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Some properties of the eigenvalues of the sum geometric arithmetic means matrix of graphs

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Abstract

In this paper, we study some properties of the eigenvalues of sum geometric arithmetic energy of graphs and we obtain the values of some coefficients of the characteristic poly nominal of the sum geometric arithmetic means matrix by using some methods. The inverse of the sum geometric arithmetic means matrix of some graphs are also obtained.

Keywords *: The sum geometric arithmetic means matrix of a graph G. The characteristic polynomial of the sum geometric arithmetic means matrix of a graph G.*
AMS Subject Classification: 05C40, 05C99

1. Introduction

The origin of graph theory can be discovered back to Leonhard Euler work on Konigsberg bridges problem (1735), which subsequently led to the concept of an eulrian graph. For more details see [12] a problem that came to be known as the "Seven Bridges of Konigsberg". In this problem, someone had to cross once all the bridges only once and in a continuous sequence, a problem the Euler proved to have no solution by representing it as a set of vertices and edges. This led to the foundation of graph theory and its subsequent improvements [23].
Graph theory is one of the areas of modern mathematics having experienced a most impressive development in recent years. In the beginning, graph theory was only a collection of recreational or challenging problems like Euler tours or the four coloring of a map, with no clear connection among them, or among techniques used to attach them [25].
The concept of energy of a graph was presented by I. Gutman [4] in the year 1978.
Algebraic graph theory is the branch of mathematics that studies graphs by using algebraic properties of associated matrices. In particular, spectral graph theory studies the relation between graph properties and the spectrum of the adjacency matrix or Laplace matrix [9].
Let G be a graph with n vertices $v_1, v_2, \ldots, v_n$ and m edges. Let $A = (a_{ij})$ be the adjacency matrix of the graph G. The eigenvalues $\lambda_1, \lambda_2, \ \ , \lambda_n$ of A, assumed in non-increasing order, are the eigenvalues of the graph G. As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ of G, is defined to be the sum of the absolute values of the eigenvalues of G.

$$E(G) = \sum_{i=1}^{i=n} |\lambda_i|$$

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, assumed in noninteracting order, are the eigenvalues of the graph G. As A is real symmetric, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct eigenvalues of G with multiplicity $m_1, m_2, \ldots, m_n$, respectively. The multiset denoted by $Spec(G)$ and defined as follows

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & .... & \lambda_n \\ m_1 & m_2 & .... & m_n \end{pmatrix},$$

of eigenvalues of $A(G)$ is called the adjacency spectrum of G, the eigenvalues of G are real with sum equal to zero.
In this paper, we will consider finite, undirected, simple (without loops and multiple edges) and connected graphs. We will start with a some important definitions. Let $G = (V(G); E(G))$ be a simple connected graph of order $n = |V(G)|$ and size $m = |E(G)|$. For a vertex $v \in V(G)$, we denote a set of neighbours of v by $N(v)$, degree. Denoted by $deg(v)$ and defined as $deg(v) = |N(v)|$, is the number of the vertices adjacent to v. For more details see [7, 8, 29].

The sum geometric arithmetic means matrix of (n, m) graph is defined via its matrix elements as

$$A_{SGAM}(G) = (a_{ij}) = \begin{cases} \sqrt{deg(u)deg(v)} + \frac{deg(u)+deg(v)}{2}, & \text{if the vertex } \quad \text{u} \quad \text{is adjacent to the vertex v} \\ 0, & \text{otherwise,} \end{cases}$$

where $deg(u), \quad deg(v)$ is degree of the vertex u and v, respectively,[22], and its eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the sum geometric arithmetic means energy of the graph G is defined as

$$E_{SGAM}(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_i \quad (i = 1, 2, \ldots, n)$ are the eigenvalues of $A_{SGAM}(G)$[22].
The energy of a graph G was defined by I. Gutman in 1978 as the sum of the absolute values of eigenvalues of G [15]. The concept of graph energy has origin in chemistry which is used to estimate the total π-electron energy of a molecule. In chemistry the conjugated hydrocarbons can be represented by a graph called molecular graph whose eigenvalues with respect to adjacency matrix A(G) represent the energy level of the electron in the molecule. In Hückel theory the sum of the energies of all the electrons in a molecule is called the π-electron energy of a molecule [24]. In spectral graph theory, the energy-like quantities such as Laplacian energy, distance energy, color energy, color Laplacian energy of a graph etc., are studied in [1, 15, 19, 5].
The concept of color energy was presented by C. Adiga et al. in [1] , P. G. Bhat and S. D'Souza [5] have studied the color Laplacian energy of a graph. Let G be a colored graph on n vertices and m edges. The color Laplacian matrix of G is defined as $Lc(G) = D(G) - Ac(G)$where $D(G) = diag(deg(v_1), deg(v_2), \ldots, deg(v_n))$ represents the diagonal matrix with vertex degrees $deg(v_1), deg(v_2), \ldots, deg(v_n)$ of $v_1, v_2, ldots, v_n$ of G and $Ac(G)$, the color matrix. The eigenvalues $\mu_1, \mu_2, ldots, \mu_n$ of $Lc(G)$ are called color Laplacian eigenvalues of the graph G. If auxiliary color eigenvalues $yi, i = 1 : n$ are defined as $yi = \mu_i - 2nm$, then color Laplacian energy of G is defined as $\sum_{i=1}^{n} |\gamma_i|$.

In 2013, Aouchiche and Hansen [3] defined the distance signless Laplacian matrix of a connected graph G as the nn matrix defined by $DQ(G) = Tr(G) + D_G$, where D_G is the distance matrix of G and $Tr(G)$ is the diagonal matrix of vertex transmissions of G. In 2015, Sehgehalli et al. [27] defined the arithmetic-geometric adjacency matrix (AG matrix) of G, denoted by $A_{ag} = (g_{ij})$, where $g_{ij} = \frac{deg(v_i)+deg(v_j)}{2\sqrt{deg(v_i)deg(v_j)}}$ if $v_iv_j \in E(G)$ and 0 otherwise. Also in 2015, E. Sampathkumar, S. V. Roopa, K. A. Vidya and M. A. Sriraj [24] have introduced the partition energy of a graph $E_{P_k}(G)$ and computed partition energy of some families of graphs with respect to a given partition.

In 2016, V. Kaladevi and A. Abinayaa [20] computed the detour distance laplacian energy of standard graphs. In 2017, B. Sharada, M. I. Sowaity and I. Gutman, [26] introduced the Laplacian sum-eccentricity matrix LS_e of a graph G, and its Laplacian sum-eccentricity energy

$LS_eE = \sum_{i=1}^{n} |\eta_i|$, where $\eta_i = \zeta_i - 2nm$ and where $\zeta_1, \zeta_2, \ldots, \zeta_n$ are the eigenvalues of LS_e. In 2017, P.G. Bhat and S. D'Souza [6] defined the new concept of color Signless Laplacian energy $LE_c^+(G)$. It depends on the underlying graph G and the colors of the vertices. Also in 2017, D. Nilanjan investigated the eccentricity version of Laplacian energy of a graph G. In 2019, Xin Guo and Yubin Gao [13] given some lower and upper bounds on arithmetic-geometric radius and arithmetic-geometric energy. Also in 2019, E. Sampathkumar, S. V. Roopa, K. A. Vidya and M. A. Sriraj [24] have introduced the concept of partition laplacian energy of a graph $lE_{P_k}(G)$ and computed partition energy of some families of graphs with respect to a given partition. In 2017, J.Guo, J. Li,P. Huang and W. Shiu given a combinatorial expression for the fifth coefficient of the (signless) Laplacian characteristic polynomial of a graph and the first five normalized Laplacian coefficients. More details on the properties of graph energy can be seen in [14, 16, 17, 18, 21].

2. Main results

3. Coefficients of the characteristic polynomial of the sum geometric arithmetic means matrix of some graphs

Definition 3.1. [8] *A* ***complete graph*** *is a simple graph in which any two vertices are adjacent and denoted by* K_n.

Example 3.1. *For the complete graph* K_3*, the characteristic polynomial is* $\lambda^3 - 48\lambda - 128 = (\lambda-8)(\lambda+4)^2$*, its clear that* $c_0 = 1$*(coefficient of* λ^3*),* $c_1 = trace(A_{SGAM}(K_3)) = 0+0+0 = 0$.
$c_2 = \frac{1}{2}trace(A_{SGAM}(K_3))-c_1I_3)) = \frac{1}{2}trace(A_{SGAM}(K_3))(A_{SGAM}(K_3)-0\times I_3)) = -\frac{1}{2}trace((A_{SGAM}(K_3)^2)$,

$$\begin{aligned} A_{SGAM}(K_3)^2 &= \begin{pmatrix} 0 & 4 & 4 \\ 4 & 0 & 4 \\ 4 & 4 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} 32 & 16 & 16 \\ 16 & 32 & 16 \\ 16 & 16 & 32 \end{pmatrix}, \end{aligned}$$

hence $c_2 = -\frac{1}{2}(32+32+32) = -48$,
$$c_3 = -\det(A_{SGAM}(K_3) = -\begin{vmatrix} 0 & 4 & 4 \\ 4 & 0 & 4 \\ 4 & 4 & 0 \end{vmatrix} = -128.$$

In general for the complete graph K_n, the coefficients are that $c_0 = 1$, $c_1 = trace(A_{SGAM}(K_n)) = 0+0+\ldots+0 = 0$, $c_2 = -\frac{1}{2}trace(A_{SGAM}(K_n)^2) = 2n(n-1)^3$, because $A_{SGAM}(K_n)^2=$

$$\begin{pmatrix} 4(n-1)^3 & 4(n-1)^2(n-2) & \ldots & 4(n-1)^2(n-2) & 4(n-1)^2(n-2) \\ 4(n-1)^2(n-2) & 4(n-1)^3 & \ldots & 4(n-1)^2(n-2) & 4(n-1)^2(n-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 4(n-1)^2(n-2) & 4(n-1)^2(n-2) & \ldots & 4(n-1)^3 & 4(n-1)^2(n-2) \\ 4(n-1)^2(n-2) & 4(n-1)^2(n-2) & \ldots & 4(n-1)^2(n-2) & 4(n-1)^3 \end{pmatrix}.$$

Hence, $c_2 = \frac{1}{2}trace(A(K_n)^2) = 4(n-1)^3+4(n-1)^3+\ldots+4(n-1)^3 = \frac{1}{2}\times 4n(n-1)^3 = 2n(n-1)^3$.

Remark 3.1. $A_{SGAM}(K_n)^3$

$$= \begin{pmatrix} 8(n-1)^4(n-2) & 8(n-1)^4+8(n-1)^3(n-2)^2 & \dots & 8(n-1)^4+8(n-1)^3(n-2)^2 \\ 8(n-1)^4(n-2) & 8(n-1)^4+8(n-1)^3(n-2)^2 & \dots & 8(n-1)^4+8(n-1)^3(n-2)^2 \\ \vdots & \vdots & \ddots & \vdots \\ 8(n-1)^4(n-2) & 8(n-1)^4+8(n-1)^3(n-2)^2 & \dots & 8(n-1)^4+8(n-1)^3(n-2)^2 \end{pmatrix}$$

Hence, $c_3 = \frac{1}{3} trace(A_{SGAM}(K_n)^3) = 8(n-1)^4(n-2) + 8(n-1)^4(n-2) + \dots + 8(n-1)^4(n-2)$ $= 8n(n-1)^4(n-2)$. *Since* $B_2 = A_{SGAM}(K_n)(B_1 - c_2 I)$ *(Farme's Method), then in this graph* K_n*, we have* $B_2 = A_{SGAM}(K_n)(B_1(K_n) - c_2 I_n)$ *and because* $B_1 = A_{SGAM}(K_n)^2$*, then*

$$B_2 = A_{SGAM}(K_n)(A_{SGAM}(K_n)^2 - c_2 A_{SGAM}(K_n)I) = A_{SGAM}(K_n)^3 - c_2 A_{SGAM}(K_n),$$

$$c_3 = \frac{1}{3} trace(B_2) = \frac{1}{3}(trace(A(K_n)^3 - c_2 trace(A(K_n))), trace(A(K_n) = 0,$$

thus

$$c_3 = \frac{1}{3}\Big(trace(A_{SGAM}(K_n)^3\Big),$$

.

3.1. **The characteristic polynomials method.** The characteristic polynomial of a graph appears explicitly or implicitly in multiple applications. Some of the applications are summarized as follows [28]

(1) The characteristic polynomial is used in the topological theory of chromaticity, in the studies on the factorization of graphs, for counting the spanning trees of labeled planar graphs and for a partial ordering of forests.
(2) The characteristic polynomial is useful in predicting the relative stabilities of conjugated hydrocarbons and in the fonnula1ion of the topological effect on molecular orbitals concept.·
(3) The characteristic polynomial has found application in quantum chemistry, chemical kinetics, dynamics of oscillatory and useful in areas other than mathematics, physics, and chemistry, for example, in computer science and biology.
(4) The characteristic polynomial is related to several other graph invariants such as spectral moments. They are auxiliary functions for counting subgraphs of various kinds belonging to a given graph. Also for the generator for the characteristic equations of a graph.

Definition 3.2. [7] *A* ***complete bipartite graph*** *is a simple bipartite graph with bipartition* (X, Y) *in which each vertex of* X *is joined to each vertex of* Y*; if* $|X| = p$ *and* $|Y| = q$*, such a graph is denoted by* $K_{p,q}$*.*
The complete bipartite graph $K_{p,q}$ *has* $p+q$ *vertices and* pq *edges. The graph* $K_{p,q}$

Example 3.2. *In the complete bipartiat graph* $K_{p,q}$*, the characteristic polynomial is*

$$p(K_{p,q}, \lambda) = \lambda^{p+q-2}\left(\lambda^2 - \left(\frac{p+q}{2}(\sqrt{pq} + pq)\right)^2\right),$$

let $y = \frac{p+q}{2}(\sqrt{pq} + pq)$*, then*

$$p(K_{p,q}, \lambda) = \lambda^{p+q} - y^2 \lambda^{p+q-2}.$$

We note that

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = -y^2, \quad c_3 = c_4 = \dots = c_{n-1} = 0 = c_n = 0.$$

Example 3.3. *The characteristic polynomials of the sum geometric arithmetic means matrix of star graph, denoted by* $p_{SGAM}(K_{1,n-1},\lambda)$ *and defined by*

$$\begin{aligned} p_{SGAM}(K_{1,n-1},\lambda) &= \lambda^{n-2}\left(\lambda^2-\left(\frac{n}{2}\sqrt{n-1}+(n-1)^2\right)\right) \\ &= \lambda^n-\lambda^{n-2}\left(\lambda^2-\left(\frac{n}{2}\sqrt{n-1}+(n-1)^2\right)\right) \\ &= c_0\lambda^n+c_1\lambda^{n-1}+c_2\lambda^{n-2}+\ldots+c_n. \end{aligned}$$

Since $c_1 = trace((A_{SGAM}(K_{1,n-1})) = 0+0+\ldots+0 = 0$ *because every element in diagonal (* $a_{ii}=0$*) in the sum geometric arithmetic means matrix of star graph.*
And

$$c_2=\sum_{i\neq j}\lambda_i\lambda_j,$$

where λ_i *and* λ_j *are eigenvalues of sum geometric arithmetic means matrix of star graph, hence* $c_2 = 0\times y+0\times y+\ldots+0\times y+0\times -y+0\times -y+\ldots+0\times -y+y\times -y=-y^2$.

Example 3.4.

$$A_{SGAM}(K_3)=\begin{pmatrix}0&4&4\\4&0&4\\4&4&0\end{pmatrix};$$

the characteristic polynomial is

$$\lambda^3-48\lambda-128=(\lambda-8)(\lambda+4)^2.$$

The eigenvalues are

$$\lambda_1=8,\lambda_2=-4,\lambda_3=-4.$$

The coefficients of the characteristic polynomials are

$$c_0=1,\quad c_1=0=trace\left(A_{SGAM}(K_3)\right),$$

$$c_2=-48=\sum_{i\neq j}\lambda_i\lambda_j=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3=8\times -4+8\times -4+-4\times -4=-48,$$

$$c_3=\lambda_1\lambda_2\lambda_3=det(A_{SGAM}(K_3))=(-1)^3\times 8\times -4\times -4=-128.$$

Example 3.5.

$$A_{SGAM}(K_4)=\begin{pmatrix}0&6&6&6\\6&0&6&6\\6&6&0&6\\6&6&6&0\end{pmatrix};$$

the characteristic polynomial is

$$\lambda^4-216\lambda^2-1728\lambda-3888=(\lambda-18)(\lambda+6)^3.$$

The eigenvalues are

$$\lambda_1=18,\lambda_2=-6,\lambda_3=-6,\lambda_4=-6.$$

The coefficients of the characteristic polynomials are

$$c_0=1,\quad c_1=0=trace\left(A_{SGAM}(k_4)\right),$$

$$\begin{aligned} c_2 &= \sum_{i\neq j}\lambda_i\lambda_j=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4 \\ &= 18\times -6+18\times -6+18\times -6+-6\times -6+-6\times -6+-6\times -6=-216. \end{aligned}$$

$$\begin{aligned} c_3 &= \lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda 4 \\ &= 18\times -6\times -6+18\times -6\times -6+18\times -6\times -6+-6\times -6\times -6=-1728. \end{aligned}$$

$$\begin{aligned} c_4 &= (-1)^4\lambda_1\lambda_2\lambda_3\lambda_4 = det\,(A_{SGAM}(k_4)) \\ &= (-1)^4 \times 18 \times -6 \times -6 \times -6 = -3888. \end{aligned}$$

Theorem 3.1. *Let λ_i $(1 \le i \le n)$ be the eigenvalues of sum geometric arithmetic means matrix of complete graph K_n, then the coefficients of the characteristic polynomial are*

(1) c_0=*1*
(2) $c_1 = 0$
(3) $c_2 = -4(n-1)^4 + 2(n-1)^3(n-2)$
(4) $c_3 = (-1)^3\left(4(n-1)^5(n-2) - \frac{4}{3}(n-1)^4(n-2)(n-3)\right)$
(5) $c_n = -1)^n \prod_{i=1}^{n} \lambda_i = \lambda_1\lambda_2\lambda_3 \ldots \lambda_n$.

Proof. From the characteristic polynomials of sum geometric arithmetic means matrix of complete graph K_n and since the eigenvalues of $A_{SGAM}(K_n)$ are

$$\lambda_1 = 2(n-1)^2; \lambda_2 = -2(n-1); \lambda_3 = -2(n-1); \lambda_4 = -2(n-1); \ldots; \lambda_n = -2(n-1).$$

We have

(1) $c_0 = 1$ from the definition of characteristic polynomials of sum geometric arithmetic means matrix of complete graph K_n.
(2) Since $c_1 = trace(A_{SGAM}(K_n))$, so
$$c_1 = 2(n-1)^2 - 2(n-1)(n-1) = 2(n-1)^2 - 2(n-1)^2 = 0.$$
(3) $c_2 = \sum_{i \ne j} \lambda_i\lambda_j$, thus
$$\begin{aligned} c_2 &= 2(n-1)^2 \times (-2(n-1)) \times (n-1) - 2(n-1) \times (-2(n-1)) \times (n-2) \\ &= -4(n-1)^4 + 2(n-1)^3(n-2). \end{aligned}$$
(4) $c_3 = (-1)^3[4(n-1)^5(n-2) - \frac{4}{3}(n-1)^4(n-2)(n-3)]$;
(5) because $c_n = (-1)^n det\,(A_{SGAM}(K_n)) = (-1)^n \prod_{i=1}^{n} \lambda_i = \lambda_1\lambda_2\lambda_3 \ldots \lambda_n$.

□

Theorem 3.2. *Let $\lambda_i (1 \le i \le n)$ be the eigenvalues of sum geometric arithmetic means matrix of complete bipartite graph $K_{p,q}$, then the coefficients of the characteristic polynomial are*

(1) c_0=*1*
(2) $c_1 = 0$
(3) $c_2 = -(\frac{p+q}{2}\sqrt{pq} + pq)^2$
(4) $c_3 = 0$
(5) $c_n = 0$.

Proof. From the the characteristic polynomials of the sum geometric arithmetic means matrix of complete bipartite graph $K_{p,q}$ is

$$p_{SGAM}(K_{p,q}, \lambda) = \lambda^{p+q-2}\left(\lambda^2 - \left(\frac{p+q}{2}(\sqrt{pq} + pq)\right)^2\right),$$

and since the eigenvalues of $A_{SGAM}(K_{p,q})$ are

$$\lambda_1 = (\frac{p+q}{2}\sqrt{pq} + pq); \quad \lambda_2 = -(\frac{p+q}{2}\sqrt{pq} + pq); \quad \lambda_3 = \lambda_4 = \lambda_5 = \ldots = \lambda_{p+q-2} = 0.$$

We have

(1) c_0=1 . It is clear that from the definition of characteristic polynomials of sum geometric arithmetic means matrix of complete bipartite graph $K_{p,q}$.
(2) Since $c_1 = trace(A_{SGAM}(K_{p,q})$, so
$$c_1 = 2(n-1)^2 - 2(n-1)(n-1) = 2(n-1)^2 - 2(n-1)^2 = 0.$$

(3) Since $c_2 = \sum_{i \neq j} \lambda_i \lambda_j$, thus

$$c_2 = 2(n-1)^2 \times (-2(n-1)) \times (n-1) - 2(n-1) \times (-2(n-1)) \times (n-2)$$
$$= -4(n-1)^4 + 2(n-1)^3(n-2).$$

(4) Since $c_3 = \sum_{i \neq jneqk} \lambda_i \lambda_j \lambda_k$, thus

$$c_3 = (-1)^3[4(n-1)^5(n-2) - \frac{4}{3}(n-1)^4(n-2)(n-3)].$$

(5) Since $c_n = (-1)^n det\,(A_{SGAM}(K_{p,q})) = (-1)^n \prod_{i=1}^n \lambda_i = \lambda_1\lambda_2\lambda_3 \ldots \lambda_n$, as $\lambda_3 = \lambda_4 = \lambda_5 = \ldots = \lambda_{p+q-2} = 0$, then $c_n = 0$.

□

Definition 3.3. [8] *A* ***star graph*** *is a complete bipartite graph $K_{p,q}$ with $|p| = 1$ or $|q| = 1$, denoted by $K_{1,n-1}$ i.e. A star graph is simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = 1$ and $|Y| = n-1$, such a graph is denoted by $K_{1,n-1}$.*

Theorem 3.3. *Let $\lambda_i \quad (1 \leq i \leq n)$ be the eigenvalues of the sum geometric arithmetic means matrix of star graph $K_{1,n-1}$, then the coefficients of the characteristic polynomial are*

(1) c_0=*1*
(2) $c_1 = 0$
(3) $c_2 = -(\frac{n}{2}\sqrt{n-1} + (n-1))^2$
(4) $c_3 = 0$
(5) $c_n = 0$.

Proof. From the characteristic polynomials of the sum geometric arithmetic means matrix of star graph $K_{1,n-1}$ is

$$p_{SGAM}(K_{1,n-1}, \lambda) = \lambda^{n-2}\left(\lambda^2 - (\frac{n}{2}\sqrt{n-1} + (n-1)^2)\right)$$

and since the eigenvalues of $A_{SGAM}(K_{1,n-1})$ are

$$\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_{n-2} = 0; \lambda_{n-1} = (\frac{n}{2}\sqrt{n-1} + (n-1)); \lambda_n = -\lambda_{n-1} = -(\frac{n}{2}\sqrt{n-1} + (n-1)).$$

We have

(1) $c_0 = 0$ from the definition of characteristic polynomials of sum geometric arithmetic means matrix of star graph $K_{1,n-1}$.
(2) Since $c_1 = trace(A_{SGAM}(K_{p,q})$, so

$$c_1 = 2(n-1)^2 - 2(n-1)(n-1) = 2(n-1)^2 - 2(n-1)^2 = 0.$$

(3) Since $c_2 = \sum_{i \neq j} \lambda_i \lambda_j$, and because $\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_{n-2} = 0$, thus

$$c_2 = 0 + \lambda_{n-1} \times \lambda_n$$
$$= (\frac{n}{2}\sqrt{n-1} + (n-1)) \times (-\frac{n}{2}\sqrt{n-1} = -(\frac{n}{2}\sqrt{n-1} + (n-1))^2.$$

(4) Since $c_3 = \sum_{i \neq jneqk} \lambda_i \lambda_j \lambda_k$, and as $\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_{n-2} = 0$, so

$$c_3 = 0.$$

(5) Since $c_n = (-1)^n det\,(A_{SGAM}(K_{p,q})) = (-1)^n \prod_{i=1}^n \lambda_i = \lambda_1\lambda_2\lambda_3 \ldots \lambda_n$, as $\lambda_3 = \lambda_4 = \lambda_5 = \ldots = \lambda_{n-2} = 0$, then $c_n = 0$.

□

Definition 3.4. [10] *The crown graph S_p^0 on $2p$ vertices for an integer $p \geq 2$ is the graph with vertex set $\{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p\}$ and edge set $\{u_iv_i : 1 \leq i, j \leq p, i \neq j\}$. S_p^0 is therefore equivalent to the complete bipartite graph $K_{p,p}$ with horizontal edges removed.*

Theorem 3.4. *Let $\lambda_i(1 \leq i \leq n$ be the eigenvalues of f the sum geometric arithmetic means matrix of crown graph $S_p^0)$, then the coefficients of the characteristic polynomial are*

(1) c_0=*1*
(2) $c_1 = 0$
(3) $c_2 = 4(p-1)^3(2-p)$
(4) $c_3 = 0$
(5) $c_{2p} == (-1)^p 2^{2p}(p-1)^2 p.$

Proof. From the characteristic polynomials of the sum geometric arithmetic means matrix of crown graph S_p^0 is and since the eigenvalues of $A_{SGAM}(S_p^0)$ are

$\lambda_1 = 2(p-1)^2; \lambda_2 = -2(p-1)^2; \lambda_3 = \lambda_4 = \lambda_5 = \ldots = \lambda_{p+1} = 2(p-1), \lambda_{p+2}\lambda_{p+3} = \ldots = \lambda_{2p} = -2(p-1).$

We have

(1) c_0=1 . Clearly, it follows from the definition of characteristic polynomials of the sum geometric arithmetic means matrix of crown graph S_p^0.
(2) Since $c_1 = trace(A_{SGAM}(S_p^0)$, so

$$c_1 = 2(p-1)^2 - 2(p-1)(p-1) = 2(p-1)^2 - 2(p-1)^2 = 0.$$

(3) $c_2 = \sum_{i \neq j} \lambda_i \lambda_j$, thus

$$c_2 = 2(p-1)^2 \times (-2(p-1)) \times (p-1) - 2(p-1) \times (-2(p-1)) \times (p-2) = -4(p-1)^4 + 2(p-1)^3(p-2).$$

(4) Since $c_3 = \sum_{i \neq jneqk} \lambda_i \lambda_j \lambda_k$, and as $\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_{p-2} = 0$, so

$$c_3 = 0.$$

(5) Since

$$\begin{aligned} c_{2p} &= (-1)^{2p} det\Big(A_{SGAM}(S_p^0)\Big) = (-1)^{2p} \prod_{i=1}^{2p} \lambda_i^* \\ &= 2(p-1)^2 \times -2(p-1)^2 \times 2p-1 \times (p-1) \times -2(p-1) \times (p-1), \end{aligned}$$

then $c_{2p} = (-1)^p 2^{2p}(p-1)^{2p}$.

□

3.2. **The inverse of the sum geometric arithmetic means matrix of some graphs.**

Definition 3.5. [11] *If a matrix B such that $AB = BA = I_n$ exists, then it is unique, and it is called the inverse of A. The matrix B is also denoted by A^{-1}. An invertible matrix is also called a nonsingular matrix, and a matrix that is not invertible is called a singular matrix.*

Example 3.6. *In the complete graph K_3, the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(K_3)$, denoted by $A_{SGAM}^{-1}(K_3)$ is equal to*

$$\begin{pmatrix} -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{pmatrix}.$$

Example 3.7. *In the complete graph K_4, the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(K_4)$, denoted by $A_{SGAM}^{-1}(K_4)$ is equal to*

$$\begin{pmatrix} -\frac{1}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} \\ \frac{1}{18} & -\frac{1}{9} & \frac{1}{18} & \frac{1}{18} \\ \frac{1}{18} & \frac{1}{18} & -\frac{1}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{1}{18} & \frac{1}{18} & -\frac{1}{9} \end{pmatrix}.$$

Example 3.8. *In the complete graph K_5, the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(K_5)$, is*

$$A^{-1}_{SGAM}(K_5) = \begin{pmatrix} -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} \\ \frac{1}{32} & -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} \\ \frac{1}{32} & \frac{1}{32} & -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} \\ \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & -\frac{3}{32} & \frac{1}{32} \\ \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & -\frac{3}{32} \end{pmatrix}.$$

Example 3.9. *In the complete graph K_5, the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(K_5)$, is*

$$A^{-1}_{SGAM}(K_5) = \begin{pmatrix} -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} \\ \frac{1}{32} & -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} \\ \frac{1}{32} & \frac{1}{32} & -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} \\ \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & -\frac{3}{32} & \frac{1}{32} \\ \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & -\frac{3}{32} \end{pmatrix}.$$

Example 3.10. *In the complete graph K_6, the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(K_6)$, is*

$$A^{-1}_{SGAM}(K_6) = \begin{pmatrix} -\frac{2}{25} & \frac{1}{50} & \frac{1}{50} & \frac{1}{50} & \frac{1}{50} & \frac{1}{50} \\ \frac{1}{50} & -\frac{2}{25} & \frac{1}{50} & \frac{1}{50} & \frac{1}{50} & \frac{1}{50} \\ \frac{1}{50} & \frac{1}{50} & -\frac{2}{25} & \frac{1}{50} & \frac{1}{50} & \frac{1}{50} \\ \frac{1}{50} & \frac{1}{50} & \frac{1}{50} & -\frac{2}{25} & \frac{1}{50} & \frac{1}{50} \\ \frac{1}{50} & \frac{1}{50} & \frac{1}{50} & \frac{1}{50} & -\frac{2}{25} & \frac{1}{50} \\ \frac{1}{50} & \frac{1}{50} & \frac{1}{50} & \frac{1}{50} & \frac{1}{50} & -\frac{2}{25} \end{pmatrix}.$$

Example 3.11. *In the complete graph K_7, we have the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(K_7)$, denoted by $A^{-1}_{SGAM}(K_7)$ and equal to*

$$\begin{pmatrix} -\frac{5}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} \\ \frac{1}{72} & -\frac{5}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} \\ \frac{1}{72} & \frac{1}{72} & -\frac{5}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} \\ \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & -\frac{5}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} \\ \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & -\frac{5}{72} & \frac{1}{72} & \frac{1}{72} \\ \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & -\frac{5}{72} & \frac{1}{72} \\ \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & -\frac{5}{72} \end{pmatrix}.$$

Example 3.12. *For the complete graph K_n, the inverse of the sum geometric- arithmetic means matrix of $A_{SGAM}(K_n)$, denoted by $A^{-1}_{SGAM}(K_n)$ and is equal to*

$$\begin{pmatrix} -\frac{n-2}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \cdots & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} \\ \frac{1}{2(n-1)^2} & -\frac{n-2}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \cdots & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} \\ \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & -\frac{n-2}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \cdots & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} \\ \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & -\frac{n-2}{2(n-1)^2} & \cdots & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \cdots & -\frac{n-2}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} \\ \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \cdots & -\frac{n-2}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} \\ \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \cdots & \frac{1}{2(n-1)^2} & -\frac{n-2}{2(n-1)^2} & \frac{1}{2(n-1)^2} \\ \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \cdots & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & \frac{1}{2(n-1)^2} & -\frac{n-2}{2(n-1)^2} \end{pmatrix}.$$

Example 3.13. *In the crown graph S_2^0 , the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(S_2^0)$, denoted by $A^{-1}_{SGAM}(S_2^0)$ and is equal to*

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix},$$

and we can write it as $A^{-1}_{SGAM}(S_2^0) = \left(\begin{array}{c|c} O_2 & A^{-1}_{SGAM}(K_2) \\ \hline A^{-1}_{SGAM}(K_2) & O_2 \end{array}\right)$, *where O_2 denotes to the zero matrix of order 2 , $A^{-1}_{SGAM}(K_2)$ denotes to the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(K_2)$.*

Example 3.14. *In the crown graph S_3^0, the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(S_3^0)$, denoted by $A^{-1}_{SGAM}(S_3^0)$ and equal to*

$$\begin{pmatrix} 0 & 0 & 0 & \frac{-1}{8} & \frac{1}{8} & \frac{1}{8} \\ 0 & 0 & 0 & \frac{1}{8} & \frac{-1}{8} & \frac{1}{8} \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{-1}{8} \\ \frac{-1}{8} & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{-1}{8} & \frac{1}{8} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{-1}{8} & 0 & 0 & 0 \end{pmatrix},$$

and we can write it as $A^{-1}_{SGAM}(S_3^0) = \left(\begin{array}{c|c} O_3 & A^{-1}_{SGAM}(K_3) \\ \hline A^{-1}_{SGAM}(K_3) & O_3 \end{array}\right)$, *where O_3 denotes to the zero matrix of order 3 , $A^{-1}_{SGAM}(K_3)$ denotes to the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(K_3)$.*

Example 3.15. *In the crown graph S_4^0, the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(S_4^0)$, denoted by $A^{-1}_{SGAM}(S_4^0)$ and is equal to*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{-1}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} \\ 0 & 0 & 0 & 0 & \frac{1}{18} & \frac{-1}{9} & \frac{1}{18} & \frac{1}{18} \\ 0 & 0 & 0 & 0 & \frac{1}{18} & \frac{1}{18} & \frac{-1}{9} & \frac{1}{18} \\ 0 & 0 & 0 & 0 & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} & \frac{-1}{9} \\ \frac{-1}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} & 0 & 0 & 0 & 0 \\ \frac{1}{18} & \frac{-1}{9} & \frac{1}{18} & \frac{1}{18} & 0 & 0 & 0 & 0 \\ \frac{1}{18} & \frac{1}{18} & \frac{-1}{9} & \frac{1}{18} & 0 & 0 & 0 & 0 \\ \frac{1}{18} & \frac{1}{18} & \frac{1}{18} & \frac{-1}{9} & 0 & 0 & 0 & 0 \end{pmatrix},$$

and we can write it as $A^{-1}_{SGAM}(S_4^0) = \left(\begin{array}{c|c} O_4 & A^{-1}_{SGAM}(K_4) \\ \hline A^{-1}_{SGAM}(K_4) & O_4 \end{array}\right)$, *where O_4 denotes to the zero matrix of order 4 , $A^{-1}_{SGAM}(K_4)$ denotes to the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(K_4)$.*

Example 3.16. *For the crown graph S_p^0 , the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(S_p^0)$, denoted by $A_{SGAM}^{-1}(S_p^0)$ and is equal to*

$$\left(\begin{array}{ccccc:ccccc}
0 & 0 & 0 & \cdots & 0 & -\frac{p-2}{2(p-1)^2} & \frac{1}{2(p-1)^2} & \frac{1}{2(p-1)^2} & \cdots & \frac{1}{2(p-1)^2} \\
0 & 0 & 0 & \cdots & 0 & \frac{1}{2(p-1)^2} & -\frac{p-2}{2(p-1)^2} & \frac{1}{2(p-1)^2} & \cdots & \frac{1}{2(p-1)^2} \\
0 & 0 & 0 & \cdots & 0 & \frac{1}{2(p-1)^2} & \frac{1}{2(p-1)^2} & -\frac{p-2}{2(p-1)^2} & \cdots & \frac{1}{2(p-1)^2} \\
0 & 0 & 0 & \cdots & 0 & \frac{1}{2(p-1)^2} & \frac{1}{2(p-1)^2} & -\frac{p-2}{2(p-1)^2} & \cdots & \frac{1}{2(p-1)^2} \\
0 & 0 & 0 & \cdots & 0 & \frac{1}{2(p-1)^2} & \frac{1}{2(p-1)^2} & -\frac{p-2}{2(p-1)^2} & \cdots & \frac{1}{2(p-1)^2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
-\frac{p-2}{2(p-1)^2} & \frac{1}{2(p-1)^2} & \frac{1}{2(n-1)^2} & \cdots & \frac{1}{2(p-1)^2} & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{2(p-1)^2} & -\frac{p-2}{2(p-1)^2} & \frac{1}{2(p-1)^2} & \cdots & \frac{1}{2(p-1)^2} & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{2(p-1)^2} & \frac{1}{2(p-1)^2} & -\frac{p-2}{2(p-1)^2} & \cdots & \frac{1}{2(p-1)^2} & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{2(p-1)^2} & \frac{1}{2(p-1)^2} & \frac{1}{2(p-1)^2} & \cdots & \frac{1}{2(p-1)^2} & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{2(p-1)^2} & \frac{1}{2(p-1)^2} & \frac{1}{2(p-1)^2} & \cdots & -\frac{p-2}{2(p-1)^2} & 0 & 0 & 0 & \cdots & 0
\end{array}\right),$$

and we can write it as $A_{SGAM}^{-1}(S_p^0) = \left(\begin{array}{c|c} O_n & A_{SGAM}^{-1}(K_n) \\ \hline A_{SGAM}^{-1}(K_n) & O_n \end{array}\right)$ *, where O_n denotes to the zero matrix of order n, $A_{SGAM}^{-1}(K_n)$ denotes to the inverse of the sum geometric arithmetic means matrix of $A_{SGAM}(K_n)$.*

Remark 3.2. *The inverse of the sum geometric arithmetic matrix of the complete bipartite graph $A_{SGAM}(K_{p,q})$ does not exist because it has zero eigenvalues (from theorem in linear algebra)(A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A)*[2].

Remark 3.3. *The inverse of the sum geometric arithmetic matrix of the star graph $A_{SGAM}(k_{1,n-1})$ does not exist because it has zero eigenvalues (from theorem in linear algebra)(A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A)*[2].

3.3. **The eigenvalues of power of the sum geometric arithmetic means matrix of some graphs.** In this section, we find the eigenvalues of power of the sum geometric arithmetic means matrix of some graphs

Definition 3.6. [2] *If A is square matrix we define $A^1 = A, A^2 = AA, A^3 = A^2A = AAA, A^n = A^{n-1}A = AA \cdots AA \quad (nfactors)$.*

Example 3.17. *The eigenvalues of $A_{SGAM}(K_3)^p$ are*

$$8^p, -4^p, -4^p$$

Example 3.18. *The eigenvalues of $A_{SGAM}(K_4)^p$ are*

$$18^p, -6^p, -6^p, -6^p$$

Example 3.19. *The eigenvalues of $A_{SGAM}(K_5)^p$ are*

$$32^p, -8^p, -8^p, -8^p$$

Remark 3.4. *In linear algebra we have important property of eigenvalues which is if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, and if $q(A)$ is polynomial of any matrix $A(q(A) = A^k)$, then the eigenvalues of $q(A)$ are $q(\lambda_1), q(\lambda_2), \ldots, q(\lambda_n)$ or ${\lambda_1}^k, {\lambda_2}^k, \ldots, {\lambda_n}^k$* [2]

Theorem 3.5. *The eigenvalues of $A_{SGAM}(K_n)^k$ are $2^k(n-1)^{2k}$, $(-2)^k(n-1)^k(n-1)$ (times).*

Proof. Since the eigenvalues of $A_{SGAM}(k_n)$ are $2(n-1)^2$, $-2(n-1)(n-1)(times)$ and by using the remark 3.4, we get the result. □

Theorem 3.6. *The eigenvalues of $A_{SGAM}(K_{p,q})^k$ are $2^p(n-1)^{2p}$, $(-2)^p(n-1)^p(n-1)$ (times).*

Proof. Since the eigenvalues of $A_{SGAM}(k_n)$ are $2(n-1)^2$, $-2(n-1)(n-1)(times)$ and by using the remark 3.4, we get the result. □

Theorem 3.7. *The eigenvalues of $A_{SGAM}(K_{1,n-1})^k$ are $2^p(n-1)^{2p}$, $(-2)^p(n-1)^p(n-1)$ (times).*

Proof. Since the eigenvalues of $A_{SGAM}(k_n)$ are $2(n-1)^2$, $-2(n-1)(n-1)(times)$ and by using the remark 3.4, we get the result. □

Theorem 3.8. *The eigenvalues of $A_{SGAM}(S_p^0)^k$ are $2^p(n-1)^{2p}$, $(-2)^p(n-1)^p(n-1)$ (times).*

Proof. Since the eigenvalues of $A_{SGAM}(S_p^0)$ are $2(n-1)^2$, $-2(n-1)(n-1)(times)$ and by using the remark 3.4, we get the result. □

References

[1] C. Adiga, E. Sampathkumar, M. A. Sriraj, Shrikanth A. S, Color energy of a graph, Proc. Jangjeon Math. Soc., 16 (2013), No. 3, 335-351.

[2] H. Anton and C. Rorres, Elementary Linear Algebra, 10^{th} ed., John Wiley and sons, 2010.

[3] M. Aouchiche,and P.Hansen,*Two Laplacians for the distance matrix of a graph*, Lin. Algebra Its Appl. 439 (2013), 21–33.

[4] R. Balakrishnan, *The energy of a graph*, Lin. Alg. Applic., 387 (2004) 287–295.

[5] P.G Bhat, S. D'Souza, *Color Laplacian energy of a graph* , Proc. Jangjeon Math. Soc., 18(2015), No.3, 321-330.

[6] P.G. Bhat and S. D'Souza, *Color signless Laplacian energy of graphs*, AKCE International Journal of Graphs and Combinatorics, 14 (2017), 142–148.

[7] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Elseyier Science Publishing Co., Inc., New York, U.S.A.,1976.

[8] J. A. Bondy and U. S. R. Murty, Graph theory, Springer, Berlin, 2008.

[9] A. Brouwer and W. Haemers, Spectra of Graphs, Springer, Amsterdam, 2011.

[10] Daoud and S.N., *Complexity of cocktail party graph and crown graph*,Am. J. Applied Sci., 9 (2)(2012), 202-207.

[11] J. Gallier, Basics of Algebra and Analysis For Computer Science, Philadelphia,U. S. A., 2012.

[12] J. Gross and J. Yellen, Handbook of Graph Theory, Disc. Math. Applic., CRC press, London, 2003.

[13] X. Guo and Y. Gao *Arithmetic-geometric spectral radius and energy of graphs*, MATCH Commun. Math. Comput. Chem., 83 (2020) 651-660.

[14] I. Gutman, *The energy of a graph: old and new results*, Alg. Combin. Applic., Springer-Verlag, Berlin, 2001, 196–211.

[15] I. Gutman and B. Zhou, *Laplacian energy of a graph*, Lin. Alg. Appl., 414 (2006), 29–37.

[16] I. Gutman, S. Z. Firoozabadi, J.A. Peña and J. Rada, *On the energy of regular graphs*, Match Commun. Math. Comput. Chem., 57 (2007), 435–442.

[17] I. Gutman, X. Li and J. Zhang, Graph energy, Analysis of Complex Networks. From Biology to Linguistics, Wiley-VCH, Weinheim, 2009, 145–174.

[18] F. Harary, Graph Theory, Addison Wesley, Reading, 1972.

[19] G. Indulal , I. Gutman, A. Vijayakumar, On distance energy of graphs, Match Commun. Math. Comput. Chem., 60(2008), 461-472.

[20] V. Kaladevi and A. Abinayaa, *On detour distance Laplacian energy*, Journal of Informatics and Mathematical Sciences, 9 (3) (2017), 721-732.

[21] S. Li, X. Li, H. Ma and I. Gutman,*On triregular graphs whose energy exceeds the number of vertices*, Match Commun. Math. Comput.Chem., 64 (2010), 201–216.

[22] M. Mohsen and S. S. Mahde, *Sum geometric arithmetic means energy of graphs*, EPH- I. J. Maths. Statics, 8 (1) (2022), 1-10.

[23] S. Monicandan, Graph Theory and Compinatorics, Manonmaiam Sundaranar University, Tirunelveli, 2016.

[24] E. Sampathkumar, S. V. Roopa, K. A. Vidya and M. A. Sriraj, *Partition Laplacian energy of a graph*, Palestine Journal of Mathematics, 8(1)(2019) , 272–284.

[25] E. Scheinerman and D. Ullman., Fractional Graph Theory, John Wiley and Sons, Paris, France, 2008.

[26] B. Sharada, M. I. Sowaity and I. Gutman,*Laplacian sum-eccentricity energy of a graph*, Mathematics Interdisciplinary Research 2 (2017) 209 - 219.

[27] V. S. Shegehall and R. Kanabur, *Arithmetic-geometric indices of path graph*, J. Math. Comput. Sci., 16 (2015), 19-24.

[28] N. Trlnajstic, Chemical Graph Theory, 2^{nd} ed., CRC Press, Ann Arbor, London, Tokyo, 1987.

[29] C.Vasudev, Graph Theorey with Applications, New Age Internationa, Daryaganj, New Delhi, 2006.