Solutions for some anisotropic nonlinear elliptic unilateral problems

Mohammed Al-Hawmi
Department of Mathematics, Faculty of Education, Arts and Sciences, University of Shaba Region, Mareb, Yemen
Email: m.alhomi2011@gmail.com,

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Abstract

In this work,, we consider the following nonlinear elliptic unilateral equations of the type

$$Au + g(x,u,\nabla u) = f \quad in \ \Omega$$

where Au is an operator of Leray-Lions type acting from $W_0^{1,\vec{p}}$ into its dual define by

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,u,\nabla u)$$

and the function $g(x,u,\nabla u)$ is a nonlinear lower order term with natural growth with respect to $|\nabla u|$, satisfying the sign condition in an anisotropic Sobolev space, and we will prove the existence of entropy solutions for our unilateral problem, where f belongs to $L^1(\Omega)$.

key words : *Anisotropic Sobolev spaces, Nonlinear elliptic unilateral problem, Entropy solutions.*

حلول لبعض مشاكل القطع الناقص الغير الخطية متماثل الخواص بجانب واحد

الملخص

في هذا العمل، نعتبر معادلة القطع الناقص الغير الخطية من جانب واحد للنموذج الاتي:

$$Au + g(x,u,\nabla u) = f \quad in \ \Omega$$

حيث Au مؤثر لاري لاينس معرف من فضاء سبوليف متماثل الخواص الى ثنائي الفضاء معرف ب

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,u,\nabla u)$$

والدالة $g(x,u,\nabla u)$ غير خطية مع التزايد الطبيعي مع ما يتعلق بالقيمة المطلقة لانحدار u $(|\nabla u|)$

وتحقق شرط الاشارة في فضاء متماثل الخواص وسوف نبرهن وجود الحلول الكونية لمشكلتنا من جانب واحد حيث الدالة f تنتمي الى فضاء $L^1(\Omega)$

الكلمات المفتاحية : *فضاءات سبوليف متماثل الخواص ، مشكلة القطع الناقص الغير الخطية من جانب واحد ، الحلول الكونية.*

1 Introduction

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$. Boccardo and Gallouët have considered in [14] the elliptic problem

$$\begin{cases} \ \text{div}\, a(x,u,\nabla u) = f & \text{in} \quad \Omega, \\ u = \ 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where f is a bounded Radon measure on Ω. They have proved the existence of solutions $u \in W_0^{1,q}(\Omega)$ for all $1 < q < \bar{q} = \frac{N(p-1)}{N-1}$. Also, they have proved some regularity results. In [11], Boccardo has studied the existence of entropy solutions for the quasilinear elliptic problem of the type

$$\begin{cases} -\text{div}\, a(x,u,\nabla u) = f - \text{div}\, \phi(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

with $f \in L^1(\Omega)$ and $\phi(\cdot) \in C^0(I\!R, I\!R^N)$. He has proved the existence and some regularity of solutions. The most studied problems were involved in the p-Laplace operators

$$\Delta_p u = \text{div}\left(|\nabla u|^{p-2}\,\nabla u\right), \tag{1.2}$$

with p constant, where the authors have proved the existence of solutions for some nonlinear elliptic unilateral problems in Sobolev spaces. (see for example [13, 19]).

In the framework of Orlicz Sobolev spaces, Aharouch and Bennouna [1] have treated the quasilinear elliptic unilateral problem

$$\begin{cases} -\text{div}\,(a(x,\nabla u)) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

where $f \in L^1(\Omega)$. They have proved the existence and uniqueness of entropy solutions $W_0^1 L_M(\Omega)$ without any restriction on the $N-$function M of the Orlicz spaces (i.e. without assuming Δ_2-condition), we refer also to [5, 10, 16] for more details.

Bendahmane and Wittbold have studied in [9] the existence and uniqueness solution for the quasilinear elliptic equation

$$\begin{cases} -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

in the Sobolev spaces with variable exponents, where the right-hand side $f \in L^1(\Omega)$ and the exponent $p(\cdot) : \overline{\Omega} \mapsto (1, +\infty)$ is continuous, for more related results we refer to [4, 6, 22].

Recently, the anisotropic Sobolev spaces have attracted the attention of many scientists and researchers (see. [7, 18, 21]), this impulse mainly comes from their physical applications in the processes of image restoration, flows of electro-rheological fluids and thermistor problem.
M. AL-hawmi, E.Azroul, H. Hjiaj and A.Touzani have studied in [2] the existence of entropy solutions for some anisotropic quasilinear elliptic unilateral problems

$$\begin{cases} Au = \mu - \text{div}\, \phi(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.5}$$

where Au is an operator of Leray-Lions type acting from $W_0^{1,\vec{p}}(\Omega)$ into its dual define by

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u),$$

where $\phi(\cdot) \in C^0(I\!R, I\!R^N)$, and $\mu = f -$ div F such that $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^{N} L^{p'_i}(\Omega)$.

AL-hawmi, A. Benkirane, H. Hjiaj and A. Touzani have studied in [3] the existence and uniqueness of Entropy Solutions for some Nonlinear Elliptic Unilateral Problems

$$\begin{cases} Au = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.6}$$

in Musielak-Orlicz-Sobolev spaces, where $f \in L^1(\Omega)$ and $A : D(A) \subset W_0^1 L_\varphi(\Omega) \mapsto W^{-1} L_\psi(\Omega)$ is the Leray-Lions operator defined as:

$$A(u) = -\text{div } a(x, \nabla u),$$

In this paper, we will prove the existence of entropy solutions for non-linear anisotropic unilateral elliptic problem of the type

$$\begin{cases} Au + g(x, u, \nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.7}$$

where Au is an operator of Leray-Lions type acting from $W_0^{1,\vec{p}}(\Omega)$ into its dual define by $Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u)$ and the function $g(x, u, \nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$ satisfying the sign condition, that is $g(x, u, \nabla u)u \geq 0$.

This paper is organized as follows. In section 2 we recall some definitions and basic properties concerning the anisotropic Sobolev spaces. We introduce in section 3 the assumptions on $a_i(x, s, \xi)$, and $g(x, s, \xi)$, for which our problem has at least one solution. Section 4 i s devoted to show the existence of entropy solutions for our anisotropic nonilinear elliptic unilateral problem (1.7).

2 Preliminary

Let Ω be an open bounded domain in $I\!R^N$ $(N \geq 2)$ with boundary $\partial\Omega$. Let $p_0, p_1, \ldots, p_N$ be $N + 1$ exponents, with $1 < p_i < \infty$ for $i =$

$0, 1, \ldots, N$. We denote

$$\vec{p} = (p_0, p_1, \ldots, p_N), \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1, \ldots, N.$$

We set

$$\underline{p} = \min\{p_0, p_1, p_2, \ldots, p_N\} \qquad \text{then} \qquad \underline{p} > 1. \tag{2.1}$$

The anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ is defined as follows

$$W^{1,\vec{p}}(\Omega) = \{u \in L^{p_0}(\Omega) \quad \text{and} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N\}$$

endowed with the norm

$$\|u\|_{1,\vec{p}} = \sum_{i=0}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{2.2}$$

The space $\left(W^{1,\vec{p}}(\Omega), \|u\|_{1,\vec{p}}\right)$ is a separable and reflexive Banach space (cf [21]).
We define also $W_0^{1,\vec{p}}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ with respect to the norm (2.2).
Now, we will present some important compacts embedding,

Lemma I: Let Ω be a bounded open set in $I\!R^N$, then the following embedding are compact

- if $\underline{p} < N$ then $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \qquad \forall q \in [\underline{p}, \underline{p}^*[$, where $\frac{1}{\underline{p}^*} = \frac{1}{\underline{p}} - \frac{1}{N}$,
- if $\underline{p} = N$ then $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \qquad \forall q \in [\underline{p}, +\infty[$,
- if $\underline{p} > N$ then $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$.

The proof of this lemma follows from the fact that the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow W_0^{1,\underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

Definition 2.1. *Let $k > 0$, we consider the truncation function $T_k(\cdot) : I\!R \longmapsto I\!R$, given by*

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we define

$$\mathcal{T}_0^{1,\vec{p}}(\Omega) := \{u : \Omega \mapsto I\!R \text{ measurable, such that } T_k(u) \in W_0^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

Proposition 2.1. *Let $u \in \mathcal{T}_0^{1,\vec{p}}(\Omega)$, for $i = 1, \ldots, N$, there exists a unique measurable function $v_i : \Omega \mapsto I\!R$ such that*

$$D^i T_k(u) = v_i . \chi_{\{|u|<k\}} \qquad a.e. \text{ in } \quad \Omega, \quad \text{for any} \quad k > 0,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivatives of u, that is $v_i = D^i u$.

The proof follows the usual techniques developed in [12] for the case of Sobolev spaces.
For more details concerning the anisotropic Sobolev spaces, we refer the reader to [8] and [15].

3 Essential assumptions

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$, we assume that the vector $\vec{p} = (p_0, p_1, \ldots, p_N)$ satisfying the requirements that $1 < p_i < \infty$ for $i = 0, 1, \ldots, N$.
Taking ψ as a measurable function on Ω with values in $\overline{I\!R}$, such that

$$\psi^+ \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega).$$

We define the convex subset K_ψ by

$$K_\psi = \{v \in W_0^{1,\vec{p}}(\Omega) \quad \text{such that} \quad v \geq \psi \text{ a.e. in } \ \Omega\}.$$

We consider a Leray-Lions operator $A : W_0^{1,\vec{p}}(\Omega) \longmapsto W^{-1,\vec{p}'}(\Omega)$ given by

$$Au = -\sum_{i=1}^{N} D^i a_i(x, u, \nabla u)$$

where $a_i : \Omega \times I\!R \times I\!R^N \longmapsto I\!R$ are Carathéodory functions, for $i = 1, \ldots, N$, (measurable with respect to x in Ω for every (s, ξ) in $I\!R \times I\!R^N$ and continuous with respect to (s, ξ) in $I\!R \times I\!R^N$ for almost every x in Ω), which satisfy the following conditions:

$$|a_i(x, s, \xi)| \leq R_i(x) + |s|^{\underline{p}-1} + |\xi_i|^{p_i-1}) \qquad \text{for} \quad i = 1, \ldots, N, \qquad (3.1)$$

$$(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0, \quad \text{for} \quad \xi_i \neq \xi_i'. \qquad (3.2)$$

There exist $d(x)$ in $L^1(\Omega)$ and a strictly positive constant σ such that, for some fixed element v_0 in $K_\psi \cap \mathrm{L}^\infty(\Omega)$

$$a_i(x,s,\xi)(\xi - \nabla v_0) \geq \sigma|\xi_i|^{p_i} - d(x) \qquad \text{for} \quad i = 1,\dots,N, \tag{3.3}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in I\!R \times I\!R^N$, where $R_i(x)$ is a nonnegative function lying in $L^{p_i'}(\Omega)$ and $\sigma > 0$.
The nonlinear term $g(x,s,\xi)$ is a *Carathéodory* function which satisfies

$$g(x,s,\xi)s \;\geq\; 0, \tag{3.4}$$

$$|g(x,s,\xi)| \leq b(|s|)\big(c(x) + \sum_{i=1}^{N} |\xi_i|^{p_i}\big), \tag{3.5}$$

with $b : I\!R^+ \to I\!R^+$ is a continuous, nondecreasing growth function and $c : \Omega \to I\!R^+$ such that $c \in L^1(\Omega)$.
We consider the anisotropic quasilinear elliptic unilateral problem

$$\begin{cases} Au + g(x,u,\nabla u) = f & \text{in } \Omega, \\ u = \;\; 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.6}$$

where

$$f \in L^1(\Omega), \tag{3.7}$$

Now, we recall some important Lemmas useful to prove our main result.

Lemma II:(see [17], Theorem 13.47 page 216) Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that

(i) $u_n \to u$ a.e. in Ω,

(ii) $u_n \geq 0$ and $u \geq 0$ a.e. in Ω,

(iii) $\int_\Omega u_n \, dx \to \int_\Omega u \, dx$,

then $u_n \to u$ in $L^1(\Omega)$.

Lemma III:(see. [7] page 6) Assuming that $(3.1) - (3.3)$ hold, and let $(u_n)_{n\in I\!N}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\Omega)$ and

$$\sum_{i=1}^{N} \int_\Omega \Big(a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u)\Big)(D^i u_n - D^i u)\, dx \longrightarrow 0, \tag{3.8}$$

then $u_n \to u$ in $W_0^{1,\vec{p}}(\Omega)$ for a subsequence.

4 Main results.

Theorem 4.1. *Assuming that* $(3.1)-(3.5)$ *and* (3.7) *hold. Then there exists at least one solution of the following unilateral problem,*

$$\begin{cases} u \in \mathcal{T}_0^{1,\vec{p}}(\Omega), \quad u \geq \psi \quad a.e. \ in \quad \Omega, \quad g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) D^i T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u) T_k(u-v)\,dx \leq \int_\Omega f T_k(u-v)\,dx, \\ \qquad\qquad for \ any \quad v \in K_\psi \cap L^\infty(\Omega), \quad \forall k > 0. \end{cases} \tag{4.1}$$

Our objective is to prove the following existence theorem.

4.1 Proof of Theorem 4.1

Step 1: Approximate problems.

Let $(f_n)_{n\in I\!N}$ be a sequence in $W^{-1,\vec{p}'}(\Omega) \cap L^1(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \leq |f|$ (for example $f_n = T_n(f)$). We consider approximate problem.

$$\begin{cases} u_n \in K_\psi, \\ \displaystyle\sum_{i=1}^{N} \int_\Omega a_i(x,T_n(u_n),\nabla u_n) D^i(u_n-v)\,dx + \int_\Omega g_n(x,T_n(u_n),\nabla u_n)(u_n-v)dx \\ \qquad \displaystyle\leq \int_\Omega f_n(u_n-v)\,dx \end{cases} \tag{4.2}$$

for any $v \in K_\psi$, and $g_n(x,T_n(s),\xi) = \dfrac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$, $g_n(x,T_n(s),\xi)s \geq 0$, $|g_n(x,s,\xi)| \leq |g(x,s,\xi)|$ and $|g_n(x,T_n(s),\xi)| \leq n$. We define the operators A_n and G_n acted from $W_0^{1,\vec{p}}(\Omega)$ into its dual $W^{-1,\vec{p}'}(\Omega)$ by $\langle A_n u, v\rangle = \displaystyle\sum_{i=1}^{N} \int_\Omega a_i(x,T_n(u),\nabla u) D^i v dx$ and $\langle G_n u, v\rangle = \displaystyle\int_\Omega g(x,T_n(u),\nabla u) D^i v dx$ $\forall u, v \in W_0^{1,\vec{p}}(\Omega)$,

Lemma IV : The operator $B_n = A_n + G_n$ from $W_0^{1,\vec{p}}(\Omega)$ into $W^{-1,\vec{p}'}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense: The bounded operator $B_n = A_n + G_n$ acted from $W_0^{1,\vec{p}}(\Omega)$ into $W^{-1,\vec{p}'}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense :

there exists $v_0 \in K_\psi$ such that

$$\frac{\langle B_n v, v - v_0\rangle}{\|v\|_{1,\vec{p}}} \longrightarrow \infty \qquad \text{as} \quad \|v\|_{1,\vec{p}} \to \infty \quad \text{for} \quad v \in K_\psi.$$

Proof of Lemma IV :
Using the Hölder's inequality and the growth condition (3.3), we can show that the operator A_n is bounded and by (3.5) we conclude that B_n is bounded. For the coercivity, let $v_0 \in K_\psi$, for any $v \in K_\psi$ we have

$$\begin{aligned}
\langle A_n v, v - v_0\rangle &= \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(v), \nabla v)(D^i v - \nabla v_0)\, dx \\
&\geq \sigma \sum_{i=0}^{N} \int_\Omega |D^i v|^{p_i}\, dx + C_1 \\
&\geq \sigma' \sum_{i=0}^{N} \int_\Omega |D^i v|^{p_i}\, dx
\end{aligned}$$

with $\sigma' = \min(\sigma, C_1)$. and

$$\begin{aligned}
\langle G_n v, v\rangle = &\int_\Omega g_n(x, T_n(v), \nabla v) v\, dx \\
&\leq \big\|g_n(x, T_n(v), \nabla v)\big\|_{p_i'} \big\| \big\|_{p_i} \\
&\leq \Big(\int_\Omega |g_n(x, T_n(v), \nabla v|)^{p_i'} + 1\Big)^{\frac{1}{p_i'}} \|v\|_{1,\vec{p}} \\
&\leq \big(n^{p_i'} \mathrm{meas}(\Omega) + 1\big)^{\frac{1}{p_i'}} \|v\|_{1,\vec{p}} \\
&\leq C_2 \|v\|_{1,\vec{p}}
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\langle G_n v, v_0\rangle = &\sum_{i=1}^{N} \int_\Omega g_n(x, T_n(v), \nabla v) D^i v_0\, dx \\
&\leq \sum_{i=1}^{N} \Big(\int_\Omega |g_n(x, T_n(v), \nabla v)|^{p_i'}\, dx\Big)^{\frac{1}{p_i'}} \|D^i v_0\|_{p_i} \\
&\leq C_3 \sum_{i=1}^{N} \Big(\int_\Omega C_4 n^{p_i'} (c(x)^{p_i'} + |D^i v|^{p_i}))\, dx\Big)^{\frac{1}{p_i'}} \|v_0\|_{1,\vec{p}}
\end{aligned}$$

$$\frac{\langle B_n v, v - v_0\rangle}{\|v\|_{1,\vec{p}}} = \frac{\langle A_n v, v - v_0\rangle}{\|v\|_{1,\vec{p}}} + \frac{\langle G_n v, v - v_0\rangle}{\|v\|_{1,\vec{p}}}$$

$$= \frac{\sigma'}{\|v\|_{1,\vec{p}}} \sum_{i=0}^{N} \int_\Omega |D^i v|^{p_i}\, dx \quad + \frac{C_2\|v\|_{1,\vec{p}}}{\|v\|_{1,\vec{p}}} \tag{4.4}$$

$$-C_3 \frac{\|v_0\|_{1,\vec{p}}}{\|v\|_{1,\vec{p}}} \sum_{i=1}^{N} \Big(\int_\Omega C_4 |n|^{p'_i} (c(x)^{p'_i} + |D^i v|^{p_i})\, dx \Big)^{\frac{1}{p'_i}} \longrightarrow \infty,$$

as $\|v\|_{1,\vec{p}} \to \infty$. On the other hand, we have then, we obtain

$$\frac{\langle B_n v, v - v_0\rangle}{\|v\|_{1,\vec{p}}} \longrightarrow +\infty \quad \text{ if } \|v\|_{1,\vec{p}} \longrightarrow +\infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \quad \text{in } W_0^{1,\vec{p}}(\Omega), \\ B_n u_k \rightharpoonup \chi \text{ in } W^{-1,\vec{p}'}(\Omega), \\ \limsup\limits_{k\to\infty} \langle B_n u_k, u_k\rangle \le \langle \chi, u\rangle. \end{cases} \tag{4.5}$$

We will prove that

$$\chi = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi, u\rangle \ \text{ as } \ k \to +\infty.$$

Firstly, since $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, then

$$u_k \to u \quad \text{in } L^{\underline{p}}(\Omega) \quad \text{for a subsequence denoted again } (u_k)_{k\in\mathbb{N}}.$$

As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,\vec{p}}(\Omega)$, then by the growth condition $(a_i(x, T_n(u_k), \nabla u_k))_{k\in\mathbb{N}}$ is bounded in $L^{p'_i}(\Omega)$.
Therefore there exists a function$\varphi_i \in L^{p'_i}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \ \text{ in } \ L^{p'_i}(\Omega) \ \text{ as } \ k \to \infty. \tag{4.6}$$

Similarly, it is easy to see that $(g_n(x, T_n(u_k), \nabla u_k))_{k\in\mathbb{N}}$ is bounded in $L^{\underline{p}'}(\Omega)$, then there exists a function $\psi_n \in L^{\underline{p}'}(\Omega)$ such that

$$g_n(x, T_n(u_k), \nabla u_k) \rightharpoonup \psi_n \ \text{ in } \ L^{\underline{p}'}(\Omega) \ \text{ as } \ k \to \infty. \tag{4.7}$$

Finally, since $u_k \to u$ in $L^{\underline{p}}(\Omega)$, by using the Lebesgue dominated convergence theorem, we deduce that: It is clear that, for all$v \in W_0^{1,\vec{p}}(\Omega)$, we get

$$\begin{aligned}\langle \chi, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i v\, dx + \lim_{k\to\infty} \int_\Omega g_n(x, T_n(u_k), \nabla u_k) v\, dx \\ &= \sum_{i=1}^{N} \int_\Omega \varphi_i\ D^i v\, dx + \int_\Omega \psi_n v\, dx.\end{aligned} \tag{4.8}$$

From relations (4.5) and (4.8), we have

$$\begin{aligned}\limsup_{k\to\infty} \langle B_n(u_k), u_k\rangle &= \limsup_{k\to\infty} \left\{ \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k\, dx + \int_\Omega g_n(x, T_n(u_k), \nabla u_k) u_k\, dx \right\} \\ &\leq \sum_{i=1}^{N} \int_\Omega \varphi_i\ D^i u\, dx + \int_\Omega \psi_n u\, dx\, dx.\end{aligned} \tag{4.9}$$

Thanks to (4.7), we have

$$\int_\Omega g_n(x, T_n(u_k), \nabla u_k) u_k\, dx \longrightarrow \int_\Omega \psi_n u\, dx. \tag{4.10}$$

Therefore

$$\limsup_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k\, dx \leq \sum_{i=1}^{N} \int_\Omega \varphi_i\ D^i u\, dx. \tag{4.11}$$

On the other hand, by (3.3), we get

$$\sum_{i=1}^{N} \int_\Omega (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u)\, dx \geq 0, \tag{4.12}$$

then

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k\, dx \\ &\qquad \geq \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u\, dx \\ &\qquad + \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u)(D^i u_k - D^i u)\, dx.\end{aligned}$$

In view of Lebesgue dominated convergence theorem, we have $T_n(u_k) \to T_n(u)$ in $L^{p_i}(\Omega)$ then $a_i(x,T_n(u_k),\nabla u) \to a_i(x,T_n(u),\nabla u)$ in $L^{p_i'}(\Omega)$, and by (4.6), we get

$$\liminf_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx \geq \sum_{i=1}^{N}\int_\Omega \varphi_i\ D^iu\,dx.$$

This implies by using (4.11) that

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx = \sum_{i=1}^{N}\int_\Omega \varphi_i\ D^iu\,dx. \qquad (4.13)$$

According to (4.8), (4.10) and (4.13), we obtain

$$\langle B_nu_k,u_k\rangle \longrightarrow \langle \chi,u\rangle \quad \text{as} \quad k\to+\infty.$$

Now, by (4.13) we can prove that

$$\lim_{k\to+\infty}\sum_{i=1}^{N}\int_\Omega (a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dx = 0.$$

and so, by vertue of Lemma III , we get

$$u_k \longrightarrow u \ \text{ in } W_0^{1,\vec{p}}(\Omega) \quad \text{and} \quad D^iu_k \longrightarrow D^iu \text{ a.e. in } \Omega,$$

then

$$a_i(x,T_n(u_k),\nabla u_k) \rightharpoonup a_i(x,T_n(u),\nabla u) \qquad \text{in} \quad L^{p_i'}(\Omega) \quad \text{for} \quad i=1,\dots,N$$

and

$$g_n(x,T_n(u_k),\nabla u_k) \rightharpoonup g_n(x,T_n(u),\nabla u) \qquad \text{in} \quad L^{p_0'}(\Omega),$$

which implies that $\chi = B_nu$, there exists at least one solution $u_n \in W_0^{1,\vec{p}}(\Omega)$ of the anisotropic nonlinear elliptic unilateral problem (4.2) (cf. [20], Theorem 8.1 page 131).

Step 2: A priori estimates.

In this step, we will prove an uniform estimate for the truncated solution $T_k(u_n)$. Let $k||v_0||_\infty$ and let $\varphi_k(s) = se^{\gamma s^2}$, where $\gamma = (\frac{b(k)}{\sigma})^2$. It is well known that,

$$\varphi_k'(s) - \frac{b(k)}{\sigma}|\varphi_k(s)| \geq \frac{1}{2}, \quad \forall s \in I\!R. \qquad (4.14)$$

Taking $v = u_n - \eta\varphi_k(T_l(u_n - v_0)(\eta = e^{\gamma l^2})$ as test function in (4.2), where $l = k + ||v||_\infty$, we obtain

$$\sum_{i=1}^{N}\int_\Omega a_i(x, T_n(u_n), \nabla u_n)D^iT_l(u_n - v_0)\varphi_k'(T_l(u_n - v_0))\,dx + \int_\Omega g_n(x, T_n(u_n), \nabla u_n)\varphi_k(T_l(u_n - v_0))$$
$$\leq \int_\Omega f_n\varphi_k(T_l(u_n - v_0))\,dx. \tag{4.15}$$

Since $g_n(x, T_n(u_n), \nabla u_n)\varphi_k(T_l(u_n - v_0)) \geq 0$ on the subset $\{x \in \Omega : |u_n(x)| > k\}$, then

$$\sum_{i=1}^{N}\int_{\{|u_n - v_0|\leq l\}} a_i(x, T_n(u_n), \nabla u_n)D^i(u_n - v_0)\varphi_k'(T_l(u_n - v_0))\,dx$$
$$\leq \int_{\{|u_n|\leq k\}} |g(x, T_n(u_n), \nabla u_n)||\varphi_k(T_l(u_n - v_0))|dx + \int_\Omega f_n\varphi_k(T_l(u_n - v_0))\,dx\,. \tag{4.16}$$

By using (3.1), (3.5) and the fact that $\{x \in \Omega, |u_n(x)| \leq k \subseteq x \in \Omega : |u_n - v_0| \leq l\}$, we get

$$\sum_{i=1}^{N}\int_{\{|u_n - v_0|\leq l\}} |D^iu_n|^{p_i}\,dx \leq \sum_{i=0}^{N}\int_\Omega |D^iT_k(u_n)|^{p_i}\,dx \leq 2C_k \tag{4.17}$$

where C_k is a positive constant depending on k and we obtain

$$\|T_k(u_n)\|_{1,\vec{p}} \leq C_1,$$

where C_1 is positive constant that does not depend on n. Thus, the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}}(\Omega)$ uniformly in n, then there exists a subsequence still denoted $(T_k(u_n))_{n\in I\!N}$ and a function $v_k \in W_0^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k & \text{weakly in} \quad W_0^{1,\vec{p}}(\Omega), \\ T_k(u_n) \to v_k & \text{strongly in} \quad L^{\underline{p}}(\Omega) \quad \text{and} \quad \text{a.e in} \quad \Omega. \end{cases} \tag{4.18}$$

On the other hand, thanks to (4.17) and Poincaré inequality, we have

$$\begin{aligned} \|\nabla T_k(u_n)\|_{\underline{p}}^{\underline{p}} &= \sum_{i=1}^{N}\int_\Omega |D^iT_k(u_n)|^{\underline{p}}\,dx \\ &\leq \sum_{i=1}^{N}\int_\Omega |D^iT_k(u_n)|^{p_i}\,dx + N|\Omega| \leq C_2k. \end{aligned} \tag{4.19}$$

Therefore, there exists a constant C_2 that does not depend on k and n, such that

$$\|\nabla T_k(u_n)\|_{\underline{p}} \leq C_2 k^{\frac{1}{\underline{p}}} \qquad \text{for} \quad k \geq 1,$$

and we obtain

$$\begin{aligned} k \text{ meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|\, dx \leq \int_\Omega |T_k(u_n)|\, dx \\ &\leq \|1\|_{\underline{p}'} \|T_k(u_n)\|_{\underline{p}} \\ &\leq C\|\nabla T_k(u_n)\|_{\underline{p}} \\ &\leq C_3 k^{\frac{1}{\underline{p}}}, \end{aligned} \tag{4.20}$$

which yields

$$\text{meas}\{|u_n| > k\} \leq C_{13} \frac{1}{k^{1-\frac{1}{\underline{p}}}} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{4.21}$$

Now, we will show that the sequence $(u_n)_n$ is a Cauchy sequence in measure. Indeed, we have for every $\delta > 0$,

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}.$$

Let $\varepsilon > 0$, in view of (4.21) we may choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{4.22}$$

Moreover, thanks to (4.18) we have

$$T_k(u_n) \longrightarrow \eta_k \quad \text{in} \quad L^{\underline{p}}(\Omega) \quad \text{and a.e. in} \quad \Omega.$$

Thus $(T_k(u_n))_{n\in I\!N}$ is a Cauchy sequence in measure, and for any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \qquad \text{for all } m, n \geq n_0(k, \delta, \varepsilon). \tag{4.23}$$

By combining (4.22) and (4.23), we conclude that : for all $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon \qquad \text{for any} \quad n, m \geq n_0.$$

It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u. Thanks to (4.18) we obtain

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{in} \quad W_0^{1,\vec{p}}(\Omega), \\ T_k(u_n) \to T_k(u) & \text{in} \quad L^{\underline{p}}(\Omega) \quad \text{and a.e. in} \quad \Omega. \end{cases} \tag{4.24}$$

Step 3: Almost everywhere convergence of the gradient.

We fix $k > ||v_0||_\infty$, and let $w_{n,h} = T_{2k}(u_n - v_0 - T_h(u_n - v_0) + T_k(u_n) - T_k(u))$ and $w_h = T_{2k}(u - v_0 - T_h(u - v_0)$, with $h > 2k$. For $\eta = e^{(-4\gamma k^2)}$, we define the following function as

$$v_{n,h} = u_n - \eta\varphi_k(w_{n,h})$$

Now, taking $v_{n,h}$ as test function in (4.1), we get

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla\omega_{n,h}\varphi'_k(\omega_{n,h})\,dx + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi_k(\omega_{n,h})\,dx \leq \int_\Omega f_n\varphi_k(\omega_{n,h})\,dx. \tag{4.25}$$

Note that, $\nabla\omega_{n,h} \geq 0$ on the set where $|un| > h + 5k$, therefore, setting $m = 5k + h$, and denoting in the sequel by $\varepsilon_i(n)$ $\;i = 1, 2, \ldots$ a various functions of real numbers which converges to 0 as n tends to infinity for any fixed value of h. By virtue of (4.25), since$u_n \to u$ a.e. in Ω and the fact that $g_n(x, u_n, \nabla u_n)\varphi_k(\omega_{n,h}) \geq 0$ on the set$x \in \Omega, |u_n(x)| > k$, we can write

$$\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))\nabla\omega_{n,h}\varphi'_k(\omega_{n,h})\,dx + \int_{\{|u_n|\leq k\}} g(x, T_m(u_n), \nabla T_m(u_n))\varphi_k(\omega_{n,h})\,dx \leq \int_\Omega f_n\varphi_k(\omega_h)\,dx + \varepsilon_h^1(n). \tag{4.26}$$

Splitting the first integral on the left hand side of (4.26) where $|u_n| \leq k$ and $|u_n| > k$, we have

$$\begin{aligned}\int_\Omega \quad & a(x, T_m(u_n), \nabla T_m(u_n))\nabla\omega_{n,h}\varphi'_k(\omega_{n,h}))\,dx \\ &= \int_{\{|u_n|\leq k\}} a(x, T_m(u_n), \nabla T_m(u_n))D^i\varphi'_k(\omega_{n,h})\,dx \\ &+ \int_{\{|u_n|>k\}} a(x, T_m(u_n), \nabla T_m(u_n))\nabla\omega_{n,h}\varphi'_k(\omega_{n,h})\,dx.\end{aligned} \tag{4.27}$$

On the one hand, we get

$$\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))(D^iT_k(u_n) - D^iT_k(u))\varphi'_k(\omega_{n,h})dx$$
$$\geq \int_\Omega (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)))(D^iT_k(u_n) - D^iT_k(u))\varphi'_k(\omega_{n,h})$$
$$-\varphi'_k(2k)\int_{\{|u-v_0|>k\}} \sigma(x))\,dx + \varepsilon_h^2(n). \tag{4.28}$$

We now turn to the second term of the left hand side of (4.26), using (3.5), we have

$$\left|\int_\Omega g_n(x, T_n(u_n), \nabla u_n)\varphi_k(\omega_{n,h})\,dx\right| \leq b(k)\int_\Omega (c(x) + \sum_{i=1}^N |D^iT_k(u_n)|^{p_i}|\varphi_k(\omega_{n,h})|)\,dx$$
$$\leq b(k)\int_\Omega c(x)|\varphi_k(\omega_{n,h})|\,dx \quad +\frac{b(k)}{\sigma}\int_\Omega \sigma(x)|\varphi_k(\omega_{n,h})|\,dx$$
$$+\frac{b(k)}{\sigma}\sum_{i=1}^N\int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n))D^iT_k(u_n|\varphi_k(\omega_n)|\,dx$$
$$-\frac{b(k)}{\sigma}\sum_{i=1}^N\int_\Omega a_i(x, T_k(u_n), \nabla T_k(u))\nabla v_0|\varphi_k(\omega_{n,h})|\,dx. \tag{4.29}$$

$$\left|\int_{\{|u_n|\leq k\}} g_n(x, T_n(u_n), \nabla u_n)\varphi_k(\omega_n)\,dx\right|$$
$$\leq \sum_{i=1}^N\int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))(D^iT_k(u_n) - D^iT_k(u))|\varphi_k(\omega_n)|\,dx$$
$$\leq b(k)\int_\Omega c(x)|\varphi_k(\omega_{n,h})|\,dx \quad +\frac{b(k)}{\sigma}\int_\Omega \sigma(x)|\varphi_k(\omega_{n,h})|\,dx$$
$$-\frac{b(k)}{\sigma}\int_\Omega h_k\nabla v_0|\varphi_k(\omega_h)|\,dx + \varepsilon_h^3(n). \tag{4.30}$$

Combining (4.28), (4.29) and (4.30), we obtain

$$\sum_{i=1}^N\int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))(D^iT_k(u_n) - D^iT_k(u))$$
$$(\varphi'_k(\omega_{n,h}) - \frac{b(k)}{\alpha}\varphi_k(\omega_{n,h}))|\varphi_k(\omega_{n,h})|\,dx \quad \leq b(k)\int_\Omega c(x)|\varphi_k(\omega_{n,h})|\,dx \quad +\frac{b(k)}{\sigma}\int_\Omega \sigma(x)|\varphi_k(\omega_h)|\,dx$$
$$-\frac{b(k)}{\sigma}\int_\Omega h_k\nabla v_0|\varphi_k(\omega_h)|\,dx + \int_\Omega f(x)\varphi_k(\omega_h)\,dx + \varepsilon_h^4(n). \tag{4.31}$$

By (4.14) we have $(\varphi_k'(\omega_{n,h}) - \frac{b(k)}{\sigma}\varphi_k(\omega_{n,h}))|\varphi_k(\omega_{n,h})| \geq \frac{1}{2}$, hence the monotonicity (3.2) gives

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega &(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n) - D^iT_k(u))\,dx\\ &\leq 2b(k)\int_\Omega c(x)|\varphi_k(\omega_h)|\,dx \quad +2\frac{b(k)}{\sigma}\int_\Omega \sigma(x)|\varphi_k(\omega_h)|\,dx\\ &-2\frac{b(k)}{\sigma}\int_\Omega h_k\nabla v_0|\varphi_k(\omega_h)|\,dx + 2\int_\Omega f(x)\varphi_k(\omega_h)\,dx + \varepsilon_h^5(n).\end{aligned} \tag{4.32}$$

Then, by letting h and n goes to infinity in (4.32), we get

$$\sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n) - D^iT_k(u))\,dx \longrightarrow 0. \tag{4.33}$$

Using the Lemma III , we deduce that

$$T_k(u_n) \longrightarrow T_k(u) \qquad \text{in} \quad W_0^{1,\vec{p}}(\Omega). \tag{4.34}$$

Therefore,

$$D^iu_n \longrightarrow D^iu \quad \text{a.e. in } \Omega.$$

Step 4: The equi-integrability of $g_n(x,T_n(u_n),\nabla u_n)$.

In order to pass to the limit in the approximate equation, we show that

$$g_n(x,T_n(u_n),\nabla u_n) \longrightarrow g(x,u,\nabla u) \quad \text{strongly in} \quad L^1(\Omega).$$

By using Vitali's Theorem, it suffices to prove that $g_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable. Indeed, taking $T_1(u_n - T_h(u_n))$ as a test function in (4.2), and using (3.2) since $T_1(u_n - T_h(u_n))$ have the same sign as u_n, we obtain

$$\begin{aligned}\int_{\{h<|u_n|\}} &g_n(x,T_n(u_n),\nabla u_n)T_1(u_n - T_h(u_n))\,dx\\ &\leq \int_{\{h<|u_n|\}} f_nT_1(u_n - T_h(u_n))\,dx\end{aligned} \tag{4.35}$$

It follows that

$$\begin{aligned}&\int_{\{h+1\le|u_n|\}}|g_n(x,T_n(u_n),\nabla u_n)|\,dx\\&\le\int_{\{h<|u_n|\}}g_n(x,T_n(u_n),\nabla u_n)T_1(u_n-T_h(u_n))\,dx\\&\le\int_{\{h<|u_n|\}}f_nT_1(u_n-T_h(u_n))\,dx\\&\le\int_{\{h<|u_n|\}}|f|\,dx,\end{aligned}$$

thus, for all $\eta>0$, there exists $h(\eta)>0$ such that

$$\int_{\{h(\eta)\le|u_n|\}}|g_n(x,T_n(u_n),\nabla u_n)|\,dx\le\frac{\eta}{2}. \tag{4.36}$$

On the other hand, for any measurable subset $E\subset\Omega$, we have

$$\begin{aligned}&\int_E|g_n(x,T_n(u_n),\nabla u_n)|\,dx\\&\le b(h(\eta))\int_E(c(x)+\sum_{i=1}^N|D^iT_{h(\eta)}(u_n)|^{p_i})\,dx\\&\quad+\int_{\{h(\eta)\le|u_n|\}}|g_n(x,T_n(u_n),\nabla u_n)|\,dx.\end{aligned} \tag{4.37}$$

From (4.34), there exists $\beta(\eta)>0$ such that, for all $E\subseteq\Omega$ with meas$(E)\le\beta(\eta)$, we have

$$b(h(\eta))\int_E(c(x)+\sum_{i=1}^N|D^iT_{h(\eta)}(u_n)|^{p_i})\,dx\le\frac{\eta}{2}\,. \tag{4.38}$$

Finally, by combining (4.36), (4.37) and (4.38), one easily has

$$\int_E|g_n(x,T_n(u_n),\nabla u_n)|\,dx\le\eta\quad\text{ for all } E \text{ such that meas}(E)\le\beta(\eta), \tag{4.39}$$

we then deduce that $(g_n(x,T_n(u_n),\nabla u_n))_n$ is equi-integrable, and by Vitali's Theorem we deduce that

$$g_n(x,T_n(u_n),\nabla u_n)\longrightarrow g(x,u,\nabla u)\quad\text{in}\quad L^1(\Omega). \tag{4.40}$$

Step 5: Passing to the limit.

By using $T_k(u_n-\varphi)$ as a test function in (4.2), with $\varphi \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, and putting $m = k + \|\varphi\|_\infty$, we get

$$\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n-\varphi)\, dx + \int_\Omega g_n(x, T_n(u_n), \nabla u_n) T_k(u_n-\varphi)\, dx = \int_\Omega f_n T_k(u_n-\varphi)\, dx. \tag{4.41}$$

On one hand, if $|u_n| > m$ then $|u_n-\varphi| \geq |u_n| - \|\varphi\|_\infty > k$, therefore $\{|u_n-\varphi| \leq k\} \subseteq \{|u_n| \leq m\}$, which implies that

$$\begin{aligned}
&\int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n-\varphi)\, dx \\
&= \int_\Omega a_i(x, T_m(u_n) \nabla T_m(u_n))(D^i T_m(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\, dx \\
&= \int_\Omega (a_i(x, T_m(u_n), \nabla T_m(u_n)) - a_i(x, T_m(u_n), \nabla\varphi))(D^i T_m(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\, dx \\
&\quad + \int_\Omega a_i(x, T_m(u_n), \nabla\varphi)(D^i T_m(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\, dx.
\end{aligned} \tag{4.42}$$

According to Fatou's Lemma, we obtain

$$\begin{aligned}
\liminf_{n\to+\infty} &\int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n-\varphi)\, dx \\
&\geq \int_\Omega (a_i(x, T_m(u), \nabla T_m(u)) - a_i(x, T_m(u), \nabla\varphi))(D^i T_m(u) - D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\, dx \\
&\quad + \lim_{n\to+\infty} \int_\Omega a_i(x, T_m(u_n), \nabla\varphi)(D^i T_m(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\, dx.
\end{aligned} \tag{4.43}$$

The second term in the right hand side of (4.43) is equal to

$$\int_\Omega a_i(x, T_m(u), \nabla\varphi)(D^i T_m(u) - D^i\varphi)\chi_{\{|u-\varphi|\leq k\}} dx.$$

Therefore, we get

$$\begin{aligned}
\liminf_{n\to+\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi)\, dx \\
&\geq \sum_{i=1}^{N} \int_\Omega a_i(x, T_m(u), \nabla T_m(u))(D^i T_M(u) - D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\, dx \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u)(D^i u - D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\, dx \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) D^i T_k(u - \varphi)\, dx.
\end{aligned}$$

On the other hand, we have $T_k(u_n - \varphi) \rightharpoonup T_k(u - \varphi)$ weak-$\star$ in $L^\infty(\Omega)$ and in view of (4.40). We obtain,

$$\int_\Omega g_n(x, T_n(u_n), \nabla u_n) T_k(u_n - \varphi)\, dx \longrightarrow \int_\Omega g(x, u, \nabla u)\ T_k(u - \varphi)\, dx \tag{4.44}$$

and

$$\int_\Omega f_n T_k(u_n - \varphi)\, dx \longrightarrow \int_\Omega f\ T_k(u - \varphi)\, dx. \tag{4.45}$$

Again, since $T_k(u_n - \varphi) \rightharpoonup T_k(u - \varphi)$ in $W_0^{1,\vec{p}}(\Omega)$ we have done completed the proof of Theorem 4.1.

References

1. Aharouch, L. and Bennouna ,J. (2010), Existence and uniqueness of solutions of unilateral problems in Orlicz spaces. *J. Nonlinear Anal*. 72(9-10), 3553 - 3565.
2. AL-hawmi, M. Azroul, E. Hjiaj, H. and Touzani, (2017), A. Existence of entropy solutions for some anisotropic quasilinear elliptic unilateral problems, *J. Afr. Mat.Springer* 28,357-378.
3. AL-hawmi, M. Benkirane,A. Hjiaj, H. and Touzani, (2017), Existence and Uniqueness of Entropy Solutions for some Nonlinear Elliptic Uni- lateral Problems in Musielak-Orlicz-Sobolev Spaces, *Journal An- nals of the University of Craiova-Mathematics and Computer Sci- ence*, 1-20
4. Azroul, E. Benboubker, M.B. and Rhoudaf, M. (2014), On some p(x-(quasilinear problem with right-hand side measure. *J. Math. Com-put. Simulation* 102,117-130.
5. Azroul, E. Benkirane, A. and Rhoudaf, M. (2010), On some strongly non-linear elliptic problems in L1-data with a nonlinearity having a constant sign in Orlicz spaces via penalization methods. Aust. *J.Math. Anal. Appl.* 7 (1), Art. 5, 1-25.
6. Azroul, E. Hjiaj,H. and Touzani, A. (2013), Existence and Regularity of Entropy solutions For Strongly Nonlinear p(x)-elliptic equations, Electronic. *J. Diff. Equ,* 68 , 1 – 27.
7. Benboubker, M. B. Hjiaj, H. and Ouaro, S. (2014), Entropy solutions to nonlinear elliptic anisotropic problem with variable exponent, *J.Appl. Anal. Comput.* 4(3), 245-270.
8. Bendahmane, M. Chrif, M. and Manouni, S. El. (2011), An Approximation Result in Generalized Anisotropic Sobolev Spaces and Application.*J. Anal. Anwend*. 3, 341-353.
9. Bendahmane, M. and Wittbold, P. (2009), Renormalized solutions for non-linear elliptic equations with variable exponents and L1 data. *J. Nonlinear Anal*. 70(2), 567-583.
10. Benkirane, A. and Elmahi, A. (1999) An existence theorem for a strongly nonlinear elliptic problem in Orlicz spaces. *J. Nonlinear Analysis*, 36(1999) 11-24.
11. Boccardo, L. (1996), *Some nonlinear Dirichlet problem in L^1 involving lower order terms in divergence form, Progress in elliptic and parabolic partial differential equations*. (Capri, 1994), Pitman Res Notes Math. Ser., 350, p. 43-57, Longman, Harlow.
12. B´enilan, Ph. Boccardo, L. Gallou¨et, T. Gariepy, R. Pierre, M. and V´azquez, J.L. (1995), An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, *.J. Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 4 , 241-273.
13. Boccardo, L. and Cirmi, G.R. (1999), Existence and uniqueness of solution of unilateral problems with L1. *J. Convex Anal.* 6(1), 206-195
14. Boccardo, L. and Gallou¨et, T. (1992), Nonlinear elliptic equations with right-hand side measures, *J. Comm. Partial Differential Equations* 17, (3-4), 641-655.
15. Nardo, R. Di. Feo, F. and Guib´e, O. (2013), Uniqueness result for nonlin-ear anisotropic elliptic equations. *J. Adv. Differential Equations*. 18, 5-6, 433-458.
16. Elmahi, A. and Meskine, D. (2004), Unilateral elliptic problems in L1 with natural growth terms. *J. Nonlinear Convex Anal.* 5(1), 97-112.
17. Hewitt, E. and Stromberg, K. (1965), Real and abstract analysis. Springer-verlng, Berlin Heidelberg New York.
18. Kone,B. Ouaro, S. Traor´e, S. (2009), Weak solutions for anisotropic non-linear elliptic equations with variable exponents, Electronic. J. Diff. Equ, 144,1-11.
19. Landes, R. (1980), On Galerkin's method in the existence theory for quasi-linear elliptic equations. *J. funct. Analysis*. 39, 123-148.
20. Lions, J. L. (1969), *Quelques m´ethodes de r´esolution des probl`emes aux lim-ites non lin´eaires*. Dunod et Gauthiers-Villars, Paris.
21. Mihailescu, M. Pucci, P. and Radulescu, V. (2008), Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent. *J. Math. Anal. Appl*. 687 - 698.
22. Sancho´n, M. and Urbano, J.M. (2009),Entropy solutions for the p(x)-Laplace equation. *J. Trans. Amer. Math. Soc.* 361, 6387- 6405.