\{\int_0^\infty e^{-ry}f_y(x,y,t)dy\right\}. \end{aligned}$$

Using integration by parts, let $\mathsf{u} = e^{-ry}$, $d\mathsf{v} = \frac{\partial f(x,y,t)}{\partial y}$, then we obtain

$$\begin{aligned} L_xA_y\mathbb{S}_t\left[\frac{\partial f(x,y,t)}{\partial x}\right] &= \frac{1}{r}\int_0^\infty e^{-px}dx\int_0^\infty e^{-\frac{s}{u}t}dt\left\{-f(x,0,t) + r\int_0^\infty e^{-ry}f(x,y,t)dy\right\} \\ &= rF(p,r,s,u) - \frac{1}{r}L_x\mathbb{S}_t[f(x,0,t)]. \end{aligned}$$

This ends the proof. □

Similarly, we can prove that:

$$\begin{aligned} L_xA_y\mathbb{S}_t\left[\frac{\partial f(x,y,t)}{\partial t}\right] &= \frac{s}{u}F(p,r,s,u) - L_xA_y[f(x,y,0)], \\ L_xA_y\mathbb{S}_t\left[\frac{\partial^2 f(x,y,t)}{\partial x^2}\right] &= p^2F(p,r,s,u) - pA_y\mathbb{S}_t[f(0,y,t)] - A_y\mathbb{S}_t[f_x(0,y,t)], \\ L_xA_y\mathbb{S}_t\left[\frac{\partial^2 f(x,y,t)}{\partial y^2}\right] &= r^2F(p,r,s,u) - L_x\mathbb{S}_t[f(x,0,t)] - \frac{1}{r}L_x\mathbb{S}_t[f_y(x,0,t)], \\ L_xA_y\mathbb{S}_t\left[\frac{\partial^2 f(x,y,t)}{\partial t^2}\right] &= \frac{s^2}{u^2}F(p,r,s,u) - \frac{s}{u}L_xA_y[f(x,y,0)] - L_xA_y[f_t(x,y,0)], \\ L_xA_y\mathbb{S}_t\left[\frac{\partial^2 f(x,y,t)}{\partial x\partial y}\right] &= prF(p,r,s,u) - \frac{p}{r}L_x\mathbb{S}_t[f(x,0,t)] - A_y\mathbb{S}_t[f_y(0,y,t)], \\ L_xA_y\mathbb{S}_t\left[\frac{\partial^2 f(x,y,t)}{\partial x\partial t}\right] &= \frac{ps}{u}F(p,r,s,u) - pL_xA_y[f(x,y,0)] - A_y\mathbb{S}_t[f_t(0,y,t)], \\ L_xA_y\mathbb{S}_t\left[\frac{\partial^2 f(x,y,t)}{\partial y\partial t}\right] &= \frac{rs}{u}F(p,r,s,u) - rL_xA_y[f(x,y,0)] - \frac{1}{r}L_x\mathbb{S}_t[f_t(x,0,t)]. \end{aligned}$$

4. Triple Laplace-Aboodh-Shehu Transform of some Elementary Functions

(1) If $f(x,y,t) = 1$, then

$$L_xA_y\mathbb{S}_t[f(x,y,t)] = \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}dxdydt = \frac{u}{pr^2s}.$$

(2) If $f(x,y,t) = xyt$, then

$$L_xA_y\mathbb{S}_t[f(x,y,t)] = \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}(xyt)dxdydt = \frac{u^2}{p^2r^3s^2}.$$

(3) If $f(x,y,t) = x^n y^m t^k,\ n,\ m\ k = 0,1,2,...,$ then

$$\begin{aligned}L_xA_y\mathbb{S}_t[f(x,y,t)] &= \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}x^n y^m t^k dxdydt\\ &= \frac{n!}{p^{n+1}}.\frac{m!}{r^{m+2}}.\left(\frac{u}{s}\right)^{k+1} k!.\end{aligned}$$

(4) If $f(x,y,t) = x^\alpha y^\beta t^\gamma,\ \alpha \geq -1,\ \beta \geq -1, \gamma \geq -1$, then

$$\begin{aligned}L_xA_y\mathbb{S}_t[f(x,y,t)] &= \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}x^\alpha y^\beta t^\gamma dxdydt\\ &= \frac{1}{r}\int_0^\infty e^{-px}x^\alpha dx\int_0^\infty e^{-ry}y^\beta dy\int_0^\infty e^{-\frac{s}{u}t}t^\gamma dt,\end{aligned}$$

let $\theta = px,\ \phi = ry$ and $\varphi = \frac{s}{u}t$

$$\begin{aligned}L_xA_y\mathbb{S}_t[f(x,y,t)] =&\frac{1}{r^{\alpha+1}}\int_0^\infty e^{-\theta}\theta^\alpha d\theta\int_0^\infty e^{-\phi}\phi^\beta d\phi\frac{1}{r^{\beta+2}}\int_0^\infty e^{-\phi}\phi^\beta d\phi\\ &= \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}.\frac{\Gamma(\beta+1)}{r^{\beta+2}}\Gamma(r+1)(\frac{u}{s})^{r+1}\end{aligned}$$

where $\Gamma(.)$ is the Euler gamma function.
(5) If $f(x,y,t) = e^{(ax+by+ct)}$, then

$$\begin{aligned}L_xA_y\mathbb{S}_t[f(x,y,t)] &= \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}e^{(ax+by+ct)}dxdydt\\ &= \frac{u}{r(p-a)(r-b)(s-cu)}.\end{aligned}$$

(6) If $f(x,y,t) = \sinh(ax+by+ct)$ or $\cosh(ax+by+ct)$.
Recall that

$$\sinh(ax+by+ct) = \frac{e^{(ax+by+ct)} - e^{-(ax+by+ct)}}{2}, \quad \cosh(ax+by+ct) = \frac{e^{(ax+by+ct)} + e^{-(ax+by+ct)}}{2}.$$

Therefore,

$$\begin{aligned}L_xA_y\mathbb{S}_t[\sinh(ax+by+ct)] &= \frac{u(cpru+bps+ars+abcu)}{r(p^2-a^2)(r^2-b^2)(s^2-c^2u^2)},\\ L_xA_y\mathbb{S}_t[\cosh(ax+by+ct)] &= \frac{u(prs+bcpu+acru+abs)}{r(p^2-a^2)(r^2-b^2)(s^2-c^2u^2)}.\end{aligned}$$

(7) If $f(x,y,t)=e^{i(ax+by+ct)}$, then

$$\begin{aligned} L_xA_y\mathbb{S}_t[f(x,y,t)] &= \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}e^{i(ax+by+ct)}dxdydt \\ &= \frac{u}{r(p-ia)(r-ib)(s-icu)} \\ &= \frac{u(prs-bcpu-acru-abs)+iu(cpru+bps+ars-abcu)}{r(p^2+a^2)(r^2+b^2)(s^2+c^2u^2)}. \end{aligned}$$

Consequently,

$$\begin{aligned} L_xA_y\mathbb{S}_t[\sin(ax+by+ct)] &= \frac{u(cpru+bps+ars-abcu)}{r(p^2+a^2)(r^2+b^2)(s^2+c^2u^2)}, \\ L_xA_y\mathbb{S}_t[\cos(ax+by+ct)] &= \frac{u(prs-bcpu-acru-abs)}{r(p^2+a^2)(r^2+b^2)(s^2+c^2u^2)}. \end{aligned}$$

(8) If $f(x,y,t)=f_1(x)f_2(y)f_3(t)$, then

$$\begin{aligned} L_xA_y\mathbb{S}_t[f(x,y,t)] &= \frac{1}{r}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+ry+\frac{s}{u}t)}\left(f_1(x)f_2(y)f_3(t)\right)dxdydt \\ &= \int_0^\infty e^{-px}f_1(x)\left\{\frac{1}{r}\int_0^\infty e^{-ry}f_2(y)\left\{\int_0^\infty e^{-\frac{s}{u}t}f_3(t)dt\right\}dy\right\}dx \\ &= L_x[f_1(x)]A_y[f_2(y)]\mathbb{S}_t[f_3(t)]. \end{aligned}$$

Therefore,

$$\begin{aligned} L_xA_y\mathbb{S}_t[\cos(ax)\cos(by)\cos(ct)] &= \frac{p}{(p^2+a^2)}\frac{1}{(r^2+b^2)}\frac{us}{s^2+c^2u^2}, \\ L_xA_y\mathbb{S}_t[\sin(ax)\sin(by)\sin(ct)] &= \frac{a}{(p^2+a^2)}\frac{b}{r(r+b^2)}\frac{cu^2}{(s^2+c^2u^2)}. \end{aligned}$$

5. Applications

In this section, we apply the triple Laplace-Aboodh-Shehu transform operator for solving some kinds of linear partial differential equations.

Example 5.1. *Consider the following diffusion equations:*

$$f_{xx}(x,y,t)+f_{yy}(x,y,t)-f_t(x,y,t)=0, \quad (x,y)\in\mathbb{R}_+^2, t>0, \tag{5.1}$$

Subject to the following initial and boundary conditions:

$$f(0,y,t)=e^{y+2t}, \qquad f_x(0,y,t)=e^{y+2t}, \tag{5.2}$$

$$f(x,0,t)=e^{x+2t}, \qquad f_y(x,0,t)=e^{x+2t} \tag{5.3}$$

$$f(x,y,0)=e^{x+y}. \tag{5.4}$$

Solution. *Applying triple Laplace-Aboodh-Shehu transform on Eq.(5.1), we have*

$$L_xA_y\mathbb{S}_t[f_{xx}(x,y,t)]+L_xA_y\mathbb{S}_t[f_{yy}(x,y,t)]-L_xA_y\mathbb{S}_t[f_t(x,y,t)]=0. \tag{5.5}$$

By linearity property and partial derivatives properties of triple Laplace-Aboodh-Shehu transform, we get

$$p^2F(p,r,s,u) - pA_y\mathbb{S}_t[f(0,y,t)] - A_y\mathbb{S}_t[f_x(0,y,t)] + r^2F(p,r,s,u) - L_x\mathbb{S}_t[f(x,0,t)] - \frac{1}{r}L_x\mathbb{S}_t[f_y(x,0,t)] - \frac{s}{u}F(p,r,s,u) + L_xA_y[f(x,y,0)] = 0. \tag{5.6}$$

Rearranging the terms, we have

$$F(p,r,s,u) = \frac{u}{up^2+ur^2-s}[pA_y\mathbb{S}_t[f(0,y,t)] + A_y\mathbb{S}_t[f_x(0,y,t)] + L_x\mathbb{S}_t[f(x,0,t)] + \frac{1}{r}L_x\mathbb{S}_t[f_y(x,0,t)] - L_xA_y[f(x,y,0)]]. \tag{5.7}$$

Using double Aboodh-Shehu transform for equations (5.2), double Laplace-Shehu transform for equations (5.3) and double Laplace-Aboodh for equation (5.4), we obtain

$$A_y\mathbb{S}_t[f(0,y,t)] = A_y\mathbb{S}_t[f_x(0,y,t)] = \frac{u}{r(r-1)(s-2u)}, \tag{5.8}$$

$$L_x\mathbb{S}_t[f(x,0,t)] = L_x\mathbb{S}_t[f_y(x,0,t)] = \frac{u}{(p-1)(s-2u)}, \tag{5.9}$$

$$L_xA_y[f(x,y,0)] = \frac{1}{r(p-1)(r-1)}. \tag{5.10}$$

Substitute equations (5.8)-(5.10) into equation (5.7) and simplify to obtain

$$\begin{aligned} F(p,r,s,u) &= \frac{u}{up^2+ur^2-s}\left\{\frac{u(p+1)}{r(r-1)(s-2u)} + \frac{u(r+1)}{r(p-1)(s-2u)} - \frac{1}{r(p-1)(r-1)}\right\} \\ &= \frac{u}{r(p-1)(r-1)(s-2u)}. \end{aligned} \tag{5.11}$$

Taking the inverse triple Laplace-Aboodh-Shehu transform of equation (5.11), we get

$$\begin{aligned} f(x,y,t) &= L_x^{-1}A_y^{-1}\mathbb{S}_t^{-1}\left[\frac{u}{r(p-1)(r-1)(s-2u)}\right] \\ &= e^{x+y+2t}. \end{aligned} \tag{5.12}$$

Example 5.2. *Consider the following nonhomogeneous Wave equation*

$$2f_{tt}(x,y,t) = f_{xx}(x,y,t)+f_{yy}(x,y,t)+24t^2+4y, \qquad (x,y) \in \mathbb{R}_+^2, \quad t > 0, \tag{5.13}$$

subject to the boundary and initial conditions

$$f(0,y,t) = t^4 + yt^2, \qquad f_x(0,y,t) = \sin y \sin t, \tag{5.14}$$

$$f(x,0,t) = t^4, \qquad f_y(x,0,t) = t^2 + \sin x \sin t, \tag{5.15}$$

$$f(x,y,0) = 0, \qquad f_t(x,y,0) = \sin x \sin y. \tag{5.16}$$

Solution. *We can rewrite the equation (5.13), as*

$$f_{tt}(x,y,t) = \frac{1}{2}\big(f_{xx}(x,y,t) + f_{yy}(x,y,t)\big) + 12t^2 + 2y, \ (x,y) \in \mathbb{R}_+^2, \ t > 0, \tag{5.17}$$

Taking the triple Laplace-Aboodh-Shehu transform to both sides of equation (5.17) and rearranging the terms, we get

$$\begin{aligned} F(p,r,s,u) = \frac{u^2}{p^2u^2+r^2u^2-2s^2}\Bigg(&\frac{p}{2}A_y\mathbb{S}_t[f(0,y,t)] + \frac{1}{2}A_yS_t[f_x(0,y,t)] \\ &+\frac{1}{2}L_x\mathbb{S}_t[f(x,0,t)] + \frac{1}{2r}L_x\mathbb{S}_t[f_y(x,0,t)] - \frac{s}{u}L_xA_y[f(x,y,0)] \\ &- L_xA_y[f_t(x,y,0)] - \frac{1}{2}\Big(\frac{48u^3}{pr^2s^3}+\frac{4u}{pr^3s}\Big)\Bigg), \end{aligned} \tag{5.18}$$

where

$$L_xA_y\mathbb{S}_t[24t^2+4y] = \frac{48u^3}{pr^2s^3}+\frac{4u}{pr^3s}.$$

Applying double Laplace-Aboodh transform, double Aboodh-Shehu transform and double Laplace-Shehu transform to the given conditions, we obtain

$$\begin{aligned} F(p,r,s,u) = \frac{u^2}{p^2u^2+r^2u^2-2s^2}\Bigg(&\frac{p}{2}\Big(\frac{24u^5}{r^2s^5}+\frac{2u^3}{r^3s^3}\Big) + \frac{u^2}{2r(r^2+1)(s^2+u^2)} \\ &+\frac{24u^5}{2ps^5}+\frac{2u^3}{2prs^3}+\frac{u^2}{2r(p^2+1)(s^2+u^2)} \\ &-\frac{1}{r(p^2+1)(r^2+1)} - \frac{24u^3}{pr^2s^3} - \frac{2u}{pr^3s}\Bigg), \end{aligned} \tag{5.19}$$

by simple computation, we get

$$F(p,r,s,u) = \frac{24u^5}{pr^2s^5}+\frac{2u^3}{pr^3s^3}+\frac{u^2}{r(p^2+1)(r^2+1)(s^2+u^2)}. \tag{5.20}$$

Applying the inverse triple Laplace-Aboodh-Shehu transform on equation (5.20), we get

$$\begin{aligned} f(x,y,t) &= L_x^{-1}A_y^{-1}\mathbb{S}_t^{-1}\left[\frac{24u^5}{pr^2s^5}+\frac{2u^3}{pr^3s^3}+\frac{u^2}{r(p^2+1)(r^2+1)(s^2+u^2)}\right] \\ &= t^4+yt^2+\sin x\sin y\sin t. \end{aligned} \tag{5.21}$$

Example 5.3. *Consider the following boundary Laplace equation*

$$f_{xx}(x,y,t)+f_{yy}(x,y,t)+f_{tt}(x,y,t)=0, \qquad (x,y,t)\in\mathbb{R}^3_+, \tag{5.22}$$

subjected to the conditions

$$f(0,y,t)=0, \qquad f_x(0,y,t)=\sin y\sinh\sqrt{2}t, \tag{5.23}$$

$$f(x,0,t)=0, \qquad f_y(x,0,t)=\sin x\sinh\sqrt{2}t, \tag{5.24}$$

$$f(x,y,0)=0, \qquad f_t(x,y,0)=\sqrt{2}\sin x\sin y. \tag{5.25}$$

Solution. *Applying triple Laplace-Aboodh-Shehu transform to the equation gives and by linearity property and partial derivatives properties of triple Laplace-Aboodh-Shehu transform, we get*

$$\begin{aligned} & p^2F(p,r,s,u) - pA_y\mathbb{S}_t[f(0,y,t)] - A_y\mathbb{S}_t[f_x(0,y,t)] \\ & + r^2F(p,r,s,u) - L_x\mathbb{S}_t[f(x,0,t)] - \frac{1}{r}L_x\mathbb{S}_t[f_y(x,0,t)] \\ & + \frac{s^2}{u^2}F(p,r,s,u) - \frac{s}{u}L_xA_y[f(x,y,0)] - L_xA_y[f_t(x,y,0)] = 0 \end{aligned} \tag{5.26}$$

Substituting

$$A_y\mathbb{S}_t[f_x(0,y,t)] = \frac{\sqrt{2}u^2}{r(r^2+1)(s^2-2u^2)}, \qquad L_x\mathbb{S}_t[f_y(x,0,t)] = \frac{\sqrt{2}u^2}{(p^2+1)(s^2-2u^2)},$$

$$L_xA_y[f_t(x,y,0)] = \frac{\sqrt{2}}{r(p^2+1)(r^2+1)},$$

in equation (5.26) and simplifying, we get

$$\begin{aligned} F(p,r,s,u) &= \frac{u^2}{(u^2p^2+u^2r^2+s^2)}\left\{\frac{\sqrt{2}(u^2p^2+u^2r^2+s^2)}{r(p^2+1)(r^2+1)(s^2-2u^2)}\right\} \\ &= \frac{\sqrt{2}u^2}{r(p^2+1)(r^2+1)(s^2-2u^2)}. \end{aligned} \tag{5.27}$$

Taking the inverse triple Laplace-Aboodh-Shehu transform of equation (5.27), we get

$$f(x,y,t) = L_x^{-1}A_y^{-1}\mathbb{S}_t^{-1}\left\{\frac{\sqrt{2}u^2}{r(p^2+1)(r^2+1)(s^2-2u^2)}\right\} \tag{5.28}$$

$$= \sin x \sin y \sinh \sqrt{2}t. \tag{5.29}$$

Which is the required solution of the considered Laplace equation.

References

[1] Abdon Atangana, *A Note on the Triple Laplace Transform and Its Applications to Some Kind of Third-Order Differential Equation*, Abstract and Applied Analysis, Hindawi, Article ID769102, (2013) 1-10.

[2] K. S. Aboodh, R. A. Fara, I. A. Almardy, F. A. ALmostafa, *Solution of partial integro-differential equations by using Aboodh and double Aboodh transform methods*, Global Journal of Pure and Applied Math., 13(8)(2017) 4347–4360.

[3] K. S. Aboodh, *The new integral transform "Aboodh transform"*, Global Journal of Pure and Applied Math., 9(1)(2013) 35–43.

[4] S. Ahmed, T. Elzaki, M. Elbadri, M. Mohamed, *Solution of Partial Differential Equations by New Double Integral Transform (Laplace - Sumudu Transform)*, Ain Shams Engineering Journal, 2020.

[5] S. Aggarwal, A. Singh, A. Kumar, N. Kumar, *Application of Laplace Transform for Solving Improper Integrals Whose Integrand Consisting Error Function*, Journal of Advanced Research in Applied Mathematics and Statistics 4(2)(2019) 1–7.

[6] S. Alfaqeih, E. Misirli, *On double Shehu transform and its properties with applications*, Inter. Journal of Analysis and Applications, 3(2020) 381-395.

[7] S. Alfaqeih, T. Ozis, *Note on triple Aboodh transform and its application*, Inter. Journal of Engineering and Information Systems, 3(3)(2019) 41-50.

[8] S. Alkaleeli, A. Mtawal, M. Hmad, *Triple Shehu transform and its properties with applications*, African Journal of Mathematics and Computer, 14(1)(2021) 4-12.

[9] R. Khalaf, F. Belgacem, *Extraction of the Laplace, Fourier, and Mellin transforms from the Sumudu transform*, AIP Proceedings, 1637(1426)(2014) 108–115.

[10] A. K. Thakur, S. Panda, "Some Properties of Triple Laplace Transform", Journal of Mathematics and Computer Applications Research (JMCAR), (2015) 2250-2408.

[11] M. Hunaiber, A. Al-Aati, *New double integral transform to solve some partial differential equation*, Albaydha University Journal, 3(3)(2021) 181–190.

[12] M. Hunaiber, A. Al-Aati, *The double Laplace-Aboodh transform and their properties with applications*, Global Scientific Journals, 10(3)(2022) 2055–2065.

[13] S. Maitama, W. Zhao, *New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations,* Inter. Journal of Analysis and Applications, 17(2)(2019) 167–190.

[14] M. S. Mechee A. J. Naeemah, *A Study of Triple Sumudu Transform for Solving Partial Differential Equations with Some Applications*, Multidisciplinary European Academic Journal, 2(2)(2020) 1–15.

[15] Y. Ouideen, A. Al-Aati, *On double Aboodh-Shehu transform and their properties with applications*, Journal of Albaydha University, 4(3)(2022) 241-251.

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