The New Adomian Technique for Solving Singular and Non-singular Partial Differential Equations of First Order

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Abstract

In this paper, the proposed modification to Adomian decomposition method (ADM) is introduced. In addition, a new modification to solve singular and nonsingular initial-value problems (IVPs) in first order partial differential equations (PDEs) is investigated.

Then, a discussion of how ADM can be used to solve singular and nonsingular IVPs is produced. Finally, some example are given to demonstrate the applicability of the results of paper.

Key words; *Adomian decomposition method; Nonlinear first-order PDE; Singular and nonsingular first-order partial differential equations.*

1 Introduction

Singular nonlinear partial differential equations (PDEs) arise in various physical phe- nomena in applied sciences and engineering from such areas as fluid mechanics and heat-

transfer, riemannian geometry, applied probability, mathematical physics, and biology [7, 10]. Recently, a great deal of interest has been focused on the convergence studies of the series solution obtained using ADM for a wide variety of stochastic and deterministic problems [2, 5, 9]. A number of researchers used the ADM method to solve PDEs by ADM. ADM for first order PDEs with unprescribed data [8], some modifications of ADM for nonlinear PDEs [3] , modified decomposition method (MDM) with new inverse differential operators for solving singular nonlinear IVPs in first-and second-order PDEs arising in fluid mechanics [7] and et al. The main objective of this paper is to introduce a new technique method for solving singular and non-singular PDEs of the first order.

2 Modified Adomian Decomposition Method

The ADM is applied for solving a wide class of linear and nonlinear ordinary differential equations, algebraic equations, difference equations, integral equations, partial differential equations and integrate differential equations as [4, 6].
Consider the singular and non-singular nonlinear first-order PDEs (in t) of the form:

$$\frac{\partial u}{\partial t}+(n+\frac{a}{t})u=F(x,u,\frac{\partial u}{\partial x}), \tag{2.1}$$

F is a nonlinear function of $x, u, and u_x$, where t and x are independent variables, u is the dependent variable, and n and a are real constants.

$$u(x,t)=\Phi(x), \tag{2.2}$$

in order to solve the PDE (2.1) with initial condition (2.2) by the ADM, at first, the linear differential operator $L_t(.)=t^{-a}e^{-nt}\frac{\partial}{\partial t}t^a e^{nt}(.)$ is defined, and the left-hand side of (2.1) is rewritten as

$$L_t(u)=t^{-a}e^{-nt}\frac{\partial}{\partial t}t^a e^{nt}(u), \tag{2.3}$$

the inverse oprator L_t^{-1} is therefore considred a tow-fold integeral operator, as below,

$$L_t^{-1}(.)=t^{-a}e^{-nt}\int_0^t t^a e^{nt}(.)dt, \tag{2.4}$$

applying L_t^{-1} of (2.1) to the third term

$$L_t^{-1}(u_t+(n+\frac{a}{t})u)=t^{-a}e^{-nt}\int_0^t t^a e^{nt}(u_t+(n+\frac{a}{t})u)dt, \tag{2.5}$$

$$=t^{-a}e^{-nt}\int_0^t [t^a e^{nt}u_t+t^a e^{nt}(n+\frac{a}{t})u]dt$$

$$=t^{-a}e^{-nt}\int_0^t [t^a e^{nt}u_t+(nt^a e^{nt}u+at^{a-1}e^{nt}u)]dt$$

$$= t^{-a}e^{-nt}\int_0^t \frac{\partial}{\partial t}(t^a e^{nt}u)dt$$
$$= t^{-a}e^{-nt}(t^a e^{nt}u)$$
$$= t^{-a+a}e^{-nt+nt}(u) = u(x,t).$$

The Adomian decomposition method introduce the solution $u(x,t)$ and the non-linear function $F(x,u,\frac{\partial u}{\partial x})$ by infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{2.6}$$

and

$$F(x,u,\frac{\partial u}{\partial x}) = \sum_{n=0}^{\infty} A_n(x,t). \tag{2.7}$$

Where the components $u_n(x,t)$ of the solution $u(x,t)$ will determine recurrently specific algorithums
Adomian polynomials [1, 9, 4].

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F(\sum_{i=0}^{\infty}\lambda^i u_i)\right]_{\lambda=0}. \qquad n = 0,1,2,3,... \tag{2.8}$$

The following algorithum:

$$\begin{aligned}
A_0(x,t) &= f(u_0(x,t)),\\
A_1(x,t) &= u_1 f'(u_0(x,t)),\\
A_2(x,t) &= u_2 f'(u_0(x,t)) + \frac{1}{2!}u_1^2 f''(u_0(x,t)),\\
A_3(x,t) &= u_3 f'(u_0(x,t)) + u_1u_2 f''(u_0(x,t)) + \frac{1}{3!}u_1^3 f'''(u_0(x,t)),\\
&\vdots
\end{aligned}$$

Adomian polynomials by using the alternative method: nonlinear polynomials, nonlinear derivatives, trigonometric nonlinearity, hyperbolic nonlinearity, exponential nonlinearity and logarithmic nonlinearity see this [10].

$$\sum_{n=0}^{\infty} u_n(x,t) = \Phi(x) + l_t^{-1}(\sum_{n=0}^{\infty} A_n(x,t)). \tag{2.9}$$

According to the ADM, all terms of $u(x,t)$ except $u_0(x,t)$ are determined by recursive relation; that is

$$\begin{aligned}
u_0(x,t) &= \Phi(x) + L_t^{-1}(g(x)),\\
u_1(x,t) &= L_t^{-1}(A_0(x,t)),\\
u_2(x,t) &= L_t^{-1}(A_1(x,t)),
\end{aligned} \tag{2.10}$$

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$$u_{n+1}(x,t) = L_t^{-1}(A_n(x,t)) \qquad n \geq 1.$$

3 Numerical illustrations

Example 1

Consider the singular nonlinear first order PDE (in t)

$$u_t + (1+\frac{1}{t})u = \frac{1}{e^t} - uu_x, \qquad u(x,0) = 0, \tag{3.1}$$

solution

Applying the new inverse differential operator $L_t^{-1}(.) = t^{-1}e^{-t}\int_0^t te^t(.)dt$, is defined with $n = 1$ and $a = 1$, and gives

$$u(x,t) = L_t^{-1}(\frac{1}{e^t}) - L_t^{-1}(uu_x) \tag{3.2}$$

Now, the dependent variable $u(x,t)$ and the nonlinear term uu_x are substituted with the infinite series as follows:

$$\begin{cases} u(x,y) = \sum_{n=1}^{\infty} u_n(x,t), \\ uu_x = \sum_{n=1}^{\infty} A_n(x,t), \end{cases}$$

$$\sum_{n=1}^{\infty} u_n(x,t) = \frac{1}{2}te^{-t} + L_t^{-1}(\sum_{n=0}^{\infty} A_n(x,t)). \tag{3.3}$$

This leads to the recursive relation

$$u_0(x,t) = \frac{1}{2}te^{-t},$$

$$u_{n+1}(x,t) = L_t^{-1}A_n, \qquad n \geq 0.$$

Where A_n are the Adomian polynomials.
The first few polynomials for the nonlinear quadratic term uu_x are given by

$$A_0 = u_0u_{0x}$$

$$A_1 = u_1 u_{0x} + u_0 u_{1x}$$

$$A_2 = u_2 u_{0x} + u_0 u_{2x} + u_1 u_{1x}$$

$$\vdots$$

Consequently, the first three terms of the solution $u(x,t)$ are given by

$$u_0(x,t) = \frac{1}{2} t e^{-t},$$

$u_1(x,t) = L_t^{-1} A_0 = L_t^{-1} u_0 u_{0x}$, since $u_{ox} = 0$ then $u_1(x,t) = 0$,
$u_2(x,t) = L_t^{-1} A_1 = L_t^{-1} u_1 u_{0x} + u_0 u_{1x}$, since $u_{1x} = u_{ox} = 0$ then $u_2(x,t) = 0$

$$\vdots$$

$$u_{n+1}(x,t) = L_t^{-1} A_n = 0,$$

thus, the infinite solution in a series form is given by

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + ...,$$

when we found u_0, u_1, u_2, and collected them, we noticed that, the solution by the new Adomian technique is the same as the exact solution.

$$u(x,t) = \frac{1}{2} t e^{-t}. \tag{3.4}$$

Example 2

Solve the singular linear first order PDE (in t)

$$u_t + (4 + \frac{2}{t})u = \frac{x+1}{e^{4t}} - 4u_{xx}, \qquad u(x,0) = 0. \tag{3.5}$$

Solution

Applying the inverse differential operator $L_t^{-1}(.) = t^{-2} e^{-4t} \int_0^t t^2 e^{4t}(.)dt$, defined with $n = 4$, and $a = 2$, gives

$$u(x,t) = L_t^{-1}(\frac{x+1}{e^{4t}}) - L_t^{-1}(4u_{xx}). \tag{3.6}$$

Now, the dependent variable $u(x,t)$ and the nonlinear term uu_x are substituted with the infinite series as follows:

$$\begin{cases} u(x,y) = \sum_{n=1}^{\infty} u_n(x,t), \\ uu_x = \sum_{n=1}^{\infty} A_n(x,t). \end{cases}$$

$$\sum_{n=1}^{\infty} u_n(x,t) = \frac{xt}{3e^{4t}} + \frac{t}{3e^{4t}} - L_t^{-1}(\sum_{n=1}^{\infty} A_n(x,t)). \tag{3.7}$$

This leads to the recursive relation

$$u_0(x,t) = \frac{xt}{3e^{4t}} + \frac{t}{3e^{4t}},$$

$$u_{n+1}(x,t) = -4L_t^{-1}A_n, \qquad n \geq 0$$

where A_n are the Adomian polynomials. The first few polynomials for the nonlinear quadratic term u_{xx} are given by

$$u_0(x,t) = \frac{xt}{3e^{4t}} + \frac{t}{3e^{4t}},$$

$$u_1(x,t) = L_t^{-1}A_0 = -4L_t^{-1}u_{0xx}, \qquad \text{since } u_{0xx} = 0 \text{ then } u_1(x,t) = 0,$$
$$u_2(x,t) = L_t^{-1}A_1 = -4L_t^{-1}(u_{1xx}), \qquad \text{since } u_{1xx} = 0 \text{ then } u_2(x,t) = 0,$$

$$\vdots$$

$$u_{n+1}(x,t) = -4L_t^{-1}A_n = 0,$$

thus, the infinite solution in a series form is given by

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + ...,$$

when we found u_0, u_1, u_2, and collected them we noticed that, the solution by the new Adomian technique is the same as the exact solution.

$$u(x,t) = \frac{xt}{3e^{4t}} + \frac{t}{3e^{4t}}. \tag{3.8}$$

4 Conclusion

In this paper, we offered the new Adomian technique for solving singular and nonsingular partial differential equations of first order. The examples presented in this paper illustrated the quality the new Adomian technique to get to the exact solution. In example 1 and 2 we got the exact solution.

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