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Global hop hub number of a graph

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Abstract

A hub set S of vertices in a graph G is a hop hub set of G if for each $v \in V(G)\setminus S$, there exists $u \in S$ such that $d(u, v) = 2$. A hop hub set h_h is a global hop hub set of G if it is a hop hub set of a graph G and its complement $\bar{G}$. The minimum cardinality of a global hop hub set of G, denoted by $g_{hh}(G)$, is called the global hop hub number of G. In this paper, we determine the global hop hub number of some standard graphs. Also, upper and lower bounds for $g_{hh}(G)$ are obtained.

Keywords: *Hub number, Hop hub number, Global domination number, Global hub number, Global hop hub number.*

1. Introduction

Throughout this paper, all graphs $G = (V, E)$, are assumed to be simple, finite and connected, and Γ is called (n, m) graph, where $|V| = n$ and $|E| = m$ In general, the degree of a vertex ϖ in a graph Γ denoted by $\delta\varepsilon\gamma(\varpi)$ is the number of edges of Γ incident with ϖ [7]. We denote the maximum degree of Γ by $\Delta(\Gamma)$, the minimum degree of Γ by $\delta(\Gamma)$. For more details we refer to [5, 6, 7].

A hub set in graph theory was introduced in 2006 by Walsh [19], a hub set S of G is a set of vertices with the property that for any pair of vertices outside of S, there is a path between them with all intermediate vertices in S. The hub number h(G) is then defined to be the minimum cardinality of hub set of G. Khalaf et al. [10], have studied the global hub set hg(G) of graphs.

In 2021, Sand and Mahde [16] have introduced the concept of hop hub set of a graph G, which is defined as follows. A hub set S ⊆ V (G) is hop hub set of G if for each $v \in V(G) - S$, there exists u ∈ S such that $d(v, u) = 2$, the minimum cardinality of a hop hub set of G, is called the hop hub number and is denoted by hh(G). Also they determined the hop hub number of some standard graphs and obtained some upper and lower bounds for hh(G). They also, studied the relationship between $h_h(G)$ and other graph parameter, for more details on the hub studies we refer to [9, 10, 11, 17].

A set $S \subseteq V(G)$ is called a dominating set of G if each vertex of $V - S$ is adjacent to at least one vertex of S. The domination number of a graph G denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G [8].

A dominating set $S \subseteq V(G)$ is a hop dominating set of G if for every $v \in V(G) - S$, there exists $u \in S$ such that $d(v, u) = 2$. The minimum cardinality of a hop dominating set of G, is called the global hop domination number of G. and is denoted by $\gamma_{gh}(G)$. The hop domination number of G was studied in [1, 2, 3, 12, 13, 14].

Salasalan et al. [15], have introduced the concept of global hop dominating set of G, which is defined as: A set $S \subseteq V(G)$ is a global hop dominating set of G if it is a hop dominating set of both G and its complement $\overline{G}$. The minimum cardinality of global hop dominating set of G, denoted by $\gamma_{gh}(G)$, is called the global hop domination number of G. Using the concept of global hub set of a graph G and the definition of the hop hub number of a graph G, motivated by this, we introduce the concept of global hop hub number of a graph G as a new parameter of a graph.

The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex set, two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G [7]. A Friendship graph F_p, for $p \geq 2$, is the graph constructed by joining p copies of K_3 with a common vertex. F_p graph has $2p + 1$ vertices and p edges [4]. The **eccentricity** $e(v)$ of a vertex $v \in V$ in a connected graph G is defined as $e(v) = max\{d(v, u) : u \in V(G)\}$. the **diameter** $diam(G)$ of a connected graph G is defined as $diam(G) = max\{e(v) : v \in V(G)\}$ [7].
The following result will be useful in the proof of our results.

Theorem 1.1. [7] *For any complete graph G, $\chi(G) = n$.*

2. **Main Results**

Definition 2.1. [16] *A hub set S is a hop hub set of G if for every $v \in V - S$, there exists $u \in S$ such that $d(v, u) = 2$, the minimum cardinality of a hop hub set of G is called a hop hub number and is denoted by $h_h(G)$.*

Definition 2.2. *A hop hub set S is a global hop hub set of G if it is a hop hub set of both G and its complement $\overline{G}$, the minimum cardinality of global hop hub set of G is called global hop hub number and is denoted by $g_{hh}(G)$.*
It is clear that $g_{hh}(G)$ is well defined for any graph G, since $V(G)$ is a global hop hub set of G. Also it is obvious that $g_{hh}(G) \geq h_h(G)$ since any global hop hub set, is global hub set of G, and for any global hop hub set of G, $g_{hh}(G) = g_{hh}(\overline{G})$.

Proposition 2.3. (1): *For any path P_n, $n \geq 4$, $g_{hh}(P_n) = n - 2$.*

(2): *For any complete graph* K_n, $g_{hh}(K_n) = n$.

(3): *For any cycle graph* C_n, $n \geq 3$, $g_{hh}(C_n) = \begin{cases} 3, & \text{if } n = 3, 5; \\ 4, & \text{if } n = 4; \\ n-3, & \text{if } n \geq 6. \end{cases}$

(4): *For the star graph* $K_{1,n-1}$, $g_{hh}(K_{1,n-1}) = n$.

(5): *For the complete bipartite graph* $K_{n,m}$, $g_{hh}(K_{n,m}) = n + m$.

(6): *For any double star* $S_{n,m}$, $n, m \geq 2$, $g_{hh}(S_{n,m}) = 2$.

(7): *For the corona graph* $(P_2 \circ K_n)$, $g_{hh}(P_2 \circ K_n) = 2$.

Proof. **1 -:** Let $n \geq 4$ and suppose that $V(P_n) = \{v_1, v_2, ..., v_n\}$ be the vertices of P_n, consider $S = \{v_2, v_3, ..., v_{n-1}\}$ is a global hop hub set of P_n such that $|S| = n - 2$.

To prove that S is a minimum global hop hub set of P_n, if we remove one vertex v_i from S, such that $2 \leq i \leq n-1$, then the set $S - \{v_i\}$, $2 \leq i \leq n-1$, is not a hub set of P_n, so the set S is a minimum global hop hub set, therefore $g_{hh}(P_n) = n - 2$.

2- Let $V(K_n) = \{v_1, v_2, ..., v_n\}$ be the vertices of K_n, consider $S = \{v_1, v_2, ..., v_n\}$ is a global hop hub set of K_n such that $|S| = n$, since $h_h(K_n) = n$ and for any graph $g_{hh}(G) \geq h_h(G)$ then $g_{hh}(K_n) = n$.

To show that S is a minimum global hop hub set of K_n, if one vertex of v_i, $1 \leq i \leq n$ is removed from set S, then the set $S - \{v_i\}$, $1 \leq i \leq n$ is not a hop hub set of K_n because $d(v_1, v_i) = 1$, therefor S is minimum global hop hub set, and hence $g_{hh}(K_n) = n$.

3- Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ be the vertices of C_n, we discuss the following cases:

Case 1: $n = 3, 4$, since $C_3 \cong K_3$ form part 2, $g_{hh}(C_3) = 3$ and if $n = 4$, consider $\{v_1, v_2, v_3, v_4\}$ is a global hop hub set of C_4 and since $\overline{C_4} = 2K_2$ we must choose all vertices as hop hub set of C_4, so $g_{hh}(C_4) = 4$.

Case 2: If $n = 5$, consider $S = \{v_1, v_2, v_3\}$ is a global hop hub set of C_5 as shown in Figure 1. It is clear S is minimum global hop hub set of C_5, therefore $g_{hh}(C_5) = 3$.

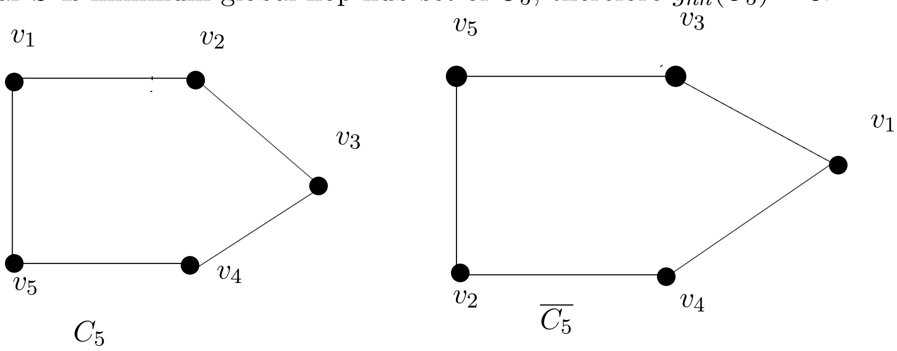

Figure 1 : C_5 and $\overline{C_5}$

Case 3 : when $n \geq 6$, consider $S = \{v_3, v_5, v_6, v_7, ..., v_n\}$ is a global hop hub set of C_n such that $|S| = n - 3$.

To prove that S is a minimum global hop hub set of C_n, if we remove one vertex v_i from S such that $3 \leq i \leq n$, $i \neq 4$ then the set $S - \{v_i\}$, $3 \leq i \leq n$, $i \neq 4$ is not hub set because there are some of vertices outside $S - \{v_i\}$ then there not exist path between them so S is a minimum global hop hub set of C_n and hence $g_{hh}(C_n) = n - 3$.

4- Let $V(K_{1,n-1}) = \{v, v_1, v_2, ..., v_{n-1}\}$ be the vertices of $K_{1,n-1}$, consider $S = V(K_{1,n-1}) = \{v, v_1, v_2, ..., v_{n-1}\}$ is a global hop hub set of $K_{1,n-1}$ and since $\overline{K_{1,n-1}} = K_{n-1} \cup K_1$ in this case we must choose all vertices of $K_{n-1} \cup K_1$ as a hop hub set, so $|S| = n$.

To prove that S is the minimum global hop hub set of $K_{1,n-1}$, if the vertex v is removed from S, $d(v_1, v_i) = 1$, $1 \leq i \leq n - 1$, and hence $S - \{v\}$ is not hop hub set of $K_{1,n-1}$. Also, if v_1 is removed from S, the set $v - \{v_1\}$ is not a hop hub set of $\overline{K_{1,n-1}}$, because $d(v_1, v_i) = 1$, and hence $g_{hh}(K_{1,n-1}) = n$.

5- Let $V(K_{n,m}) = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_m\}$ be the vertices of $k_{n,m}$, consider $S = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_m\}$ is a global hop hub set of $K_{n,m}$, and since $\overline{K_{n,m}} = K_n \cup K_m$ in this case, we get the result from part 2, so $|S| = n + m$. It is clear that S is a minimum global hop hub set of $K_{n,m}$.

6- Let $V(S_{n,m}) = \{v, v_1, v_2, ..., v_{n-1}, u, u_1, u_2, ..., u_{n-1}\}$ be the vertices of $S_{n,m}$, consider $S = \{v, u\}$ is a global hop hub set of $S_{n,m}$, since $h_h(S_{n,m}) = 2$ and since for any graph G $g_{hh}(G) \geq h_h(G)$, then $|S| = 2$.

To prove that S is a minimum global hop hub set of $S_{n,m}$, if the vertex u or v is removed from S then set $S = \{v\}$ or $S = \{u\}$ is not a hub set of $S_{n,m}$ and some vertices in $\overline{S_{n,m}}$ such that $d(u, v_{n-1}) = 1$, $1 \leq i \leq n - 1$ or $d(v, u_{m-1}) = 1, 1 \leq i \leq m - 1$ so S is a minimum global hop hub set of $S_{n,m}$, therefor $g_{hh}(S_{n,m}) = 2$.

7- Let $V(P_2 \circ K_n) = \{u, u_1, u_2, ..., u_n, v, v_1, v_2, ..., v_n\}$ be the vertices of $P_2 o K_n$, consider $S = \{u, v\}$ is a global hop hub set of $P_2 \circ K_n$, since $h_h(P_2 \circ K_n) = 2$ and since for any graph G, $g_{hh}(G) \geq h_h(G)$ then $|S| = 2$. It is clear that S is a minimum global hop hub set of $P_2 \circ K_n$.

□

Theorem 2.4. *For the wheel graph $W_{1,n-1}$, $n \geq 4$*

$$g_{hh}(W_{1,n-1}) = \begin{cases} 4, & \textit{if } n = 4, 6; \\ 5, & \textit{if } n = 5; \\ 2r + 1, & \textit{if } n = 6r, 6r - 1, 6r + 1; \\ 2r + 2, & \textit{if } n = 6r + s, 2 \leq s \leq 4 \ . \end{cases}$$

Proof. Let $V(W_{1,n-1}) = \{v, v_1, v_2, ..., v_{n-1}\}$ be the vertices of $W_{1,n-1}$, we have the following cases:

Case 1: $n = 4$. Then $S = \{v, v_1, v_2, v_3\}$ is the unique minimum a global hop hub set of $W_{1,3}$, and $\overline{W_{1,3}} = 4\overline{K_1}$, so $|S| = 4$.

Case 2: $n = 5$. Then $S = \{v, v_1, v_2, v_3, v_4\}$ is the only minimum global hop hub set of $W_{1,4}$ and $\overline{W_{1,4}} = 2K_2 \bigcup K_1$, so that $|S| = 5$.

Case 3:$n = 6$, consider $S = \{v, v_1, v_4, v_5\}$ is a global hop hub set of $W_{1,5}$ such that $|S| = 4$, so $g_{hh}(W_{1,5}) = 4$.

Case 4: when $n = 6r, 6r - 1, 6r + 1$ we have the following cases:

Case 4.1: when $n = 6r$, where $r \geq 2$, consider $S = \{v, v_1, v_4, v_7, ..., v_{6r-2}\}$ is a global hop hub set of $W_{1,n-1}$ such that $|S| = 2r + 1$.

Case 4.2: when $n = 6r - 1$, where $r \geq 2$, consider $S = \{v, v_1, v_4, v_7, ..., v_{6r-2}\}$ is a global hop hub set of $W_{1,n-1}$ such that $|S| = 2r + 1$.

Case 4.3: when $n = 6r + 1$, where $r \geq 1$, consider $S = \{v, v_1, v_4, v_7, ..., v_{6r-2}\}$ is a global hop hub set of $W_{1,n-1}$ such that $|S| = 2r + 1$.

To prove S is a minimum global hop hub set of $W_{1,n-1}$, if we remove one vertex v_i from set S, then there exists one vertex of $S - \{v_i\}$ there is not a path between them, so S is a minimum global hop hub set of $W_{1,n-1}$, then $g_{hh}(W_{1,n-1}) = 2r + 1$.

Case 5: when $n = 6r + s$, $2 \leq s \leq 4$ we have the following cases:

Case 5.1: when $n = 6r + 2$, where $r \geq 1$, consider $S = \{v, v_1, v_4, v_7, ..., v_{6r+1}\}$ is a global hop hub set of $W_{1,n-1}$ such that $|S| = 2r + 2$.

Case 5.2: when $n = 6r + 3$, where $r \geq 1$, consider $S = \{v, v_1, v_4, v_7, ..., v_{6r+1}\}$ is a global hop hub set of $W_{1,n-1}$ such that $|S| = 2r + 2$.

Case 5.3: when $n = 6r + 4$, where $r \geq 1$, consider $S = \{v, v_1, v_4, v_7, ..., v_{6r+1}\}$ is a global hop hub set of $W_{1,n-1}$ such that $|S| = 2r + 2$.

To prove that S is a minimum global hop hub set of $W_{1,n-1}$, if we remove one vertex v_i from set S, then there exist vertices $v_j \in \overline{G} - S$, $1 \leq j \leq n - 1$, such that $d(v_i, v_j) = 1$, $i \neq j$. So S is a minimum global hop hub set of $W_{1,n-1}$. Therefore $g_{hh}(W - 1, n - 1) = 2r + 2$. □

Proposition 2.5. *For the friendship graph F_p, $p \geq 2$. Then $g_{hh}(F_p) = p + 1$.*

Proof. Let $V(F_p) = \{v, v_1, v_2, v_3, ..., v_{2p}\}$ be a vertex set of F_p, suppose that $S = \{v, v_2, v_4, v_6, ..., v_{2p}\}$ is a global hop hub set of F_p, such that $| S |= p + 1$.

To prove that S is a minimum global hop hub set of F_p, if the vertex v is removed from S then for any vertex of the set $\{v_1, v_3, v_5, ..., v_{2p-1}\}$ there is no exist path between them. Therefore, $g_{hh}(F_p) = p + 1$. □

Theorem 2.6. *For any graph G, $2 \leq g_{hh}(G) \leq n$. The lower bound is sharp for $G = S_{n,m}$ and the upper bound is sharp for $G = K_n$.*

Theorem 2.7. *For any connected graph G, if $g_{hh}(G) = 2$, then $diam(G) \leq 3$.*

Proof. Let $g_{hh}(G) = 2$, to prove that $diam(G) \leq 3$, suppose that $diam(G) > 3$, then by definition of diam(G), there exists a path between five vertices at least and $g_{hh}(P_5) = 3$ which is a contradiction to $g_{hh}(G) = 2$ and hence diam$(G) \leq 3$.
The converse is not true, for example $G = K_3$, with diam$(K_3) = 1$ and $g_{hh}(K_3) = 3$. □

Theorem 2.8. *For any connected graph G, $g_{hh}(G) = 2$ if and only if $G \cong S_{n,m}$ and $G \cong (P_2 \circ K_n)$.*

Corollary 2.9. *Let $G = C_n$ with order $n \geq 6$, or $G = P_n$ with order $n \geq 4$, and $\Delta(G) = 2$. Then $g_{hh}(G) = h_h(G) = h_g(G) = h(G)$.*

Theorem 2.10. *For any connected graph G $g_{hh}(G) = n$ if and only if $G \cong K_n$ or $G \cong K_{n,m}$ and $G \cong W_5$.*

Proof. Consider $g_{hh}(G) = n$, this means that all vertices of a graph G are adjacent, and in this case $G \cong K_n$, or complement of G is $pK_n \cup qK_m, n, m \geq 1, p, q \geq 1$ and this is achieved if $G \cong K_{n,m}$ and $G \cong W_5$.
Conversely, let $G \cong K_n, K_{n,m}$ or W_5. The proof is clear. □

Remark 2.1. *If G be double star graph $S_{n,m}$, then*

$$g_{hh}(G) = h_h(G) = h_g(G) = h(G).$$

Theorem 2.11. *For any connected graph G, $g_{hh}(G) \geq h_g(G)$.*

Proof. Every hop hub set is a hub set, we have $g_{hh}(G) \geq h_g(G)$. □

Proposition 2.12. *If G is complete graph, then $\chi(G) = g_{hh}(G)$.*

Proof. From Theorem 1.1 and Proposition 2.3, we get the result. □

Remark 2.2. *The converse of theorem 2.12 is not true, for example $G = S_{5,5}$ as shown in figure 2.*

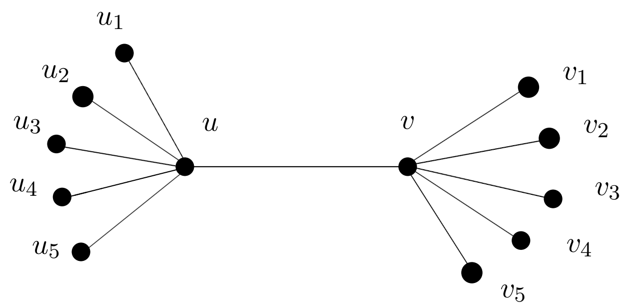

Figure 2: double star $S_{5,5}$

Note that $\chi(G) = 2$ *and* $g_{hh}(G) = 2$*, but it is not complete.*

3. **Conclusion**

In this paper, we introduced the concept of global hop hub number of connected graphs of order $n \geq 2$. We obtained the bounds and some properties for global hop hub number of graphs. The global hop hub number of several other families of graphs is an open problem.

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