Sum Geometric Harmonic Means Index of Graphs

Sultan Senan Mahde, Ebrahim Mohammed Abdallah
Department of Mathematics, Faculty of Education and Science, Rada'a, Albaydha University, Yemen
sultan.mahde@gmail.com, ebrahim.m.a.n.s.155@gmail.com
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Abstract
Let G be a connected graph, the sum geometric - arithmetic means index of G is defined as $\mathrm{SGAM(G)} = \sum_{\mathrm{uv \in E(G)}} \left(\sqrt{\mathrm{d(u)d(v)}} + \frac{\mathrm{d(v)+d(v)}}{2} \right)$. In this paper, the concept of sum geometric harmonic means index of a graph G, denoted by $\mathrm{SG_{hm}(G)}$ is introduced and $\mathrm{SG_{hm}(G)}$ of few families of graphs is computed. Further, we establish the bounds for sum geometric harmonic means index.
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1. Introduction

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$and edge set E(G). Its order is $|V(G)|$, denoted by n, and its size is $|E(G)|$, denoted by m.
The degree of a vertex v of a simple graph $G = (V, E)$, denoted by $d(v)$, is the number of vertices adjacent to v. We denote the maximum degree of G by $\Delta(G)$, the minimum degree of G by $\delta(G)$, where $\Delta(G) = max\{d(v): v \in V(G)\}$ } and $\delta(G) = min\{d(v): v \in V(G)\}$ [4].

Definition 1.1. *A molecular graph is a simple graph whose vertices correspond to the atoms and whose edges correspond to the bonds. It can be described in different ways, suth as by a drawing, a polynomial, a sequence of numbers, a matrix or by a derived number called a topological indices[1].*

Topological indices are numerical parameters of chemical graphs associated with quantitative structure property relationship (QSPR) and quantitative structure activity relationship (QSAR).

The study of topological indices is a subject of increasing interest, both in pure and applied mathematics. Hundreds of topological indices have been introduced and studied, starting with the seminal work by Wiener in which he used the sum of all shortest-path distances of a (molecular) graph for modeling physical properties of alkanes. The first topological index was introduced in 1947 by Harry Wiener and used for correlation with boiling points of alkanes. Wiener's index is related to the distances in molecular graphs [7].

There are various topological indices for example distance based topological indices and degree based topological indices ABC index, ABC4 index, Randic connectivity index and sum connectivity index. For more detail we refer to [19].

The concept of geometric - arithmetic index was introduced in the chemical graph theory recently and defined, as

$$GA_1(G) = \sum_{uv \in E(G)} \left(\frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} \right),$$

where $d(u), d(v)$ denotes the degree of a vertex u and the degree of a vertex v in G respectively. In chemical graph theory, we have many different topological index of arbitrary molecular graph G. A topological index of graphs is a member related to a graph which is invariant under graph automorphism obviously, every topological index defines a counting polynomial and vice versa [12]. In 2012, Zhong presented the minimum and maximum values of the harmonic index for simple connected graphs and trees and characterized the corresponding extremal graphs [23]. In 2012, Das and Trinajstic compared the geometric - arithmetic indices for chemical trees, starlike trees and general trees [3]. In 2014, Rodrıguez and Sigarreta [17] studied the geometric - arithmetic index GA_1 from an algebraic viewpoint.

In 2018, Deng et al., obtained the first and second maximum values of geometric – arithmetic index for all tricyclic graphs on n vertices and the corresponding extremal graphs [4]. In 2022, Mohsen and Mahde [14] have introduced the concept of sum geometric - arithmetic means index of graphs, they computed the sum geometric - arithmetic means index $SGAM(G)$ of several classes of graphs, also they detemined bounds in terms of other graph parameters, The Sum geometric - arithmetic means index was defined as

$$SGAM(G) = \sum_{uv \in E(G)} \left(\sqrt{d(u)d(v)} + \frac{d(u)+d(v)}{2} \right).$$

For history and further results on this family of topological indices, please refer to [6, 7, 12, 13, 19, 22].

In this paper, we consider another variant of the sum geometric index, called the sum geometric harmonic means index $SG_{hm}(G)$, which is defined as

$$SG_{hm}(G) = \sum_{uv \in E(G)} \left(\sqrt{d(u)d(v)} + \frac{2}{\frac{1}{d(u)} + \frac{1}{d(v)}} \right).$$

2 Main Results

In this section, we investigate the sum geometric harmonic means index of some standard graphs.

Calculate the index by using the edge partition of G on the basis of the degrees of end vertices of each edge method.

Definition 2.1. *[20] The cycle graph* $(C_n), n \geq 3,$ *consists of* n *vertices* $\{v_1, v_2, ..., v_n\}$ *and edges* $\{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$.

Theorem 2.2. *For the cycle graph* (C_n), $n \geq 3$,

$$SG_{hm}(C_n) = 4n.$$

Proof. Since C_n is regular graph, of order 2, the number of edges is n and because its vertex set can not be partitioned into subsets, and so has only one set, then the number of edges of C_n on the basis of the degrees of the vertices of each edge is equal to the number of edges of C_n and because

$$SG_{hm}(G) = \sum_{uv \in E(G)} \left(\sqrt{d(u)d(v)} + \frac{2}{\frac{1}{d(u)} + \frac{1}{d(v)}} \right),$$

then

$$\begin{aligned} SG_{hm}(C_n) &= n \left(\sqrt{4} + \frac{2}{\frac{1}{2} + \frac{1}{2}} \right) \\ &= n \left(2 + \frac{2}{1} \right) \\ &= n(2+2) = 4n. \end{aligned}$$

□

Definition 2.3. *[20] The graph obtained from the cycle graph (C_n), $n \geq 3$, by removing an edge is called the path graph of n vertices, it is denoted by P_n.*

Theorem 2.4. *For the path graph $(P_n), n \geq 3$, the SG_{hm} index is equal to the following*

$$SG_{hm}(P_n) = 2\sqrt{2} + \frac{8}{3} + 4(n-3).$$

Proof. Suppose that $V(P_n) = \{u_1, u_2, ..., u_n\}, E(P_n) = \{e_1, e_2, ..., e_{n-1}, e_i = u_i u_{i+1} (1 \leq i \leq n-1)\}$. The number of edges of P_n is $n-1$ in which, there are two types of edges, in the first type $d(u) = 1, d(v) = 2$, and number of edges is 2 and in the second type $d(u) = d(v) = 2$ and number of edges is $n-3$ and because

$$SG_{hm}(G) = \sum_{uv \in E(G)} \left(\sqrt{d(u)d(v)} + \frac{2}{\frac{1}{d(u)} + \frac{1}{d(v)}} \right),$$

we have,

$$\begin{aligned} SG_{hm}(P_n) &= 2 \left(\sqrt{2} + \frac{4}{3} \right) + (n-3) \left(\sqrt{4} + \frac{2}{1} \right) \\ &= 2\sqrt{2} + \frac{8}{3} + 4(n-3). \end{aligned}$$

□

Definition 2.5. *[9] The double star graph $(S_{r,s})$ is the graph constructed from $(K_{1,r-1})$ and $(K_{1,s-1})$ by joining their centers v_0 and u_0. A vertex set $V(S_{r,s}) = V(K_{1,r-1}) \bigcup V(K_{1,s-1}) = \{v_0, v_1, ..., v_{r-1}, u_0, u_1, ..., u_{s-1}\}$ and edge set $E(S_{r,s}) = \{v_0u_0, v_0v_i, u_0u_j | 1 \leq i \leq r-1, 1 \leq j \leq s-1\}$.*

Theorem 2.6. *For the double star graph $(S_{r,s})$ for an integer $(r, s) \geq 3$,*

$$SG_{hm}(S_{r,s}) = (r-1)\sqrt{r} + (s-1)\sqrt{s} + \sqrt{rs} + 2\left(\frac{r(r-1)}{1+r} + \frac{s(s-1)}{1+s} + \frac{rs}{r+s}\right).$$

Proof. By categorizing the edge in $(S_{r,s})$ edge partition according to the degrees of their connecting vertices, we can identify three distinct edge cases:
Case 1: $d(u_0) = r, d(u_1) = 1$ and the number of edges is $r-1$,
Case 2: $d(v_0) = s, d(v_1) = 1$ and the number of edges is $s-1$,
Case 3: $d(u_0) = r, d(v_0) = s$ and the number of edges is 1, and because

$$SG_{hm}(G) = \sum_{uv \in E(G)} \left(\sqrt{d(u)d(v)} + \frac{2}{\frac{1}{d(u)} + \frac{1}{d(v)}}\right),$$

then

$$\begin{aligned} SG_{hm}(S_{r,s}) &= (r-1)\left(\sqrt{r} + \frac{2}{\frac{1}{r} + \frac{1}{1}}\right) \\ &+ (s-1)\left(\sqrt{s} + \frac{2}{\frac{1}{s} + \frac{1}{1}}\right) + \left(\sqrt{rs} + \frac{2}{\frac{1}{r} + \frac{1}{s}}\right) \\ &= (r-1)\sqrt{r} + (s-1)\sqrt{s} + \sqrt{rs} + 2\left(\frac{r(r-1)}{1+r} + \frac{s(s-1)}{1+s} + \frac{rs}{r+s}\right). \end{aligned}$$

□

Definition 2.7. *[20] A simple graph G is said to be complete if every vertex in G is connected with every other vertex, i.e if G contains exactly one edge between each pair of distinct vertices, and is usually denoted by K_n and exactly $\frac{n(n-1)}{2}$ edges.*

Theorem 2.8. *For the complete graph (K_n), the SG_{hm} index is equal to the following*

$$SG_{hm}(K_n) = n(n-1)^2.$$

Proof. Since K_n is a regular graph of order $n-1$ and the edge partition of K_n on the basis of the degrees of the end vertices of edge, and because its vertex set can not be partitioned into subsets, and so has only one set, then the number of edges of the vertices of each edge is equal to the number of edges of K_n and because

$$SG_{hm}(G) = \sum_{uv \in E(G)} \left(\sqrt{d(u)d(v)} + \frac{2}{\frac{1}{d(u)} + \frac{1}{d(v)}} \right),$$

it follows that

$$\begin{aligned} SG_{hm}(K_n) &= \frac{n(n-1)}{2} \left(\sqrt{(n-1)(n-1)} + \frac{2}{\frac{1}{n-1} + \frac{1}{n-1}} \right) \\ &= \frac{n(n-1)}{2} \left(\sqrt{(n-1)^2} + \frac{2(n-1)}{2} \right) \\ &= \frac{n(n-1)}{2} (n-1+n-1) \\ &= \frac{n(n-1)}{2} \quad (2n-2) \\ &= \frac{n(n-1)}{2} \quad 2\,(n-1) \\ &= n(n-1)(n-1) = n(n-1)^2. \end{aligned}$$

□

Definition 2.9. *[11] A graph G is called a bipartite graph if the vertex set V can be partitioned into subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. Further more, if every vertex of V_1 is joined to every vertex of V_2, then G is a complete bipartite graph. The complete bipartite graph with two partite sets V_1 and V_2 of vertices such that $|V_1| = r$ and $|V_2| = s$ is denoted by $(K_{r,s})$ the graph $(K_{1,n-1})$ is a star.*

Theorem 2.10. *For the complete bipartite graph $(K_{r,s})$,*

$$SG_{hm}(K_{r,s}) = (rs)\left(\sqrt{rs} + \frac{2rs}{r+s} \right).$$

Proof. By using the edge partition of $K_{r,s}$ on the basis of the degrees of the vertices of each edge because it is vertex set can be partitioned into subsets, namely X and

Y such that $\forall u \in X$, $d(u) = s$ and $\forall v \in Y$, $d(v) = r$ where $\mid E(K_{r,s}) \mid = rs$. then the number of edges of $K_{r,s}$ on the basis of the degrees of the vertices of each edge is such that $d(u) = s$ and $d(v) = r$ is equal to the number of edges, of $K_{r,s}$. and because

$$SG_{hm}(G) = \sum_{uv \in E(G)} \left(\sqrt{d(u)d(v)} + \frac{2}{\frac{1}{d(u)} + \frac{1}{d(v)}} \right),$$

this implies that

$$\begin{aligned} SG_{hm}(k_{r,s}) &= (rs)\left(\sqrt{rs} + \frac{2}{\frac{1}{r} + \frac{1}{s}}\right) \\ &= (rs)\left(\sqrt{rs} + \frac{2rs}{r+s}\right) \\ &= rs\left(\sqrt{rs} + \frac{2rs}{r+s}\right). \end{aligned}$$

□

Corollary 2.11. *For the star graph* $(K_{1,n-1})$,

$$SG_{hm}(K_{1,n-1}) = (n-1)^{\frac{3}{2}} + \frac{2(n-1)^2}{n}.$$

Definition 2.12. *[16] A Friendship graph F_p for an integer $p \geq 2$, is the graph constructed by joining p copies of K_3 graph with common vertex. F_p graph has $n = 2p + 1$ vertices and has $m = 3p$ edges.*

Theorem 2.13. *For the friendship graph F_p for an integer $p \geq 2$,*

$$SG_{hm}(F_p) = 4p\left(\sqrt{p} + \frac{2p}{p+1} + 1\right).$$

Proof. By categorizing the edge in F_p edge partition according to the degrees of their connecting vertices, we can identify two distinct edge types:
in the first type $d(u) = d(v) = 2$ and number of edges is p and
in the second type $d(u) = 2, d(v) = 2p$ and number of edges is $2p$ and because

$$SG_{hm}(G) = \sum_{uv \in E(G)} \left(\sqrt{d(u)d(v)} + \frac{2}{\frac{1}{d(u)} + \frac{1}{d(v)}} \right),$$

we have,

$$\begin{aligned} SG_{hm}(F_p) &= 2p\left(\sqrt{2(2p)} + \frac{2}{\frac{1}{2p}+\frac{1}{2}}\right) + p\left(\sqrt{4} + \frac{2}{\frac{1}{2}+\frac{1}{2}}\right) \\ &= 2p\left(2\sqrt{p} + \frac{2(2p)}{p+1}\right) + 4p \\ &= 4p\sqrt{p} + \frac{4p(2p)}{p+1} + 4p \\ &= 4p\left(\sqrt{p} + \frac{2p}{p+1} + 1\right). \end{aligned}$$

□

Definition 2.14. *[20] The wheel graph W_n is obtained when an additional vertex to the cycle C_{n-1}, for $n \geq 4$, and connect this new vertex to each of the $n-1$ vertices in (C_{n-1}), by new edges.*

Theorem 2.15. *For the wheel graph $(W_n), n \geq 4$,*

$$SG_{hm}(W_n) = (n-1)\left(\sqrt{3(n-1)} + \frac{6(n-1)}{n+2} + 6\right).$$

Proof. Let $V(W_n) = \{v_1, v_2, ..., v_n\}$, $E(W_n) = \{e_1, e_2, ..., e_{2n-2}\}$. By using the edge partition of W_n on the basis of the degrees of the vertices of each edge, there are two types of edges, in the first type $d(u) = 3, d(v) = n-1$ and number of edges is $n-1$ and in the second type $d(u) = d(v) = 3$ and number of edges is $n-1$ and because

$$SG_{hm}(G) = \sum_{uv \in E(G)} \left(\sqrt{d(u)d(v)} + \frac{2}{\frac{1}{d(u)} + \frac{1}{d(v)}}\right),$$

this implies that

$$\begin{aligned} SG_{hm}(W_n) &= (n-1)\left(\sqrt{3(n-1)} + \frac{2}{\frac{1}{n-1}+\frac{1}{3}}\right) + \left(\sqrt{(3)(3)} + \frac{2}{\frac{1}{3}+\frac{1}{3}}\right)(n-1) \\ &= (n-1)\left(\sqrt{3(n-1)} + \frac{6(n-1)}{n+2}\right) + (n-1)(6) \\ &= (n-1)\left(\sqrt{3(n-1)} + \frac{6(n-1)}{n+2} + 6\right). \end{aligned}$$

□

Definition 2.16. *[15] The crown graph (S_r^0) for an integer $r \geq 2$ is the graph with vertex set $\{u_1, u_2, ..., u_r, v_1, v_2, ..., v_r\}$ and edge set $\{u_i v_j : 1 \leq i, j \leq r, i \neq j\}$. S_r^0 is therefore equivalent to the complete bipartite graph $K_{r,s}$ with horizontal edges removed.*

Theorem 2.17. *For the crown graph (S_r^0), $r \geq 2$,*

$$SG_{hm}(S_r^0) = 2r(r-1)^2.$$

Proof. Suppose that $V(S_r^0) = \{u_1, u_2, ..., u_r, v_1, v_2, ..., v_r\}$ and edge set $E(S_r^0) = \{u_i v_j : 1 \leq i, j \leq r-1, i \neq j\}$. Since in every edge e_{ij} in S_r^0 has $d(u_i) = d(v_j) = r-1$, thus the edge partition of S_r^0 on the basis of the degrees of the vertices can not be partitioned into more one subset and equal to the number of edges of S_r^0 and because

$$SG_{hm}(G) = \sum_{uv \in E(G)} \left(\sqrt{d(u)d(v)} + \frac{2}{\frac{1}{d(u)} + \frac{1}{d(v)}} \right),$$

therefore

$$\begin{aligned} SG_{hm}(S_r^0) &= r(r-1) \left(\sqrt{(r-1)(r-1)} + \frac{2}{\frac{1}{r-1} + \frac{1}{r-1}} \right) \\ &= r(r-1) \left(\sqrt{(r-1)^2} + \frac{2}{\frac{2}{r-1}} \right) \\ &= r(r-1) \left(r - 1 + \frac{2(r-1)}{2} \right) \\ &= r(r-1)\,(r-1+r-1) \\ &= r(r-1)(2)(r-1) \\ &= 2r(r-1)^2. \end{aligned}$$

□

Theorem 2.18. *Let G be a simple connected graph of order n with m edges, Δ and δ be the maximum and minimum degree of the graph, respectively. Then*

$$2\delta m \leq SG_{hm}(G) \leq 2\Delta m.$$

The equality holds if and only if G is a regular graph.

Proof. Since

$$\delta \leq \sqrt{d(u)d(v)} \leq \Delta, \delta \leq \frac{2}{\frac{1}{d(u)} + \frac{1}{d(v)}} \leq \Delta,$$

therefore

$$2\delta \leq \sqrt{d(u)d(v)} + \frac{2}{\frac{1}{d(u)} + \frac{1}{d(v)}} \leq 2\Delta.$$

Because

$$SG_{hm}(G) = \sum_{uv \in E(G)} \left(\sqrt{d(u)d(v)} + \frac{2}{\frac{1}{d(u)} + \frac{1}{d(v)}} \right),$$

we get,

$$\sum_{uv \in E(G)} 2\delta \leq SG_{hm}(G) \leq \sum_{uv \in E(G)} 2\Delta.$$

Since $|E(G)| = m$, it follows that

$$2\delta m \leq SG_{hm}(G) \leq 2\Delta m.$$

□

Theorem 2.19. *For any connected graph G,*

$$SGAM(G) \geq SG_{hm}(G),$$

with equality holding if and only if G is a regular graph.

Proof. To prove this relationship, we assume the series with only two observations $d(u)$ and $d(v)$ (where $d(u)$ and $d(v)$ are positive vertices degree) for sake of simplicity and clarity although the relationship holds for any number of observations

since,

$$SGAM(G) = \sum_{uv \in E(G)} \left(\sqrt{d(u)d(v)} + \frac{d(u) + d(v)}{2} \right),$$

and

$$\sum_{uv\in E(G)}\left(\sqrt{d(u)d(v)}+\frac{2}{\frac{1}{d(u)}+\frac{1}{d(v)}}\right),$$

we want to show that

$$\frac{d(u)+d(v)}{2}\geq\frac{2}{\frac{1}{d(u)}+\frac{1}{d(v)}},$$

we know that

$$\begin{aligned}
&\left(\sqrt{d(u)}-\sqrt{d(v)}\right)^2\geq 0\\
=&\left(\sqrt{d(u)}\right)^2-2\sqrt{d(u)}\sqrt{d(v)}+\left(\sqrt{d(v)}\right)^2\geq 0\\
=&d(u)-2\sqrt{d(u)d(v)}+d(v)\geq 0\\
=&d(u)+d(v)-2\sqrt{d(u)d(v)}\geq 0\\
=&d(u)+d(v)\geq 2\sqrt{d(u)d(v)}\\
=&\frac{d(u)+d(v)}{2}\geq\sqrt{d(u)d(v)}.
\end{aligned}\tag{1}$$

We have,

$$\begin{aligned}
&\left(\sqrt{d(u)}-\sqrt{d(v)}\right)^2\geq 0\\
=&d(u)-2\sqrt{d(u)d(v)}+d(v)\geq 0\\
&d(u)+d(v)\geq 2\sqrt{d(u)d(v)}.
\end{aligned}$$

Dividing both sides by $d(u)d(v)$

$$\begin{aligned}
=&\frac{d(u)+d(v)}{d(u)d(v)}\geq\frac{2\sqrt{d(u)d(v)}}{d(u)d(v)}\\
=&\frac{d(u)}{d(u)d(v)}+\frac{d(v)}{d(u)d(v)}\geq\frac{2\sqrt{d(u)d(v)}}{d(u)d(v)}\\
=&\frac{1}{d(v)}+\frac{1}{d(u)}\geq\frac{2}{(d(u)d(v))^{1-\frac{1}{2}}}\\
=&\frac{1}{d(v)}+\frac{1}{d(u)}\geq\frac{2}{\sqrt{d(u)d(v)}}
\end{aligned}$$

Dividing both sides by $d(u)+d(v)$ and multiply both sides by $\sqrt{d(u)d(v)}$

$$\Rightarrow \sqrt{d(u)d(v)} \geq \frac{2}{\frac{1}{d(u)}+\frac{1}{d(v)}}. \tag{2}$$

From (1) and (2), we get

$$\frac{d(u)+d(v)}{2} \geq \frac{2}{\frac{1}{d(u)}+\frac{1}{d(v)}}.$$

We add both sides $\sqrt{d(u)d(v)}$,

$$\sqrt{d(u)d(v)} + \frac{d(u)+d(v)}{2} \geq \sqrt{d(u)d(v)} + \frac{2}{\frac{1}{d(u)}+\frac{1}{d(v)}}.$$

Therefor

$$SGAM(G) \geq SG_{hm}(G).$$

□

3 Sum geometric harmonic means index of some chemical graphs

Definition 3.1. *[13] An alkane graph is a tree in which vertices correspond to atoms and edges to carbon - carbon or hydrogen - carbon bonds in a chemical alkane such that the carbon atoms represent the vertices and the hydrogen atoms are the edges i.e Alkanes having* n *carbon atoms and* $2n+2$ *hydrogen atoms we denote by* C_nH_{2n+2} *as shown in Fig. 1.*

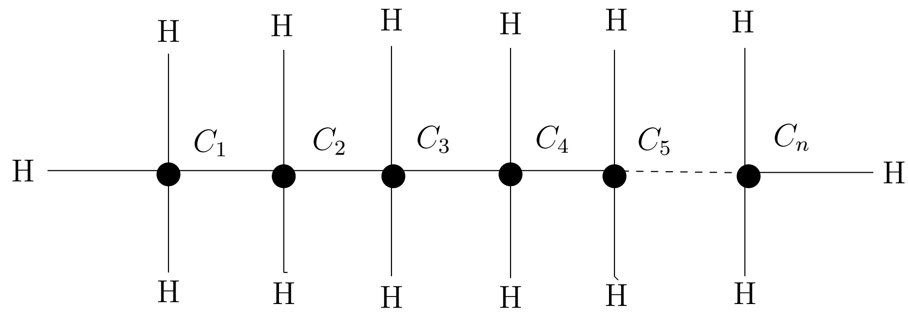

Figure 1: Alkanes

Theorem 3.2. *For the alkane graph* (C_nH_{2n+2}),

$$SG_{hm}(C_nH_{2n+2}) = 4\left(\frac{9}{5}(n+1)+2(n-1)\right).$$

Proof. By using the edge partition of C_nH_{2n+2} on the basis of the degrees of the vertices of each edge, we have two types of edges depending on the vertices of carbon atoms, first type, all edges have one vertex of degree one and another vertex of degree four, and the number of edges of this type is $2n+2$. In the second type, all edges have two vertices of degree four, so the number of edges of this type is $n-1$ and because

$$SG_{hm}(G) = \sum_{uv\in E(G)}\left(\sqrt{d(u)d(v)}+\frac{2}{\frac{1}{d(u)}+\frac{1}{d(v)}}\right),$$

this implies that

$$\begin{aligned} SG_{hm}(C_nH_{2n+2}) &= \left(\sqrt{(1)(4)}+\frac{2}{1+\frac{1}{4}}\right)(2n+2)+\left(\sqrt{(4)(4)}+\frac{2}{\frac{1}{4}+\frac{1}{4}}\right)(n-1)\\ &= \left(2+\frac{8}{5}\right)(2n+2)+8(n-1)\\ &= \frac{18}{5}(2n+2)+8(n-1)\\ &= 4\left(\frac{9}{5}(n+1)+2(n-1)\right). \end{aligned}$$

□

Definition 3.3. *[18] Cycloalkanes are hydrocarbones in which all the carbon - carbon bonds are single bonds,all or some of the carbon atoms are arranged in a ring* *and have the general formula C_nH_{2n} as shown in Fig. 2.*

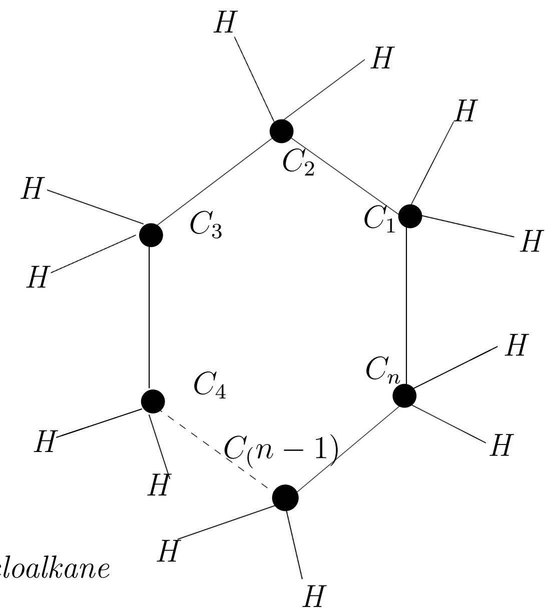

Figure 2: A cycloalkane

Theorem 3.4. *For the cycloalkane graph $(C_nH_{2n}), n \geq 3$, the SG_{hm} index is equal to the following*

$$SG_{hm}(C_nH_{2n}) = \frac{36}{5}(n) + 8(n).$$

Proof. By using the edge partition of C_nH_{2n} on the basis of the degrees of the vertices of each edge, we have two types of edges depending on the vertices of carbon atoms, first type, all edges have one vertex of degree one and another vertex of degree four, and the number of edges of this type is $2n$,

in the second type, all edges have two vertices of degree four, and the number of edges of this type is n and because

$$SG_{hm}(G) = \sum_{uv \in E(G)} \left(\sqrt{d(u)d(v)} + \frac{2}{\frac{1}{d(u)} + \frac{1}{d(v)}} \right),$$

we have,

$$\begin{aligned} SG_{hm}(C_nH_{2n}) &= \left(\sqrt{(1)(4)} + \frac{2}{1+\frac{1}{4}}\right)(2n) + \left(\sqrt{(4)(4)} + \frac{2}{\frac{1}{4}+\frac{1}{4}}\right)(n) \\ &= \left(2 + \frac{8}{5}\right)(2n) + 8(n) \\ &= \frac{18}{5}(2n) + 8(n) \\ &= \frac{36}{5}(n) + 8(n). \end{aligned}$$

□

Definition 3.5. *[14] Cycloalkene, We denote a cycloalkene having* r *carbon atoms and* $2r-2$ *hydrogen atoms by* C_r^{2r-2}*. The molecular graphs of them are obtained by attaching* $2r-2$ *pendant vertices corresponding to hydrogen atoms to vertices of a cycle corresponding to carbon atoms as shown in Fig.3.*

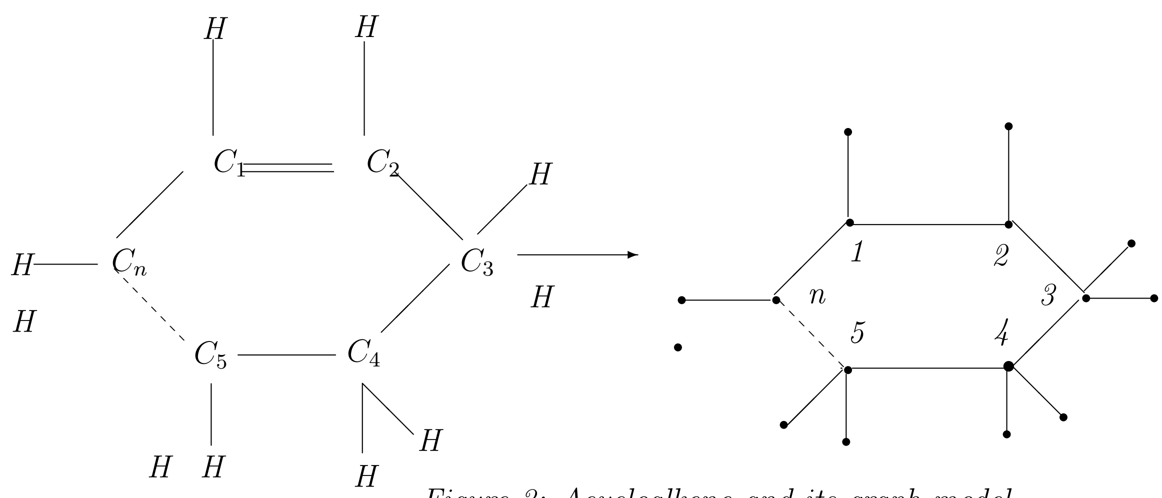

Figure 3: Acycloalkene and its graph model

Theorem 3.6. *For* $r \geq 3$*, the* SG_{hm} *index is equal to the following*

$$SG_{hm}(C_r^{2r-2}) = 532r - 789 + 210\sqrt{3}\ .$$

Proof. The cycloalkene molecular graph C_r^{2r-2} has $3r-2$ vertices including two vertices (namely,C_1 and C_2) of degree three, $r-2$ vertices $C_3, C_4, ..., C_r$ of degree four and correspond to the carbon atoms of cycloalkenes and the remaining $2r-2$ vertices (namely,H's) are end vertices and they correspond to hydrogen atoms of cycloalkenes. thus we have the following: on the basis of degrees of the vertices we divide the edge set into a partition

$$E_1 = uv \in E(C_r^{2r-2}) \mid d(u) = d(v) = 4,;$$

$$E_2 = uv \in E(C_r^{2r-2}) \mid d(u) = d(v) = 3,;$$

$$E_3 = uv \in E(C_r^{2r-2}) \mid d(u) = 3, d(v) = 4,;$$

$$E_4 = uv \in E(C_r^{2r-2}) \mid d(u) = 1, d(v) = 3,;$$

$$E_5 = uv \in E(C_r^{2r-2}) \mid d(u) = 1, d(v) = 4.$$

So there are five types of edges, where $\mid E_1 \mid= r-3$, $\mid E_2 \mid= 1$, $\mid E_3 \mid= 2$, $\mid E_4 \mid= 2$, $\mid E_5 \mid= 2r-4$, and because

$$SG_{hm}(G) = \sum_{uv\in E(G)} \left(\sqrt{d(u)d(v)} + \frac{2d(u)d(v)}{d(u)+d(v)} \right),$$

it follows that,

$$\begin{aligned}
SG_{hm}(C_r^{2r-2}) &= (r-3)\left(\sqrt{(4)(4)} + \frac{2}{\frac{1}{4}+\frac{1}{4}}\right) + (1)\left(\sqrt{(3)(3)} + \frac{2}{\frac{1}{3}+\frac{1}{3}}\right) \\
&+ (2)\left(\sqrt{(3)(4)} + \frac{2}{\frac{1}{4}+\frac{1}{3}}\right) + (2)\left(\sqrt{(1)(3)} + \frac{2}{\frac{1}{1}+\frac{1}{3}}\right) \\
&+ (2r-4)\left(\sqrt{(1)(4)} + \frac{2}{\frac{1}{1}+\frac{1}{4}}\right) \\
&= 8(r-3) + 6 + 2\left(\sqrt{12} + \frac{24}{7}\right) + 2\left(\sqrt{3} + \frac{3}{2}\right) \\
&+ (2r-4)(2 + \frac{8}{5}) \\
&= 8r - 24 + 6 + 2\sqrt{12} + \frac{48}{7} + 2\sqrt{3} + 3 \\
&+ 4r - 8 + \frac{16r}{5} - \frac{32}{5} \\
&= 12r + \frac{16r}{5} - 23 + 6\sqrt{3} + \frac{48}{7} - \frac{32}{5} \\
&= 76r - 115 + 30\sqrt{3} + \frac{16}{7} \\
&= 532r - 789 + 210\sqrt{3}\ .
\end{aligned}$$

□

4 Conclusion

In this paper, we have computed the concept of sum geometric harmonic means index of some standard graphs. Also, sum geometric harmonic means index of a graph in chemistry is given. The sum geometric harmonic means index of several other families of graphs is an open problem.

References

[1] M. S. Abdelgader, C. Wang and S. A. Mohammed, Computation of topological indices of some special graph, Maths., 6 (33) (2018), 1-15.

[2] D. M. Cvetkovic and I. Gutman (Eds.), Applications of graph spectra, Math. Inst., Belgrade, 2009.

[3] K. Ch. Das and N. Trinajstic, *Comparison between geometric - arithmetic indices*, Croat. Chem. Acta, 85 (3) (2012), 353 -357.

[4] H. Deng, S. Elumalai, and S. Balachandran, *Maximum and second Maximum of geometric - arithmetic index of tricyclic graphs*, MATCH Commun. Math. Comput. Chem., 79 (2018), 467-475.

[5] N. Deo, Graph theory with applications to engineering and computer science,prentice Hall of India, 1990.

[6] M. R. Farahani, *Computing Randic, geometric - arithmetic and atom-bond connectivity indices of circumcoronene series of benzenoid*, Int. J. Chem. Model, 5 (4) (2013), 485-493.

[7] B. Furtula and I. Gutman, *Geometric - arithmetic indices*, Discrete Applied Mathematics, 2010.

[8] J. L. Gross, J. Yellen and P. Zhang, Handbook of Graph Theory, CRC Press, Robin J. Wilson, 2013.

[9] J. W. Grossman, F. Harary and M. Klawe, *Generalized ramsey theorem for graphs*, X: Double stars, Discrete Mathematics, 28(1979), 247-254.

[10] I. Gutman, X. Li, and Y. Shi, Graph Energy, Springer, New York Heidelberg Dordrecht, London, 2012.

[11] F. Harary, Graph Theory, Narosa Publishing House, New Delhi, 2001.

[12] R. Kanabur and Shegehalliv, Arithmetic - geometric Indices of Some Class of Graph, Researchgate, 2016.

[13] R. E. Merrified, H. E. Simmons, Topological Methods in Chemistry. New York: Wiley, 1989.

[14] M. M. Mohsen and S. S. Mahde, *Sum geometric - arithmetic means index of graphs*, International Journal of Computer Applications, 187(17)(2022), 0975 - 8887.

[15] M. R. Rajesh Kanna and B. N. Dharmendra, *Minimum covering distance energy of a graph*, Appl. Math. Sci., 7(111)(2013), 5525 - 5536.

[16] P. S. K. Reddy, K. N. Prakasha and Gavirangaiah, *Minimum dominating color energy of a graph*, Int. J. Math. Combin., 3(2017), 22-31.

[17] J. M. Rodrıguez and J. M. Sigarreta, *Spectral study of the geometric - arithmetic index*, MATCH Commun. Math. Comput. Chem , 74 (2015), 121-135.

[18] T. W. G. Solomons and C. B. Fryhle, Organic Chemistry,Wiley india Pvt. Ltd., New Delhi, 2010.

[19] A. Sudarman, M. Jauhari and F. Ali, *Survey on topological indices and graphs associated with a commutative ring*, Journal of Physics: Conference Series,1562 (2020), 012 - 008.

[20] C.Vasudey,Graph Theory with Applications, New Age Int.(p)Ltd., New Delhi, 2007.

[21] D. B. West, Introduction to Graph Theory, PHI Learning Private Ltd., New Delhi, 2009.

[22] L. Xiao, S. Chen, Z. Guo, Q. Chen, *The geometric - arithmetic index of benzenoid systems and phenylenes*, Int. J. Contemp. Math. Sciences, 5 (45) (2010), 2225-2230.

[23] L. Zhong, *The arithmetic index for graphs*, Applied Mathematics Letters, 25 (2012), 561 - 566.

[24] V. Zverovich, Research Topics in Graph Theory and Its Applications, Cambridge Scholars, British, 2019.