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An Atlas of k-Distance Neighborhood Polynomials of Graphs with at most Six Vertices

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Abstract

In a connected graph $G = (V, E)$, the distance from a vertex v to a vertex u, denoted $d(u, v)$, is the length of the shortest path joins them. For a vertex v in G, the eccentricity $e(v)$ is the distance to the farthest vertex from v and for $0 \le k \le e(v)$, the k-distance degree (k-degree) of v, is $d_k(v) = \{u \in V(G): d(v, u) = k\}$. The N_k-polynomial of a graph G is a distance-degree-based topological polynomial and is denoted by $N_k(G, x)$. It is a polynomial with the coefficient of the term k, and is equal to the sum of $d_k(v)$, for every $v \in V(G)$. The roots of an N_k-polynomial of a graph are called the N_k-roots of G and denoted by $Z(N_k(G, x))$. In this paper, we compute the N_k-polynomial of all graphs of order $n \le 6$, and present it in a table. The complement graph $\overline{G}$ of every graph is found and presented directly in the same row of the table. Moreover, the roots $Z(N_k(G, x))$ of every N_k-polynomial are estimated. The classes of graphs with the same N_k-polynomial are found. Finally, the relationship between the coefficients of N_k-polynomial and graph connectivity is presented.

Keywords: *Second Degree (of vertex), $distance$, graph polynomial.* **MSC 2010 Classifications:** *05C07, 05C12, 05C76.*

1. Introduction

All the graphs $G=(V,E)$ considered here are finite undirected with no loops and multiple edges. As usual, we denote by $n=|V|$ and $m=|E|$ to the number of vertices and edges in a graph G, respectively, unless we refer otherwise. The distance $d(u)$ between any two vertices u and v of G is the length of the minimum path that join them. For a vertex $v\in V(G)$ and a positive integer k, the open k-distance neighborhood of v in a graph G, denote $N_k(v)$, is $N_k(v)=\{u\in V(G): d(u,v)=k\}$ and the k-distance degree of a vertex v, denote $d_k(v)$, is $d_k(v)=|N_k(v)|$. It is clearly that $d_1(v)=d(v)$. A graph G with only one vertex is called a trivial, otherwise is called a nontrivial. The complement of a graph G, denoted $\overline{G}$, is a graph with vertex set $V(\overline{G})=V(G)$ and edge set $\overline{E}$, such that $uv\in\overline{E}$, if and only if $uv\notin E(G)$. A graph G is self-complementary if G is isomorphic to its complement. $\overline{K_n}$ is the empty or total disconnected graph with n vertices, i.e., the graph with n vertices no two of which are adjacent. A graph G is called d-regular graph if the degree $d_1(v)$ of each vertex v in G is equal to d. For a vertex v of G, the eccentricity $e(v)=\max\{d(v,u): u\in V(G)\}$. The radius of G is $rad(G)=\min\{e(v): v\in V(G)\}$ and the diameter of G is $diam(G)=\max\{e(v): v\in V(G)\}$. For any terminology or notation not mentioned here, the readers refered to [2, 6].

In (2016), Naji and Soner [12], have been introduced a new type of graph topological polynomial, based on distance and degree, called k-distance neighborhood polynomial of a graph. Which, for simplicity of notion, referred as N_k-polynomial and defined by

$$N_k(G,x)=\sum_{k=0}^{e(v_i)}\left(\sum_{i=1}^{n} d_k(v_i)\right)x^k.$$

Since, for each $v\in V(G)$, $e(v)\le diam(G)\le n-1$, and $d_k(v)=0$, for $k\ge e(v)$. Then we can rewrite N_k-polynomial as

$$N_k(G,x)=\sum_{k=0}^{n-1}\left(\sum_{v\in V(G)} d_k(v)\right)x^k.$$

The authors in the papers [13, 7, 12], obtained some basic properties of N_k-polynomial of graphs and they presented the exact formulas for the N_k-polynomial of some well-known graphs (namely, a path P_n, a cycle C_n, a complete K_n, a star $K_{1,n-1}$, a wheel W_n, a complete bipartite $K_{r,s}$). They also established the N_k-polynomial for some graph operations namely cartesian product, join, union, corona product of graphs and for the complement graphs and splitting of some graphs.

Two graphs G_1 and G_2 are said to be N_k-equivalent, written as $G_1 \sim^{N_k} G_2$, if

$$N_k(G_1,x)=N_k(G_2,x)$$

It is evident that the relation $\sim^{N_k}$ is an equivalence relation on the family of graphs $\mathbb{G}_n$, and thus $\mathbb{G}_n$ is partitioned into equivalence classes, called the N_k-equivalence classes. Given $G\in\mathbb{G}_n$, let $[G]=\{H\in\mathbb{G}_n: H\sim^{N_k} G\}$, we call $[G]$, the N_k-equivalence class determined by G.

A graph G is said to be N_k-unique if there is no graph H such that $G\sim^{N_k} H$, i.e., if $[G]=\{G\}$.

As each graph polynomial typys, the analysis of the N_k-polynomial of graphs can give us many information about graph. Motivated by the domination polynomial of a graph [1], we establish an atlas for the N_k-polynomial for every graphs with order at most 6, so also for its complement as well as we find their roots. Another important reason to study the N_k-polynomials of graphs with at most six vertices, is topological indices. A topological index (or a graph invariant) is a fixed invariant number for two isomorphic graphs and gives some information about the graph under consideration. These indices are especially useful in the study of molecular graphs. Some of the topological indices are defined by means of the vertex degrees. For instance, Zagreb indices [4, 5], Leap Zagreb indices [8, 9, 10, 11], etc. There are many papers on the degree-based topological indices of graphs. For instance, the readers referred to [3], and the references cited in.

In this article, we establesh N_k-polynomial and the N_k-roots of all connected graphs of order less than or equal to six, and we present them directly in a table. Furthermore, the complement $\overline{G}$ of every graph is found and any one can easily compute its N_k-polynomial from the table. Finally, we deduced new results from the table.

Let $\mathbb{G}_6$ be denote to the family of all connected graphs with $n \leq 6$ vertices and as its presented in the following table. Hence, we have 142 graphs, i.e. $|\mathbb{G}_6| = 142$ with at most 6 vertices. Since for any graph G with n vertices, $|V(\overline{G})| = |V(G)| = n$. Therefor, $\overline{G} \in \mathbb{G}_6$, for every $G \in \mathbb{G}_6$. However, if $\overline{G}$ is disconnected, then it consist of two or more components all of them in $\overline{G}$. Thus, in the table 0, we added a serial number in the first column. Then, in the second column, we set the figures of every graph $G_i \in \mathbb{G}_6$, where G_i, for $1 \leq i \leq 142$, is a graph whose figure presented in the row number i. For instance, $G_4 = P_3$ is the graph shown in forth row and $\overline{G_4} = G_1 \cup G_2$, where $G_1 = K_1$, as shown in the first row and $G_2 = P_2$, as shown in second row.

For the disconnected graphs one can use the following Lemma.

Lemma 1.1 *[7] If a graph G consists of $t > 2$ components $G_1, G_2 \dots, G_t$, then*

$$N_k(G,x) = N_k(G_1,x) + N_k(G_1,x) + \cdots + N_k(G_1,x).$$

The following are some fundamental results which will be required for many of our arguments in this paper and which are finding in [7, 12].

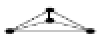
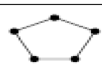
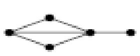
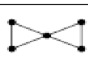
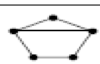
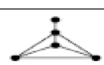
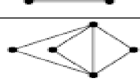
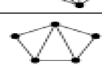
Proposition 1.2 *Let G be a graph of order n, size m and $diam(G) \geq 1$ and let the N_k-polynomial of G is*

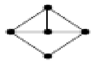
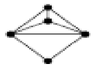
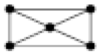
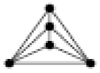
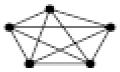
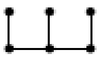
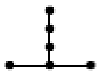
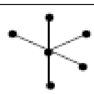
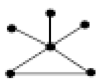
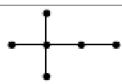
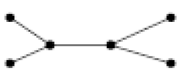
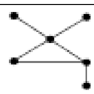
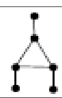
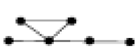
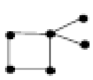
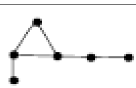
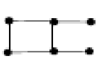
$$N_k(G,x) = a_p x^p + a_{p-1} x^{p-1} + \ldots + a_1 x + a_0.$$

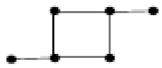
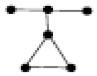
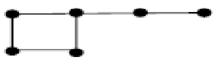
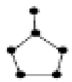
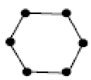
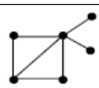
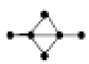
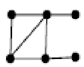
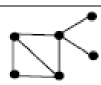
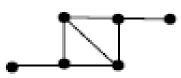
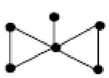
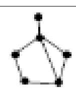
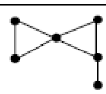
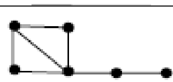
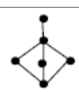
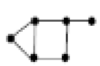
Then

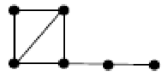
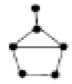
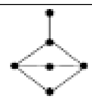
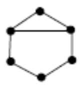
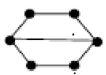
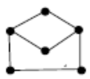
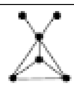
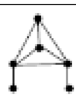
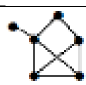
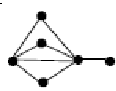
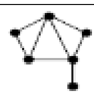
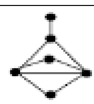
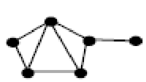
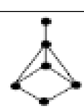
1. $a_0 = n$.
2. $a_1 = 2m$.
3. a_i, for every $i = 1,2,\ldots,p$, is an even integer number.
4. The degree p of $N_k(G,x)$ is equal to $diam(G)$.

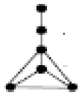
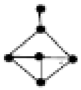
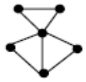
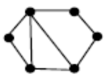
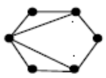
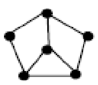
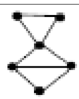
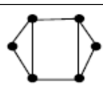
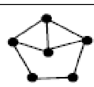
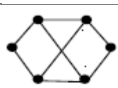
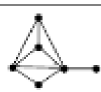
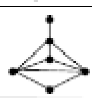
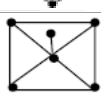
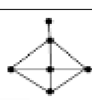
i	G_i	$N_k(G_i,x)$	$\overline{G_i}$	$Z(N_k(G_i,x))$
1		1	$G = \overline{G}$	NA

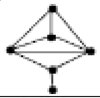
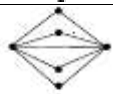
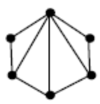
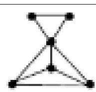
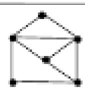
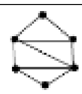
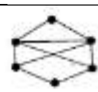
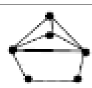
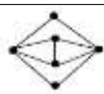
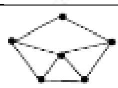
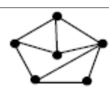
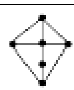
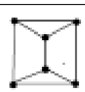
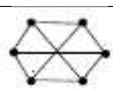
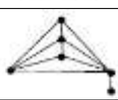
2		$2x+2$	$2G_1$	-1
3		$6x+3$	$3G_1 \; - 0.5$	-0.5
4		$2x^2+4x+3$	$G_1 \cup G_2$	$-1.0 \pm 0.7071i$
5		$2x^3+4x^2+6x+4$	$G=\overline{G}$	$-1, -0.50 \pm 1.3229i$
6		$6x^2+6x+4$	$G_1 \cup G_3$	$-0.50 \pm 0.6455i$
7		$4x^2+8x+4$	$G_1 \cup G_4$	-1
8		$4x^2+8x+4$	$2G_2$	-1
9		$2x^2+10x+4$	$2G_1 \cup G_2$	$-0.4385, -4.5616$
10		$12x+4$	$4G_1$	-0.3333
11		$2x^4+4x^3+6x^2+8+5$	G_{22}	$0.0559 \pm 1.3995i$, $-1.0559 \pm 0.3995i$
12		$4x^3+8x^2+8x+5$	G_{16}	-1.2013, $-0.3993 \pm 0.9386i$
13		$2x^3+8x^2+10x+5$	$G=\overline{G}$	-2.2972, $-0.8514 \pm 0.6028i$
14		$4x^3+6x^2+10x+5$	G_{15}	-0.6413, $-0.4294 \pm 1.3285i$
15		$2x^3+8x^2+10x+5$	G_{14}	-2.2972, $-0.8514 \pm 0.6028i$
16		$2x^3+6x^2+12x+5$	G_{12}	-0.5338, $-1.2331 \pm 1.7785i$
17		$12x^2+8x+5$	$G_1 \cup G_{10}$	$-0.3333 \pm -0.5528i$
18		$10x^2+10x+5$	$G_1 \cup G_9$	$-0.50 \pm 0.50i$
19		$10x^2+10x+5$	$G=\overline{G}$	$-0.50 \pm 0.50i$
20		$8x^2+12x+5$	$G_1 \cup G_7$	$-0.75 \pm 0.25i$
21		$8x^2+12x+5$	$G_1 \cup G_8$	$-0.75 \pm 0.25i$
22		$8x^2+12x+5$	G_{11}	$-0.75 \pm 0.25i$
23		$8x^2+12x+5$	$G_2 \cup G_3$	$-0.75 \pm 0.25i$
24		$6x^2+14x+5$	$G_1 \cup G_6$	$-1.8931, -0.4402$
25		$6x^2+14x+5$	$2G_1 \cup G_3$	$-1.8931, -0.4402$
26		$6x^2+14x+5$	$G_1 \cup G_5$	$-1.8931, -0.4402$

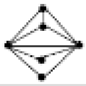
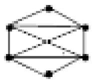
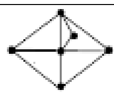
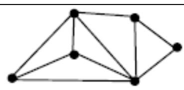
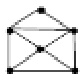
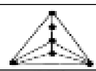
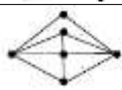
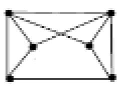
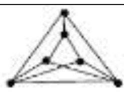
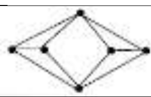
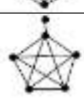
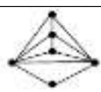
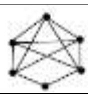
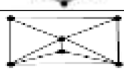
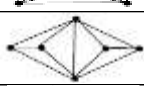
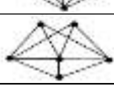
27		$6x^2 + 14x + 5$	$G_2 \cup G_4$	$-1.8931, -0.4402$
28		$4x^2 + 16x + 5$	$2G_1 \cup G_4$	$-3.6583, -0.3417$
29		$4x^2 + 16x + 5$	$G_1 \cup 2G_2$	$-3.6583, -0.3417$
30		$2x^2 + 18x + 5$	$3G_1 \cup G_2$	$0.2869, -8.7131$
31		$20x + 5$	$5G_1$	-0.25
32		$2x^5 + 4x^4 + 6x^3 + 8x^2 + 10x + 6$	G_{123}	$-1, 0.4057 \pm 1.2928i, -0.9057 \pm 0.902i$
33		$2x^4 + 8x^3 + 10x^2 + 10x + 6$	G_{119}	$-1, -2.6717, -0.1642 \pm 1.0469i$
34		$4x^4 + 6x^3 + 10x^2 + 10x + 6$	G_{120}	$-0.7689 \pm 0.5647i, -0.0190 \pm 1.2836i$
35		$20x^2 + 10x + 6$	$G_1 \cup G_{31}$	$-0.25 \pm 0.4873i$
36		$18x^2 + 12x + 6$	$G_1 \cup G_3$	$-0.3333 \pm 0.4714i$
37		$6x^3 + 14x^2 + 10x + 6$	G_{113}	$-1.6987, -0.3173 \pm 0.6986i$
38		$8x^3 + 12x^2 + 10x + 6$	G_{116}	$-1.0, -0.25 \pm 0.8292i$
39		$4x^3 + 14x^2 + 12x + 6$	G_{93}	$-2.5558, -0.4721 \pm 0.6034i$
40		$6x^3 + 12x^2 + 12x + 6$	G_{99}	$-1, -0.5 \pm 0.8661i$
41		$6x^3 + 12x^2 + 12x + 6$	G_{95}	$-1, -0.5 \pm 0.8661i$
42		$4x^3 + 14x^2 + 12x + 6$	G_{96}	$-2.5558, -0.4721 \pm 0.6034i$
43		$2x^4 + 6x^3 + 10x^2 + 12x + 6$	G_{102}	$-1, -1.3925, -0.3040 \pm 1.4360i$
44		$6x^3 + 12x^2 + 12x + 6$	G_{103}	$-1, -0.5000 \pm 0.8661i$

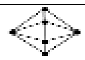
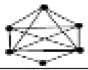
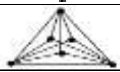
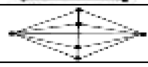
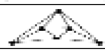
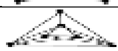
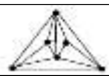
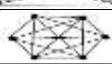
45		$2x^4+4x^3+12x^2+12x+6$	G_{105}	$-0.3733 \pm 2.0591i, -0.6267 \pm 0.5407i$
46		$8x^3+10x^2+12x+6$	G_{104}	$-0.6746, -0.2877 \pm 1.0144i$
47		$4x^4+6x^3+8x^2+12x+6$	G_{108}	$-1, -0.8401, 0.1701 \pm 1.3254i$
48		$2x^4+6x^3+10x^2+12x+6$	G_{109}	$-1, -1.3926, -0.3037 \pm 1.4359i$
49		$4x^3+14x^2+12x+6$	G_{107}	$-2.5558, -0.4721 \pm 0.6034i$
50		$6x^3+12x^2+12x+6$	G_{110}	$-1, -0.50 \pm 0.8661i$
51		$16x^2+14x+6$	$G_1 \cup G_{28}$	$-0.4375 \pm 0.4285i$
52		$2x^3+14x^2+14x+6$	G_{71}	$-5.8997, -0.5502 \pm 0.4537i$
53		$4x^3+12x^2+14x+6$	G_{75}	$-1, -1 \pm 0.7071i$
54		$4x^3+12x^2+14x+6$	G_{76}	$-1, -1 \pm 0.7071i$
55		$2x^4+4x^3+10x^2+14x+6$	G_{82}	$-1, -0.7832, -0.1084 \pm 1.9541i$
56		$16x^2+14x+6$	$G_1 \cup G_{29}$	$-0.4375 \pm 0.4285i$
57		$2x^3+14x^2+14x+6$	G_{77}	$-5.8997, -0.5502 \pm 0.4537i$
58		$4x^3+12x^2+14x+6$	G_{78}	$-1, -1 \pm 0.7071i$
59		$6x^3+10x^2+14x+6$	G_{72}	$-0.5887, -0.5390 \pm 1.1867i$
60		$2x^3+14x^2+14x+6$	G_{79}	$-5.8997, -0.5502 \pm 0.4537i$
61		$4x^3+12x^2+14x+6$	G_{84}	$-1, -1 \pm 0.7071i$

62		$2x^4+6x^3+8x^2+14x+6$	G_{86}	$-2.3350, -0.5398, -0.0625 \pm 1.5420i$
63		$4x^3+12x^2+14x+6$	G_{85}	$-1, -1 \pm 0.7071i$
64		$4x^3+12x^2+14x+6$	G_{88}	$-1, -1 \pm 0.7071i$
65		$4x^3+12x^2+14x+6$	G_{80}	$-1, -1 \pm 0.7071i$
66		$2x^3+14x^2+14x+6$	G_{87}	$-5.8997, -0.5502 \pm 0.4537i$
67		$4x^3+12x^2+14x+6$	G_{89}	$-1, -1 \pm 0.7071i$
68		$16x^2+14x+6$	G_{90}	$-0.4375 \pm 0.4285i$
69		$8x^3+8x^2+14x+6$	G_{91}	$-0.50, -0.25 \pm 1.1990i$
70		$14x^2+16x+6$	$G_1 \cup G_{25}$	$-0.5714 \pm 0.3194i$
71		$2x^3+12x^2+16x+6$	G_{52}	$-1, -0.6972, -4.3028$
72		$2x^3+12x^2+16x+6$	G_{59}	$-1, -0.6972, -4.3028$
73		$14x^2+16x+6$	$G_1 \cup G_{24}$	$-0.5714 \pm 0.3194i$
74		$14x^2+16x+6$	$G_1 \cup G_{26}$	$-0.5714 \pm 0.3194i$
75		$2x^3+12x^2+16x+6$	G_{53}	$-1, -0.6972, -4.3028$
76		$4x^3+10x^2+16x+6$	G_{54}	$-1, -1 \pm 1.4142i$
77		$4x^3+10x^2+16x+6$	G_{57}	$-1, -1 \pm 1.4142i$
78		$2x^3+12x^2+16x+6$	G_{58}	$-1, -0.6972, -4.3028$

79		$6x^3 + 8x^2 + 16x + 6$	G_{60}	-0.4398, $-0.4468 \pm 1.4402i$
80		$4x^3 + 10x^2 + 16x + 6$	G_{65}	$-1, -1 \pm 1.4142i$
81		$14x^2 + 16x + 6$	$G_1 \cup G_{27}$	$-0.5714 \pm 0.3194i$
82		$14x^2 + 16x + 6$	G_{55}	$-0.5714 \pm 0.3194i$
83		$14x^2 + 16x + 6$	$G_2 \cup G_{10}$	$-0.5714 \pm 0.3194i$
84		$2x^3 + 12x^2 + 16x + 6$	G_{61}	$-1, -0.6972, -4.3028$
85		$14x^2 + 16x + 6$	G_{63}	$-0.5714 \pm 0.3194i$
86		$14x^2 + 16x + 6$	G_{62}	$-0.5714 \pm 0.3194i$
87		$14x^2 + 16x + 6$	G_{66}	$-0.5714 \pm 0.3194i$
88		$4x^3 + 10x^2 + 16x + 6$	G_{64}	$-1, -1 \pm 1.4142i$
89		$2x^3 + 12x^2 + 16x + 6$	G_{67}	$-1, -0.6972, -4.3028$
90		$14x^2 + 16x + 6$	G_{68}	$-0.5714 \pm 0.3194i$
91		$2x^3 + 12x^2 + 16x + 6$	G_{69}	$-1, -0.6972, -4.3028$
92		$12x^2 + 18x + 6$	$G_1 \cup G_{20}$	$-1, -0.5$
93		$2x^3 + 10x^2 + 18x + 6$	G_{39}	-0.4253, $-2.2874 \pm 1.3500i$
94		$12x^2 + 18x + 6$	$G_1 \cup G_{29}$	$-1, -0.5$
95		$2x^3 + 10x^2 + 18x + 6$	G_{41}	-0.4253, $-2.2874 \pm 1.3500i$

96		$4x^3 + 8x^2 + 18x + 6$	G_{42}	-0.3870, $0.8065 \pm 1.7959i$
97		$12x^2 + 18x + 6$	$2G_1 \cup G_{10}$	$-1, -0.5$
98		$12x^2 + 18x + 6$	$G_1 \cup G_{16}$	$-1, -0.5$
99		$12x^2 + 18x + 6$	G_{40}	$-1, -0.5$
100		$12x^2 + 18x + 6$	$G_1 \cup G_{22}$	$-1, -0.5$
101		$12x^2 + 18x + 6$	$G_1 \cup G_{23}$	$-1, -0.5$
102		$12x^2 + 18x + 6$	G_{43}	$-1, -0.5$
103		$2x^3 + 10x^2 + 18x + 6$	G_{44}	-0.4253, $-2.2874 \pm 1.3500i$
104		$2x^3 + 10x^2 + 18x + 6$	G_{46}	-0.4253, $-2.2874 \pm 1.3500i$
105		$12x^2 + 18x + 6$	G_{45}	$-1, -0.5$
106		$12x^2 + 18x + 6$	$G_2 \cup G_9$	$-1, -0.5$
107		$12x^2 + 18x + 6$	G_{49}	$-1, -0.5$
108		$12x^2 + 18x + 6$	G_{47}	$-1, -0.5$
109		$12x^2 + 18x + 6$	G_{48}	$-1, -0.5$
110		$12x^2 + 18x + 6$	G_{50}	$-1, -0.5$
111		$12x^2 + 18x + 6$	$2G_3$	$-1, -0.5$
112		$12x^2 + 18x + 6$	$G_1 \cup G_{18}$	$-1, -0.5$

113		$2x^3 + 8x^2 + 20x + 6$	G_{37}	-0.3430, $-1.8285 \pm 2.3243i$
114		$10x^2 + 20x + 6$	$2G_1 \cup G_9$	$-1.6325, -0.3675$
115		$10x^2 + 20x + 6$	$G_1 \cup G_{13}$	$-1.6325, -0.3675$
116		$2x^3 + 8x^2 + 20x + 6$	G_{38}	-0.3430, $-1.8285 \pm 2.3243i$
117		$10x^2 + 20x + 6$	$G_1 \cup G_{14}$	$-1.6325, -0.3675$
118		$10x^2 + 20x + 6$	$G_1 \cup G_{15}$	$-1.6325, -0.3675$
119		$10x^2 + 20x + 6$	G_{33}	$-1.6325, -0.3675$
120		$10x^2 + 20x + 6$	G_{34}	$-1.6325, -0.3675$
121		$10x^2 + 20x + 6$	$G_2 \cup G_7$	$-1.6325, -0.3675$
122		$10x^2 + 20x + 6$	$G_1 \cup G_{19}$	$-1.6325, -0.3675$
123		$10x^2 + 20x + 6$	G_{32}	$-1.6325, -0.3675$
124		$10x^2 + 20x + 6$	$G_3 \cup G_4$	$-1.6325, -0.3675$
125		$10x^2 + 20x + 6$	$G_2 \cup G_{19}$	$-1.6325, -0.3675$
126		$8x^2 + 22x + 6$	$G_1 \cup G_{17}$	$-0.307, -2.443$
127		$8x^2 + 22x + 6$	$2G_1 \cup G_7$	$-0.307, -2.443$
128		$8x^2 + 22x + 6$	$G_2 \cup G_6$	$-0.307, -2.443$
129		$8x^2 + 22x + 6$	$G_1 \cup G_{11}$	$-0.307, -2.443$
130		$8x^2 + 22x + 6$	$2G_2 \cup G_8$	$-0.307, -2.443$
131		$8x^2 + 22x + 6$	$\cup_{i=1}^{3} G_i$	$-0.307, -2.443$

132		$8x^2 + 22x + 6$	$G_2 \cup G_5$	$-0.307, -2.443$
133		$8x^2 + 22x + 6$	$2G_4$	$-0.307, -2.443$
134		$6x^2 + 24x + 6$	$2G_1 \cup G_6$	$-3.7321, -0.2680$
135		$6x^2 + 24x + 6$	$3G_1 \cup G_3$	$-3.7321, -0.2680$
136		$6x^2 + 24x + 6$	$2G_1 \cup G_5$	$-3.7321, -0.2680$
137		$6x^2 + 24x + 6$	$G_2 \cup G_4$	$-3.7321, -0.2680$
138		$6x^2 + 24x + 6$	$3G_2$	$-3.7321, -0.2680$
139		$4x^2 + 26x + 6$	$3G_1 \cup G_4$	$-0.2396, -6.2604$
140		$4x^2 + 26x + 6$	$2G_1 \cup 2G_2$	$-0.2396, -6.2604$
141		$2x^2 + 28x + 6$	$4G_1 \cup G_2$	$-0.2396, -6.2604$
142		$30x + 6$	$6G_1$	-0.2

Remark 1.3 *From the table 0, for any graph $G_i \in \mathbb{G}_n$, $i = 1, \ldots, 142$, we have $N_k(G, x)$ as presented in the third column, and for its complement $\overline{G_i}$, we can directly deduce $N_k(\overline{G_i}, x)$ from the table, where we have two cases:*

Case 1: $\overline{G_i}$ is connected, for instance, G_{12} (the graph whose figure shown in the row 12),

$$N_k(G_{12}, x) = 4x^3 + 8x^2 + 8x + 5$$

and $\overline{G_{12}} = G_{16}$, hence

$$N_k(\overline{G_{12}}, x) = N_k(G_{16}, x) = 2x^3 + 6x^2 + 12x + 5.$$

Case 2: $\overline{G_i}$ is disconnected, let G_i be G_{17} in row 17. Then

$$N_k(G_{17}, x) = 12x^2 + 8x + 5$$

and since, $\overline{G_{17}} = G_1 \cup G_{10}$, hence by Lemma 1.1,

$$\begin{aligned} N_k(\overline{G_{17}}, x) &= N_k(G_1, x) + N_k(G_{10}, x) \\ &= 1 + (12x + 4) \\ &= 12x + 5. \end{aligned}$$

Proposition 1.4 *Let the N_k-polynomial of a connected graph G with n vertices be defined as*

$$N_k(G,x) = a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0.$$

Then,

$$\sum_{i=0}^{p} a_i = n^2.$$

Proof. Let G be a connected graph with n vertices, m edges and $diam(G) = p$. Since for any vertex $v \in V(G)$, we get

$$\sum_{k=0}^{p} d_k(v) = n$$

and since

$$N_k(G,x) = \sum_{k=0}^{p} \left(\sum_{v\in V(G)} d_k(v)\right) x^k$$

$$= (\sum_{v\in V(G)} d_0 x^0) + (\sum_{v\in V(G)} d_1 x) + \ldots + (\sum_{v\in V(G)} d_p x^p)$$

Then, the coefficients a_i, for any $i = 1, . . , d$, of $N_k(G,x)$ are defined as

$$a_i = \sum_{v\in V(G)} d_i(v)$$

Therefore,

$$\sum_{i=1}^{p} a_i = \sum_{i=1}^{p} (\sum_{v\in V(G)} d_i(v)) = \sum_{v\in V(G)} (\sum_{i=1}^{p} d_i(v)) = \sum_{v\in V(G)} n = n^2.$$

Corollary 1.5 *For a connected graph G with n vertices, if the N_k-polynomial of G is*

$$N_k(G,x) = a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0.$$

Then,

1. $\sum_{i=1}^{p} a_i = n(n-1)$.
2. If $G = P_n$, then $a_i = 2(n-i)$, for every $i = 1,2, \ldots, n-1$
3. If $G = C_n$, then $a_i = 2n$, for every $i = 1,2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$ and

$$a_{\lfloor \frac{n}{2} \rfloor} = \begin{cases} 2n, & if\ n\ is\ odd; \\ n, & if\ n\ is\ even. \end{cases}$$

In the following result, we investigate the relationship between the N_k-polynomial of a graph and the graph connectivity.

Theorem 1.6 *Let G be a graph with n vertices and let*

$$N_k(G,x) = a_p x^p + \ldots + a_1 x + a_0.$$

Then the graph G is disconnected, if and only if

$$\sum_{i=1}^{p} a_i \neq n^2.$$

Proof. From Proposition 1.4, if G is connected then $\sum_{i=0}^{p} a_i = n^2$.
Conversely, suppose, without loss of generality, that G is a disconnected and consists of two components G_1 and G_2 with n_1 and n_2 vertices, respectively, such that $n_1 + n_2 = n$. Let

$$N_k(G_1, x) = \sum_{i=1}^{r} a_i x^i$$

and

$$N_k(G_2, x) = \sum_{i=1}^{t} a_i x^i$$

where, $diam(G_1) = r$ and $diam(G_2) = t$. Then by Lemma 1.1, we get

$$N_k(G, x) = \sum_{i=1}^{p} a_i x^i = N_k(G_1, x) + N_k(G_2, x)$$

and the degree of $N_K(G, x)$, $p = \max\{s, t\}$.
Therefore, from Proposition 1.4, we obtained

$$\sum_{i=1}^{p} a_i = \sum_{i=1}^{s} a_i + \sum_{i=1}^{t} a_i = n_1^2 + n_2^2 < (n_1 + n_2)^2 = n^2.$$

Proposition 1.7 *For a positive integer n, The complete graph K_n, the graph constructed from a complete graph by delete an edge $K_n - e$, the path P_n and the star $K_{1,n-1}$ are N_k-unique.*
Proof. The proof immediately consequences to the definition of N_k-polynomial of graph.

Let a subset $U \subset \mathbb{G}_6$ be the set of all N_k-unique graphs. Then

$$U = \{G_i \in \mathbb{G}_6 : i = 1 - 6, 9, 11, 12, 14, 16, 17, 30 - 38, 45 - 47, 55, 59, 62, 69, 79, 141, 142\}.$$

That mean we have 31 N_k-unique graphs from 142 graph with $n \leq 6$ vertices.

Remark 1.8 *All trees with $n \leq 6$ vertices is a N_k-unique.*

Open Problem: Prove or disprove, any tree with n vertices is N_k-unique.
The N_k-equivalence classes of $\mathbb{G}_6$ are as shown in the following:
1) $n \leq 3$,
there is no N_k-equivalence graphs.
2) $n = 4$,
there is only one class is $[G_7] = \{G_7, G_8\}$.
3) $n = 5$,

we have five classes as following:

- $[G_{13}] = \{G_{13}, G_{15}\}$,
- $[G_{18}] = \{G_{18}, G_{19}\}$,
- $[G_{20}] = \{G_{20}, G_{21}, G_{22}, G_{23}\}$,
- $[G_{24}] = \{G_{24}, G_{25}, G_{26}, G_{27}\}$,
- $[G_{28}] = \{G_{28}, G_{29}\}$

4) $n = 6$,

we have 16 classes as following:

- $[G_{39}] = \{G_{39}, G_{42}, G_{49}\}$,
- $[G_{40}] = \{G_{40}, G_{41}, G_{44}, G_{50}\}$,
- $[G_{43}] = \{G_{43}, G_{48}\}$,
- $[G_{51}] = \{G_{51}, G_{56}, G_{68}\}$,
- $[G_{52}] = \{G_{52}, G_{57}, G_{60}, G_{66}\}$,
- $[G_{53}] = \{G_{53}, G_{54}, G_{58}, G_{61}, G_{63}, G_{64}, G_{65}, G_{67}\}$,
- $[G_{70}] = \{G_{70}, G_{73}, G_{74}, G_{81}, G_{82}, G_{83}, G_{85}, G_{86}, G_{87}, G_{90}\}$,
- $[G_{71}] = \{G_{71}, G_{72}, G_{75}, G_{78}, G_{84}, G_{89}, G_{91}\}$,
- $[G_{76}] = \{G_{76}, G_{77}, G_{80}, G_{88}\}$,
- $[G_{92}] = \{G_{92}, G_{94}, G_{97}, G_{98}, \cdots, G_{102}, G_{105}, \cdots, G_{112}\}$,
- $[G_{93}] = \{G_{93}, G_{95}, G_{103}, G_{104}\}$,
- $[G_{113}] = \{G_{113}, G_{116}\}$,
- $[G_{114}] = \{G_{114}, G_{115}, G_{117}, \cdots, G_{125}\}$,
- $[G_{126}] = \{G_{126}, \cdots, G_{133}\}$,
- $[G_{134}] = \{G_{134}, \cdots, G_{138}\}$,
- $[G_{139}] = \{G_{139}, G_{140}\}$.

Remark 1.9 *For any two graphs* G *and* H, *if* $G \sim^{N_k} H$, *then not need* $\overline{G} \sim^{N_k} \overline{H}$. *For instance,* $G_7 \sim^{N_k} G_8$, *where,* $N_k(G_7, x) = N_k(G_8, x) = 4x^2 + 8x + 4$, $\overline{G_7} = G_1 \cup G_4$ *and* $\overline{G_8} = 2G_2$. *Hence*

$$N_k(\overline{G_7}, x) = 2x^2 + 4x + 4$$

but $N_k(\overline{G_8}, x) = 2x + 4$.

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