The Solution of some Kinds of Higher Order Differential Equations by Adomian Decomposition Method

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Abstract
In this paper, we presented a new differential operator to solve singular initial value problems of higher order differential equations by utilizing Adomian decomposition method (ADM). The suggested approach is applicable to linear as well as non linear problems. Some exampls are used in testing and the obtained results demonstrate that the suggested strategy is effective.
Keywords: Adomian decomposition method; singular initial value problems ; higher order diferential equations

1. Introduction

Adomian Decomposition Method (ADM) [1-3] presented by G. Adomian is a powerful and reliable method for solving various kinds of problems arising in applied sciences. The main advantage of ADM is that it can be applied directly to solve all types of differential equations, ordinary [14], partial [11], fractional [8], both linear and non- linear [6], and systems [7]. The method gives approximate solution which converges rapidly to accurate [9,12]. A large amount of literature developed concerning ADM [1,2,4,5] and the related modifications [10,13,15], to investigate various scientific models. The aim of this paper is to use ADM for solving some higher order differential equations. For this reason, a new differential operator which can be used for solving singular initial problems of some higher order differential equation, is used.

2 . Building Differential Equations of Higher Order

We consider the equation in form

$$Ly + f(x,y) = g(x), \tag{1}$$

where $f(x,y)$, $g(x)$ are real functions and L is linear operator,
we use Eq. (1), to derive higher order differential equation and we suggest L is differential operator as below:

$$L(.) = x^{-n}\frac{d^{n-2}}{dx^{n-2}}x^{n-m}\frac{d^2}{dx^2}x^m(.), \tag{2}$$

where $m \geq n, m \geq 3$, $n \geq 2$, and the inverse operator L^{-1} as

$$L^{-1}(.) = x^{-m}\int_0^x\int_0^x x^{m-n}\int_0^x ...\int_0^x\int_0^x x^n(.)dxdxdx...dxdx. \tag{3}$$

The Eq. (1), it can be rewritten as

$$x^{-n}\frac{d^{n-2}}{dx^{n-2}}x^{n-m}\frac{d^2}{dx^2}x^m(y) + f(x,y) = g(x). \tag{4}$$

To determine such different equations of higher order, we set n to different values, when putting $n = 2$ in(4), we obtain

$$y^{''} + \frac{2m}{x}y^{'} + \frac{m(m-1)}{x^2}y + f(x,y) = g(x),$$

when putting $n = 3$ in Eq. (4), we obtain

$$y^{(3)} + \frac{(3+2m)}{x}y^{''} + \frac{(m^2+3m)}{x^2}y^{'} + \frac{(m^2-m)}{x^3}y + f(x,y) = g(x),$$

when putting $n = 4$ in Eq. (4), we obtain

$$y^{(4)}+\frac{(8+2m)}{x}y^{(3)}+\frac{(m^2+11m+12)}{x^2}y^{''}+\frac{(4m^2+8m)}{x^3}y^{'}+\frac{(2m^2-2m)}{x^4}y+f(x,y) = g(x),$$

by continuing with the same procedure we get

$$\sum_{r=0}^{n-2}\binom{n-2}{r}\frac{n!}{(n-r)!}x^{-r}y^{(n-r)} + 2m\sum_{r=0}^{n-2}\binom{n-2}{r}\frac{(n-1)!}{(n-r-1)!}x^{-r-1}y^{(n-r-1)}$$

$$+m(m-1)\sum_{r=0}^{n-2}\binom{n-2}{r}\frac{(n-2)!}{(n-r-2)!}x^{-r-2}y^{(n-r-2)} + f(x,y) = g(x). \tag{5}$$

3 . The Adomian Decmposition Method

We consider the equation in form (5) with conditions

$y(0) = b_0, y^{'}(0) = b_1, y^{''}(0) = b_2, ..., y^{(n-1)}(0) = b_{n-1}$,
where $b_0, b_1, ..., b_{n-1}$ are constants.
According to the ADM, we rewrite Eq. (5)in the form as

$$Ly = g(x) - f(x, y), \tag{6}$$

by applyingL^{-1} on (6), we obtain

$$y(x) = \phi(x) + L^{-1}g(x) - L^{-1}f(x, y), \tag{7}$$

where $\phi(x) = 0$, the constants of integration, are determined by the initial or boundary conditions according to the problem as initial-value problem or boundary-value problem.

The ADM represents the solution $y(x)$ and the non-linear function $f(x, y)$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{8}$$

and

$$f(x, y) = \sum_{n=0}^{\infty} A_n, \tag{9}$$

where the elements $y_n(x)$ of the solution $y(x)$ will be determined recurrently by algorithm [1,2,3]. And the A_n are the Adomain polynomials, which are obtained as in the following formula :

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F(\sum_{i=0}^{\infty} \lambda^i y_i)\right]_{\lambda=0}, n = 0, l, ... \tag{10}$$

where λ is a parameter.
From Eq. (10) we have

$$\begin{aligned}
&A_0 = F(y_0),\\
&A_1 = y_1 F^{'}(y_0),\\
&A_2 = y_2 F^{'}(y_0) + \frac{1}{2} y_1^2 F^{''}(y_0),\\
&A_3 = y_3 F^{'}(y_0) + y_1 y_2 F^{''}(y_0) + \frac{1}{3i} y_1^3 F^{'''}(y_0),\\
&....
\end{aligned} \tag{11}$$

Substituting Eq. (8)and(9)into Eq. (7), we have

$$\sum_{n=0}^{\infty} y_n = \phi(x) + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} A_n, \qquad (12)$$

we get the components y_n which can be specified as

$$\begin{aligned} y_0 &= \phi(x) + L^{-1}g(x), \\ y_{n+1} &= -L^{-1}A_n, n \geq 0, \end{aligned} \qquad (13)$$

which gives

$$\begin{aligned} y_0 &= \phi(x) + L^{-1}g(x), \\ y_1 &= -L^{-1}A_0, \\ y_2 &= -L^{-1}A_1, \\ y_3 &= -L^{-1}A_2, \\ &.... \end{aligned}$$

Using the equations (11) and (13) we can determine the components y_n, and therefore, we can immediately obtain series solutions of $y(x)$ in (7). In addition, and for numerical reasons, we can use the n-term approximate

$$\varphi_n = \sum_{k=0}^{n-1} y_k(x),$$

which can be used to approximate the exact solution.

4 . Illustrative Examples

Example 1. We consider the non-linear initial problem:

$$y^{''} + \frac{6}{x}y^{'} + \frac{6}{x^2}y = (x^4 + 20) - y^2, \qquad (14)$$

$y(0) = 0, y^{'}(0) = 0,$
with exact solution $y(x) = x^2$, Eq. (14) which can be written as

$$Ly = (x^4 + 20) - y^2, \qquad (15)$$

where differential operator as

$$L(.) = x^{-3}\frac{d^2}{dx^2}x^3(.),$$

and the inverse operator

$$L^{-1}(.) = x^{-3}\int_0^x \int_0^x x^3(.)dxdx,$$

by applying L^{-1} on both sides of (15), and using the initial conditions at $x = 0$, yields

$$y(x) = L^{-1}(x^4 + 20) - L^{-1}y^2, \tag{16}$$

substituting the decomposition series y_n for $y(x)$ into (16) gives

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}(x^4 + 20) - L^{-1}y^2, \tag{17}$$

$$y_0 = L^{-1}(x^4 + 20),$$
$$y_{n+1} = -L^{-1}A_n, n \geq 0, \tag{18}$$

were the non linear term y^2 has the first few Adomain polynomials A_n are given by

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_0y_1, \\ y_2 &= y_1^2 + 2y_0y_2, \\ &\ldots \end{aligned}$$

Using (18), the first several calculated solution components are

$$\begin{aligned} y_0 &= x^2 + \frac{x^6}{72}, \\ y_1 &= -\frac{x^6}{72} - \frac{x^{10}}{5616} - \frac{x^{14}}{1410048}, \\ y_2 &= \frac{x^{10}}{5616} + \frac{150x^{14}}{54991872} + \frac{350x^{18}}{23096586240} + \frac{x^{22}}{3045703600}, \end{aligned}$$

the series solution by ADM is given by

$$y(x) = y_0 + y_1 + y_2 = x^2 + \frac{111x^{14}}{54991872} + \frac{350x^{18}}{23096586240} + \frac{x^{22}}{3045703600},$$

that converges to the exact solution $y(x) = x^2$.

Example 2. We assume the linear initial problem:

$$y''' + \frac{11}{x}y'' + \frac{28}{x^2}y' + \frac{12}{x^3} = f(x)e^x + xe^x - xy, \tag{19}$$

$y(0) = 1, y^{'}(0) = 1, y^{''}(0) = 1,$
with exact solution $y(x) = e^x$,
where $f(x) = \dfrac{x^4 + x^3 + 11x^2 + 28x + 12}{x^3}$.
Eq.(19) which can be written as

$$Ly = f(x)e^x + xe^x - xy, \tag{20}$$

where differential operator

$$L(.) = x^{-3}\frac{d}{dx}x^{-1}\frac{d^2}{dx^2}x^4(.),$$

and the inverse operator

$$L^{-1}(.) = x^{-4}\int_0^x\int_0^x x\int_0^x x^3(.)dxdxdx,$$

by applying L^{-1} on both sides of (20), and using the initial condition at $x = 0$, yields

$$y(x) = L^{-1}(f(x)e^x + xe^x) - L^{-1}xy, \tag{21}$$

substituting the decomposition series y_n for $y(x)$ into (21)gives

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}(f(x)e^x + xe^x) - L^{-1}xy, \tag{22}$$

$$\begin{aligned} y_0 &= L^{-1}(f(x)e^x + xe^x),\\ y_{n+1} &= L^{-1}xy_n, n \geq 0, \end{aligned} \tag{23}$$

by using Taylor series of e^x and Eq. (23), we obtain

$$\begin{aligned} y_0 &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ... + \frac{x^4}{390} + \frac{x^5}{432} + \frac{x^6}{1260} + \frac{x^7}{5280} + ..,\\ y_1 &= -\frac{x^4}{390} - \frac{x^5}{432} - \frac{x^6}{1260} - \frac{x^7}{5280} - .. - \frac{x^8}{463320} - \frac{x^9}{673990} - \frac{x^{10}}{2522520} - \frac{x^{11}}{13305600} + ...,\\ y_2 &= \frac{x^8}{463320} + \frac{x^9}{673990} + \frac{x^{10}}{2522520} + \frac{x^{11}}{13305600} + .. + \frac{x^{12}}{1445558400} + \frac{13}{2566287360} + \frac{x^{14}}{11578366800} + ..., \end{aligned}$$

the solution series by ADM is given by

$$y(x) = y_0 + y_1 + y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ... + \frac{x^{12}}{1445558400} + \frac{x^{13}}{2566287360} + \frac{x^{14}}{11578366800} + ..$$

Which is Taylor series of exact solution $y(x) = e^x$.

Example 3. We consider the non-linear initial problem:

$$y'''' + \frac{18}{x}y''' + \frac{72}{x^2}y'' + \frac{140}{x^3}y' + \frac{40}{x^4}y = f(x)e^{2x} + e^{6x} - y^3, \tag{24}$$

$y(0) = 1, y'(0) = 2, y''(0) = 4, y'''(0) = 8,$
with exact solution $y(x) = e^{2x}$,

where $f(x) = \dfrac{16x^4 + 144x^3 + 288x^2 + 280x + 140}{x^4}$,

Eq. (24) which can be written as

$$Ly = f(x)e^{2x} + e^{6x} - y^3, \tag{25}$$

where differential operator as

$$L(.) = x^{-4}\frac{d^2}{dx^2}x^{-2}\frac{d^2}{dx^2}x^6(.),$$

and the inverse operator

$$L^{-1}(.) = x^{-6}\int_0^x\int_0^x x^2\int_0^x\int_0^x x^4(.)dxdxdxdx,$$

by applying L^{-1} on both sides of (25), and using the initial condition at $x = 0$, yields

$$y = L^{-1}f(x)e^{2x} + L^{-1}e^{6x} - L^{-1}y^3, \tag{26}$$

substituting the decomposition series y_n for $y(x)$ into (26) gives

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}f(x)e^{2x} + L^{-1}e^{6x} - L^{-1}y^3, \tag{27}$$

by modification of ADM in [15], we have

$$\begin{aligned} y_0 &= L^{-1}f(x)e^{2x}, \\ y_1 &= L^{-1}e^{6x} - L^{-1}A_0, \\ y_{n+2} &= -L^{-1}A_n, n \geq 0, \end{aligned} \tag{28}$$

where the Adomian polynomail A_n to non-linear term y^3 gives

$$\begin{aligned} A_0 &= y_0^3, \\ A_1 &= 3y_0^2y_1, \\ A_1 &= 3y_0^2y_2 + 3y_0y_1^2, \\ &\ldots.. \end{aligned}$$

by using (28), we have

$$y_0 = e^{2x},$$
$$y_1 = 0,$$
$$y_{n+1} = 0, n \geq 1,$$

the series solution by ADM given by
$y(x) = y_0 = e^{2x}$, which is the exact solution .

5 Conclusion

The ADM is a powerful and strong to solve higher order differential equations both linear or non linear. Using a new operator has been successful for solving singular initial value problems, therefore, obtaining approximation solutions rabidly converges to exact solution without large computations in some cases can be obtained directly exact solution.

References

[1] K. Abbaoui, Y. Cherruault, Convergence of Adomian's Method Applied to Differential Equations, Comput. Math. Appl. ,1994; 28(5): 103-109.

[2] G. Adomian, A review of the decomposition method and some recent results for nonlinear equation, Math. Comput. Modelling, 1990;13(7):17-43.

[3] G. Adomian, Solving frontier problems of physics: The Decomposition Method, Kluwer, Boston, MA, 1994.

[4] G. Adomian, R. Rach, Noise terms in decomposition series solution, Comput. Math. Appl., 1992;24(11): 61-64.

[5] G. Adomian, R. Rach, Inversion of nonlinear stochastic operators, J. Math. Anal. Appl., 1983; 39-46 .

[6] S. A. Alaqel, Y. Q. Hasan, A new application of Adomian Decomposition method, IOSR-JM., 2020;16:11-15.

[7] E. Babolian, J. Biazar and R. Islam, Solution of the system of ordinary differential equations by Adomian decomposition method, Appl. Math. Comput., 2004;147: 713-719.

[8] N. M. Dabwan, Y. Q. Hasan, Using Adomian decomposition method in solving fractional Lane-Emden type equations, R. A. Journal, 2020;3(8):274-283.

[9] Y. Q. Hasan, The numerical solution of third order boundary value problems by the modifed decomposition method, Advances in Intelligent Transportation Systems, 2012;1(3):71-74.

[10] Y. Q. Hasan, Modified Adomain decomposition method for solving ordinary differential equations, Advances in Intelligent Transportation Systems, 2012;1(4):86-89.

[11] A. M. Wazwaz, Partial differential equations and solitary waves theory, HEP and Springer, Pejing and Berlin, 2009.

[12] A. M. Wazwaz, The numerical solution of fifth-order BVP by the decomposition method, J .Comput. Appl. Math., 2001;136:59-270.

[13] A.M.Wazwaz, A new algorithm for calculating Adomian polynomials for nonlinear operators, Appl. Math. Comput., 2000;111(1):53-69.

[14] A. M. Wazwaz, A new method for solving singular initial value problems in the second-order ordinary differential equations, Appl. Math. Comput., 2002;128:45-57.

[15] A. M. Wazwaz, A reliable modification of Adomian decomposition method, Appl. Math. Comput., 1999;102: 77-86.