King type modification of Gamma operators based on the (p,q)-integers which preserve x^2

A. A. H. Al-Abied

Department of Mathematics ، Faculty of Education & Sciences at Rada'a, Albaydha University - Albaydha - Yemen

abied1979@gmail.com.

Abstract

By using (p; q)-calculus, we introduce the Kings type modication of (p; q)- Gamma operators, which reproduce the test function x^2 . Then, we estimate the rate of convergence for the constructed operators by using the modulus of continuity in terms of a Lipschitz class function and by means of Peetres K-functionals based on Korovkin theorem.

Keywords: *(p; q)-Beta function, (p; q)-Gamma operators, weighted approximation, rate of convergence.*

تعديل نوع لمشغلي جاما بناء علىالأعداد الصحيحة (كيو, بي) التي تحافظ على X^2

DOI: https://doi.org/10.56807/buj.v2i1.35

الملخص

باستخدام التفاضل والتكامل (كيو,بي) نقدم تعديل من نوع الكشف (كيو,بي) - مشغل جاما بحيث تعيد إنتاج الدالة X^2 ثم نقدم معدل التقارب للمشغلين المبنيين باستخدام معامل الإستمرارية من حيث دالة فئة بتشز وبواسطة دالة كي بيترز على أساس نظرية كورفكن

AMS Subject Classification (2010): 41A10, 41A25, 41A36.

1. Introduction and Preliminaries

In the field of approximation theory, since its inception, the q-calculus has been an area of intensive research. The q-calculus has attracted attention of many researchers because of its applications in various fields such as numerical analysis, computer-aided geometric design, differential equations etc. New applications and generalizations are being discovered constantly. King-type modification of positive linear operators preserves the test function $e_2(x) = x^2$ and so provides better error estimations on some appropriate domains than the classical ones. Such a modification was first noticed by (King, 2003), for the classical Bernstein polynomials has presented an example of linear and positive operators $V_n : C([0,1]) \to C([0,1])$, are defined by

$$(V_n f\,)(x) = \sum_{k=0} \begin{pmatrix} n \\ k \end{pmatrix} (r_n^*(x))^k (1-r_n^*(x)) f\begin{pmatrix} k \\ n \end{pmatrix}, \qquad f \in C([0,1]), x \in [0,1],$$

where $r_n^*(x) : [0,1] \to [0,1]$, given as follows

$$r_n^*(x) = \begin{cases} x^2 & (n=1) \\ -\dfrac{1}{2(n-1)} + \sqrt{\dfrac{n}{n-1}x^2 + \dfrac{1}{4(n-1)^2}} & (n = 2, 3, ...). \end{cases}$$

This sequence preserves two test functions e_0, e_2 and $(V_n e_1)(x) = r_n^*(x)$ holds.

Later, this idea was applied to some other well-known approximating operators, such as, the Szász-Mirakjan operators (Duman and Özarslan, 2007), the Baskakov operators (Rempulska and Tomczak, 2009), the Meyer-König and Zeller operators (Özarslan and Duman, 2007), the Bernstein-Chlodovsky operators (Agratini, 2006), the q-Stancu-Beta operators (Mursaleen and Ansari, 2017), and more general summation-type positive linear operators (Agratini, 2006).

Approximation properties of q-analogue of some more operators are studied recently in (Mursaleen et al, 2017; Mursaleen and Khan, 2013; Mursaleen and Khan, 2013).

Most recently, the (p,q)-analogue of some more operators have been studied in (Mursaleen et al, 2017; Mursaleen et al, 2016; Mursaleen et al, 2016).

It has been observed that most of the approximating operators, L_n, preserve $e_i(x) = x_i (i = 0,1)$, i.e. $L_n(e_0;x) = e_0(x)$ and $L_n(e_1;x) = e_1(x)$, $n \in \mathbb{N}$. These conditions hold specially, for the Bernstein polynomials, Szász-Mirakjan operators, and the Baskakov operators. For each of these operators, $L_n(e_2;x) = e_2(x) = x^2$.

Let f be a function defined on $[0,\infty)$ and satisfy the following growth condition:

$$\mid f(x) \mid \leq Me^{-\beta x}(M > 0; \beta \geq 0; x \to \infty). \tag{1.1}$$

In (Zeng, 2005), investigated and studied some approximation properties of the following sequence of linear positive operators (named Gamma operators)

$$G_n(f;x) = \frac{1}{x^n \Gamma(n)} \int_0^\infty f\left(\frac{t}{n}\right) t^{n-1} e^{-\frac{t}{x}} dt. \tag{1.2}$$

In (Cai, 2014), introduced and studied q-analogue of Gamma operators as follows: For $f \in C[0,\infty)$ satisfies (1.1), $q \in (0,1)$ and $n \in \mathbb{N}$, the q-Gamma operators is defined by

$$G_{n,q}(f;x) = \frac{1}{x^n \Gamma_q(n)} \int_0^{\infty/A} f\left(\frac{t}{[n]_q}\right) t^{n-1} E_q\left(-\frac{qt}{x}\right) d_q t.$$

Stancu generalization of $G_{n,q}(f;x)$ was discussed and studied in (Özge and Örkcü, 2016).

The (p,q)-integer was introduced to generalize or unify several forms of q-oscillator algebras well known in the Physics literature related to the representation theory of single parameter quantum algebras. The (p,q)-integer is defined by

$$[n]_{p,q} = p^{n-1} + qp^{n-2} + \cdots + pq^{n-2} + q^{n-1} = \begin{cases} \dfrac{p^n - q^n}{p-q} & (p \neq q \neq 1) \\ \dfrac{1-q^n}{1-q} & (p = 1) \\ n & (p = q = 1) \end{cases} \tag{1.3}$$

The (p,q)-Binomial expansion is

$$(ax+by)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k,$$

$$(x+y)^n_{p,q} := (x+y)(px+qy)(p^2x+q^2y)\cdots(p^{n-1}x+q^{n-1}y),$$

$$(1-x)^n_{p,q} := (1-x)(p-qx)(p^2-q^2x)\cdots(p^{n-1}-q^{n-1}x).$$

The (p,q)-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

The improper (p,q)-integrals of a function f is defined by

$$\int_0^\infty f(x) d_{p,q}t = (p-q) \sum_{k=-\infty}^{\infty} f\left(\frac{q^k}{p^{k+1}}\right) \frac{q^k}{p^{k+1}}, \qquad 0 < \frac{q}{p} < 1.$$

There are two (p,q)-analogues of the classical exponential function defined as follows

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!},$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!},$$

which satisfy the equality $e_{p,q}(x)E_{p,q}(-x) = 1$. For $p = 1$, $e_{p,q}(x)$ and $E_{p,q}(x)$ reduce to q-exponential functions.

For $m, n \in \mathbb{N}$, the (p,q)-Beta and the (p,q)-Gamma functions are defined by

$$B_{p,q}(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} d_{p,q}x,$$

and

$$\Gamma_{p,q}(n) = \int_0^\infty p^{\frac{n(n-1)}{2}} E_{p,q}(-qx) d_{p,q}x, \qquad \Gamma_{p,q}(n+1) = [n]_{p,q}!.$$

The (p,q)-Gamma function of the second kind was defined in (Sadjang, 2017) as follows

$$\gamma_{p,q}(z) = \int_0^\infty q^{\frac{z(z-1)}{2}} t^{z-1} e_{p,q}(-pt) d_{p,q}t, \qquad \Re(z) > 0.$$

For any nonnegative integer $n > 0$, the following relation holds $\gamma_{p,q}(n+1) = [n]_{p,q}!$.

In the recent paper, (Cheng et al, 2019), introduced (p,q)-Gamma operators using the (p,q)-Gamma function of the second kind preserving linear functions as:

For $f \in C[0,\infty)$ satisfies (1.1), $0 < q < p \leq 1$ and $n \in \mathbb{N}$, the (p,q)-analogue of Gamma operators (1.2) are defined as:

$$G_{n,p,q}(f;x) = \frac{1}{x^n \gamma_{p,q}(n)} \int_0^\infty f\left(\frac{q^n t}{[n]_{p,q}}\right) q^{\frac{n(n-1)}{2}} t^{n-1} e_{p,q}\left(-\frac{pt}{x}\right) d_{p,q}t. \qquad (1.4)$$

Motivated by it, we propose the King type modification of (p,q)-analogue of Gamma operators which preserves x^2 as:

$$G^*_{n,p,q}(f;x) = \frac{1}{(r_n(x))^n \gamma_{p,q}(n)} \int_0^\infty f\left(\frac{q^n t}{[n]_{p,q}}\right) q^{\frac{n(n-1)}{2}} t^{n-1} e_{p,q}\left(-\frac{pt}{r_n(x)}\right) d_{p,q}t \quad (1.5)$$

where

$$r_n(x) = x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}}, \ r_n(x) \geq 0, \ x \geq 0.$$

2. Auxiliary results

In this section we give the following lemmas, which we need to prove our theorems.

Lemma 2.1. *For $x \in [0,\infty), 0 < q < p \leq 1,$ and $k = 0, 1, ...,$ we have*

$$G^*_{n,p,q}(t^k;x) = \frac{r_n^k(x) q^{-\frac{k(k-1)}{2}} [n+k-1]_{p,q}!}{[n]^k_{p,q}[n-1]_{p,q}!}.$$

Lemma 2.2. *For $x \in [0,\infty), 0 < q < p \leq 1,$ we have*

$$\begin{aligned}
(i)\ G_{n,p,q}(1;x) &= 1 \\
(ii)\ G_{n,p,q}(t;x) &= x \\
(iii)\ G_{n,p,q}(t^2;x) &= \left(1 + \frac{p^n}{q[n]_{p,q}}\right)x^2 \\
(iv)\ G_{n,p,q}(t^3;x) &= \left(1 + \frac{p^n([2]_{p,q}+q)}{q^2[n]_{p,q}} + \frac{p^{2n}}{q^3[n]^3_{p,q}}\right)x^3 \\
(v)\ G_{n,p,q}(t^4;x) &= \left(1 + \frac{p^n(q^2 + q[2]_{p,q} + [3]_{p,q})}{q^3[n]_{p,q}}\right. \\
&\quad \left. + \frac{p^{2n}([2]_{p,q}[3]_{p,q} + q[3]_{p,q} + q^2[2]_{p,q})}{q^5[n]^2_{p,q}} + \frac{[2]_{p,q}[3]_{p,q}p^{3n}}{q^6[n]^3_{p,q}}\right)x^4.
\end{aligned}$$

Lemma 2.3. *Let $e_r(t) = t^r$, $r \in \mathbb{N} \cup \{0\}$. For $x \in [0,\infty)$, $0 < q < p \leq 1$, we have*

$$\begin{aligned}
(i)\ G^*_{n,p,q}(e_0;x) &= 1 \\
(ii)\ G^*_{n,p,q}(e_1;x) &= x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}} \\
(iii)\ G^*_{n,p,q}(e_2;x) &= x^2 \\
(iv)\ G^*_{n,p,q}(e_3;x) &= \left(1 + \frac{p^n([2]_{p,q}+q)}{q^2[n]_{p,q}} + \frac{p^{2n}}{q^3[n]^3_{p,q}}\right) r^3_n(x) \\
(v)\ G^*_{n,p,q}(e_4;x) &= \left(1 + \frac{p^n(q^2+q[2]_{p,q}+[3]_{p,q})}{q^3[n]_{p,q}}\right. \\
&\quad \left. + \frac{p^{2n}([2]_{p,q}[3]_{p,q}+q[3]_{p,q}+q^2[2]_{p,q})}{q^5[n]^2_{p,q}} + \frac{[2]_{p,q}[3]_{p,q}p^{3n}}{q^6[n]^3_{p,q}}\right) r^4_n(x).
\end{aligned}$$

Proof. From Lemma (2.1) , we get the equalities of (i), (ii), (iii), (iv) and (v) easily. □

Lemma 2.4. *For $x \in [0,\infty)$, $0 < q < p \leq 1$, we have*

$$\begin{aligned}
(i)\ G^*_{n,p,q}((t-x);x) &= r_n(x) - x \\
&\leq x(\sqrt{q} - 1) \\
(ii)\ G^*_{n,p,q}((t-x)^2;x) &= 2x(x - r_n(x)) \\
&\leq 2x^2(1 - \sqrt{q}).
\end{aligned}$$

3. Quantitative results

In this section, we present local approximation theorem for the operators $G^*_{n,p,q}$. By $C_B[0,\infty)$, we denote the space of real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\| f \| = \sup_{0\leq x<\infty} | f(x) | .$$

Further let us consider the following K-functional:

$$K_2(f,\delta) = \inf_{g\in W^2}\{\| f-g \| + \delta \| g'' \|\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By Theorem (2.4) of (Devore and Lorentz, 1993), there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}) \tag{3.1}$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} | f(x+2h) - 2f(x+h) + f(x) |$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$. The usual modulus of continuity of $f \in C_B[0,\infty)$ is given by

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} \mid f(x+h) - f(x) \mid .$$

Let $B_m[0,\infty)$ be the set of all functions satisfying the condition $\mid f(x) \mid \leq M_f(1+x^m)$, $x \in [0,\infty), m > 0$ and M_f is some constant depending on f. Further, let $C_m[0,\infty) = B_m[0,\infty) \cap C[0,\infty)$ endowed with the norm

$$\| f \|_m = \sup_{x\geq 0} \frac{\mid f(x) \mid}{1+x^m}$$

and

$$C_m^*[0,\infty) = \left\{ f \in C_m[0,\infty) : \lim_{x\to\infty} \frac{\mid f(x) \mid}{1+x^m} < \infty \right\}.$$

Theorem 3.1. *Let $q = q_n \in (0,1)$, $p = p_n \in (q,1]$ such that $q_n \to 1$, $p_n \to 1$ as $n \to \infty$. Then for each function $f \in C_2^*[0,\infty)$ we get*

$$\lim_{n\to\infty} \| G^*_{n,p_n,q_n} f - f \|_2 = 0.$$

Proof. According to weighted Korovkin theorem proved in (Gadzhiev, 1974), it is sufficient to verify the following three conditions of (3.2)

$$\lim_{n\to\infty} \| G^*_{n,p_n,q_n} e_i - e_i \|_2 = 0, \qquad i = 0,1,2, \tag{3.2}$$

where $e_i(t) = t^i$, $i = 0,1,2$. By Lemma (2.3) (i) and (iii), (3.2) holds for $i = 0,2$. Finally, for $i = 1$ by Lemma (2.3) (ii) we have

$$\begin{aligned} \| G^*_{n,p_n,q_n} e_1 - e_1 \|_2 &= \sup_{x\geq 0} \frac{\mid r_n(x) - x \mid}{1+x^2} \\ &\leq \sup_{x\geq 0} \frac{x}{1+x^2} \left(\frac{1}{\sqrt{1+\dfrac{p_n^n}{q_n[n]_{p_n,q_n}}}} - 1 \right) \\ &\leq \sqrt{q_n} - 1 \end{aligned}$$

we get

$$\lim_{n\to\infty} \| G^*_{n,p_n,q_n} e_1 - e_1 \|_2 = 0.$$

Then the proof of (3.2) is completed. □

Theorem 3.2. *Let $f \in C_B[0,\infty)$ and $0 < q < p \leq 1$. Then for all $n \in \mathbb{N}$, there exists an absolute constant $C > 0$ such that*

$$\mid G^*_{n,p,q}(f;x) - f(x) \mid \leq C\omega_2(f, \gamma_{n,p,q}(x)) + \omega(f, \delta_{n,p,q}(x)),$$

where

$$\gamma_{n,p,q}(x) = \sqrt{G^*_{n,p,q}((t-x)^2;x) + (\delta_{n,p,q}(x))^2}, \quad \delta_{n,p,q}(x) = G^*_{n,p,q}((t-x);x).$$

Proof. For $x \in [0, \infty)$, we consider the auxiliary operators $\bar{G}_n^*$ defined by

$$\bar{G}_n^*(f;x) = G_{n,p,q}^*(f;x) + f(x) - f\left(x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}}\right). \tag{3.3}$$

From Lemma (2.3) (i)(ii) and (3.3), we observe that the operators $\bar{G}_n^*(f;x)$ are linear and reproduce the linear functions. Hence

$$\begin{aligned}
\bar{G}_n^*(1;x) &= G_{n,p,q}^*(1;x) + 1 - 1 = 1 \\
\bar{G}_n^*(t;x) &= G_{n,p,q}^*(t;x) + x - x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}} = x \\
\bar{G}_n^*((t-x);x) &= \bar{G}_n^*(t;x) - x\bar{G}_n^*(1;x) = 0.
\end{aligned}$$

Let $x \in [0, \infty)$ and $g \in W^2$. Using the Taylor's formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\mathrm{d}u.$$

Applying $\bar{G}_n^*$ to both sides of the above equation, we have

$$\begin{aligned}
\bar{G}_n^*(g;x) - g(x) &= g'(x)\bar{G}_n^*((t-x);x) + \bar{G}_n^*\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right) \\
&= G_{n,p,q}^*\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right) \\
&\quad - \int_x^{x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}}} \left(x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}} - u\right) g''(u)\mathrm{d}u.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
\left|\int_x^t (t-u)g''(u)\mathrm{d}u\right| &\leq \int_x^t \mid t-u \mid\mid g''(u) \mid \mathrm{d}u \\
&\leq \parallel g'' \parallel \int_x^t \mid t-u \mid \mathrm{d}u \leq (t-x)^2 \parallel g'' \parallel
\end{aligned}$$

and

$$\left|\int_x^{x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}}} \left(x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}} - u\right) g''(u)\mathrm{d}u\right|$$

$$\leq \left(x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}} - x\right)^2 \parallel g'' \parallel.$$

We conclude that

$$\begin{aligned}\left|\bar{G}_n^*(g;x)-g(x)\right| &\le \left|G_{n,p,q}^*\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right)\right.\\&\quad\left.-\int_x^{x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}}}\left(x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}}-u\right)g''(u)\mathrm{d}u\right|\\&\le \| g''\| G_{n,p,q}^*((t-x)^2;x)+\| g''\| \left(x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}}-x\right)^2\\&= \| g''\| \gamma_{n,p,q}^2(x).\end{aligned}$$

Now, taking into account boundedness of $\bar{G}_n^*$, we have

$$|\bar{G}_n^*(f;x)| \le |G_{n,p,q}^*(f;x)|+2\| f\|\le 3\| f\|.$$

Therefore

$$\begin{aligned}|G_{n,p,q}^*(f;x)-f(x)| &\le |\bar{G}_n^*(f-g;x)-(f-g)(x)|\\&\quad+\left|f\left(x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}}\right)-f(x)\right|+|\bar{G}_n^*(g;x)-g(x)|\\&\le |\bar{G}_n^*(f-g;x)|+|(f-g)(x)|\\&\quad+\left|f\left(x\sqrt{\frac{q[n]_{p,q}}{[n+1]_{p,q}}}\right)-f(x)\right|+|\bar{G}_n^*(g;x)-g(x)|\\&\le 4\| f-g\|+\omega(f,\delta_{n,p,q}(x))+\gamma_{n,p,q}^2(x)\| g''\|.\end{aligned}$$

Hence, taking the infimum on the right-hand side over all $g\in W^2$, we have the following result

$$|G_{n,p,q}^*(f;x)-f(x)| \le 4K_2(f,\gamma_{n,p,q}^2(x))+\omega(f,\delta_{n,p,q}(x)).$$

In view of the property of K-functional, we get

$$|G_{n,p,q}^*(f;x)-f(x)| \le C\omega_2(f,\gamma_{n,p,q}(x))+\omega(f,\delta_{n,p,q}(x)).$$

Consequently, the proof is finished. □

Theorem 3.3. *Let $q=q_n\in(0,1), p=p_n\in(q,1]$ such that $q_n\to1,\ p_n\to1,$ as $n\to\infty$. If $f\in C_m^*(0,\infty]$ and let $f^*(z)=f(z^2), z\in[0,\infty)$. For all $t>0$ and $x\ge0$, we have*

$$|G_{n,p_n,q_n}^*(f;x)-f(x)| \le 2\omega\left(f^*,\sqrt{2x(1-\sqrt{q_n})}\right).$$

Proof. Let $f\in C_m^*(0,\infty]$ is fixed. By definition of f^* we get

$$G_{n,p_n,q_n}^*(f;x) = G_{n,p_n,q_n}^*(f^*(\sqrt{\cdot});x).$$

Now, we can write

$$\begin{aligned}
\mid G^*_{n,p_n,q_n}(f;x)-f(x)\mid &= \mid G^*_{n,p_n,q_n}(f^*(\sqrt{\cdot});x)-f^*\sqrt{x}\mid \\
&= \left|\Big(f^*(\sqrt{\frac{q_n^n t}{[n]_{p_n,q_n}}})-f^*(\sqrt{x})\Big)S_n(p_n,q_n;r_n(x))\right| \\
&\leq \left|f^*(\sqrt{\frac{q_n^n t}{[n]_{p_n,q_n}}})-f^*(\sqrt{x})\right|S_n(p_n,q_n;r_n(x)) \\
&\leq \omega\Big(f^*;\left|\sqrt{\frac{q_n^n t}{[n]_{p_n,q_n}}}-\sqrt{x}\right|\Big)S_n(p_n,q_n;r_n(x)) \\
&= \omega\Big(f^*;\frac{\left|\sqrt{\frac{q_n^n t}{[n]_{p_n,q_n}}}-\sqrt{x}\right|}{G^*_{n,p_n,q_n}(\mid\sqrt{\cdot}-\sqrt{x}\mid;x)} \\
&\quad \times G^*_{n,p_n,q_n}(\mid\sqrt{\cdot}-\sqrt{x}\mid;x)\Big)S_n(p_n,q_n;r_n(x))
\end{aligned}$$

where

$$S_n(p_n,q_n;r_n(x)) = \frac{1}{(r_n(x))^n\gamma_{p_n,q_n}(n)}\int_0^\infty q_n^{\frac{n(n-1)}{2}}t^{n-1}e_{p_n,q_n}\Big(-\frac{p_n t}{r_n(x)}\Big)d_{p_n,q_n}t.$$

Using the property of modulus of continuity

$$\omega(f^*;\alpha\delta) \leq (1+\alpha)\omega(f^*;\delta), \qquad \alpha,\delta\geq 0,$$

we obtain

$$\begin{aligned}
\left|G^*_{n,p_n,q_n}(f;x)-f(x)\right| &\leq \omega\Big(f^*;G^*_{n,p_n,q_n}(\mid\sqrt{\cdot}-\sqrt{x}\mid;x)\Big) \\
&\quad \times\Big(1+\frac{\left|\sqrt{\frac{q_n^n t}{[n]_{p_n,q_n}}}-\sqrt{x}\right|}{G^*_{n,p_n,q_n}(\mid\sqrt{\cdot}-\sqrt{x}\mid;x)}\Big)S_n(p_n,q_n;r_n(x)) \\
&= 2\omega\Big(f^*;G^*_{n,p_n,q_n}(\mid\sqrt{\cdot}-\sqrt{x}\mid;x)\Big).
\end{aligned}$$

Since $\dfrac{1}{\sqrt{\frac{q_n^n t}{[n]_{p_n,q_n}}}+\sqrt{x}} \leq \dfrac{1}{\sqrt{x}}$ and using Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
G^*_{n,p_n,q_n}(|\sqrt{\cdot}-\sqrt{x}|;x) &= \left|\sqrt{\frac{q_n^n t}{[n]_{p_n,q_n}}}-\sqrt{x}\right| S_n(p_n,q_n;r_n(x)) \\
&= \frac{\left|\dfrac{q_n^n t}{[n]_{p_n,q_n}}-x\right|}{\sqrt{\dfrac{q_n^n t}{[n]_{p_n,q_n}}}+\sqrt{x}} S_n(p_n,q_n;r_n(x)) \\
&\leq \frac{1}{\sqrt{x}}\left|\frac{q_n^n t}{[n]_{p_n,q_n}}-x\right| S_n(p_n,q_n;r_n(x)) \\
&\leq \frac{1}{\sqrt{x}}\sqrt{\left|\frac{q_n^n t}{[n]_{p_n,q_n}}-x\right|^2 S_n(p_n,q_n;r_n(x))} \\
&\leq \frac{1}{\sqrt{x}}\sqrt{G^*_{n,p_n,q_n}((\cdot-x)^2;r_n(x))} \\
&\leq \sqrt{2x(1-\sqrt{q_n})}.
\end{aligned}
$$

Thus the proof is completed. □

4. **Approximation properties in weighted spaces**

First, let us recall the definitions of weighted spaces and corresponding modulus of continuity. Let $C[0,\infty)$ be the set of all continuous functions f defined on $[0,\infty)$ and $B_\rho[0,\infty)$ the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \leq M_f\rho(x)$ where M_f is a constant depending only on f and $\rho(x)$ is a weight function.

Let $C_\rho[0,\infty)$ be the space of all continuous functions in $B_\rho[0,\infty)$ with the norm

$\|f\|_\rho = \sup\limits_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}$ and

$$C_\rho^0 = \left\{ f \in C_\rho[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} < \infty \right\}.$$

In what follows, we assume the weight function as $\rho(x) = 1 + x^2$.

Next we give the following theorem to approximate all functions in C_ρ^0. This type of result is discussed in (Lenze, 1988), for locally integrable functions.

Theorem 4.1. *Let $0 < q = q_n < p = p_n \leq 1$ such that $q_n \to 1$, $p_n \to 1$, $q_n^n \to 1$, $p_n^n \to 1$ as $n \to \infty$. For each $f \in C_\rho^0$ and $a > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|G^*_{n,p_n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+a}} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\begin{aligned}
\sup_{x\in[0,\infty)} \frac{\mid G^*_{n,p_n,q_n}(f;x) - f(x) \mid}{(1+x^2)^{1+a}} &\leq \sup_{x\leq x_0} \frac{\mid G^*_{n,p_n,q_n}(f;x) - f(x) \mid}{(1+x^2)^{1+a}} \\
&+ \sup_{x\geq x_0} \frac{\mid G^*_{n,p_n,q_n}(f;x) - f(x) \mid}{(1+x^2)^{1+a}} \\
&\leq \|G^*_{n,p_n,q_n}(f;x) - f(x)\|_{C[0,x_0]} \\
&+\|f\|_\rho \sup_{x\geq x_0} \frac{\mid G^*_{n,p_n,q_n}(1+t^2;x) \mid}{(1+x^2)^{1+a}} \\
&+ \sup_{x\geq x_0} \frac{\mid f(x) \mid}{(1+x_0^2)^{1+a}} \\
&= I_1 + I_2 + I_3. \qquad (4.1)
\end{aligned}$$

Since $\mid f(x) \mid\leq \|f\|_\rho(1+x^2)$, we have

$$I_3 = \sup_{x\geq x_0} \frac{\mid f(x) \mid}{(1+x^2)^{1+a}} \leq \sup_{x\geq x_0} \frac{\|f\|_\rho}{(1+x^2)^a} \leq \frac{\|f\|_\rho}{(1+x_0^2)^a}.$$

Let $\epsilon > 0$ be arbitrary.
There exists $n_1 \in \mathbb{N}$ such that

$$\begin{aligned}
\|f\|_\rho \frac{\mid G^*_{n,p_n,q_n}(1+t^2;x) \mid}{(1+x^2)^{1+a}} &< \frac{1}{(1+x^2)^{1+a}} \|f\|_\rho \left((1+x^2) + \frac{\epsilon}{3\|f\|_\rho} \right), \forall n \geq n_1 \\
&< \frac{\|f\|_\rho}{(1+x^2)^a} + \frac{\epsilon}{3} \qquad \forall n \geq n_1. \qquad (4.2)
\end{aligned}$$

Hence

$$\|f\|_\rho \sup_{x\geq x_0} \frac{\mid G^*_{n,p_n,q_n}(1+t^2;x) \mid}{(1+x^2)^{1+a}} < \frac{\|f\|_\rho}{(1+x_0^2)^a} + \frac{\epsilon}{3}, \qquad \forall n \geq n_1.$$

Thus

$$I_2 + I_3 < \frac{2\|f\|_\rho}{(1+x_0^2)^a} + \frac{\epsilon}{3}, \qquad \forall n \geq n_1.$$

Now, let us choose x_0 to be so large that $\dfrac{\|f\|_\rho}{(1+x^2)^a} < \dfrac{\epsilon}{6}$.
Then,

$$I_2 + I_3 < \frac{2\epsilon}{3}, \qquad \forall n \geq n_1. \qquad (4.3)$$

$$I_1 = \|G^*_{n,p_n,q_n}(f) - f\|_{C[0,x_0]} < \frac{\epsilon}{3}, \qquad \forall n \geq n_2. \qquad (4.4)$$

Let $n_0 = \max(n_1, n_2)$. Then, combining (4.1)-(4.4)

$$\sup_{x\in[0,\infty)} \frac{\mid G^*_{n,p_n,q_n}(f;x) - f(x) \mid}{(1+x^2)^{1+a}} < \epsilon, \qquad \forall n \geq n_0.$$

This completes the proof. $\square$

Now we present ordinary approximation in terms of Lipschitz constant defined by

$$lip_M(\beta) = \left\{ f \in C_B[0,\infty) : \mid f(t) - f(x) \mid \leq M \frac{\mid t - x \mid^\beta}{(t+x)^{\frac{\beta}{2}}} \right\}, \tag{4.5}$$

where M is a positive constant and $0 < \beta \leq 1$.

In the following theorem, we obtain the rate of convergence of the operators $G^*_{n,p,q}$ for functions $lip_M(\beta)$.

Theorem 4.2. *Let be $f \in C_B[0,\infty)$, $0 < q < p \leq 1$, then for any $x \in (0,\infty)$, the following inequality holds:*

$$|G^*_{n,p,q}(f;x) - f(x)| \leq M \left(\frac{\alpha_{n,p,q}(x)}{x} \right)^{\frac{\beta}{2}},$$

where $\alpha_{n,p,q}(x) = G^*_{n,p,q}((t-x)^2;x)$.

Proof. First, we prove that the result is true for $\beta = 1$. Then, for $f \in lip_M(\beta)$, we obtain

$$\begin{aligned} |G^*_{n,p,q}(f;x) - f(x)| &\leq \frac{1}{(r_n(x))^n \gamma_{p,q}(n)} \int_0^\infty q^{\frac{n(n-1)}{2}} t^{n-1} e_{p,q}\Big(-\frac{pt}{r_n(x)} \Big) \\ &\quad \times \Big| f(\frac{q^n t}{[n]_{p,q}}) - f(x) \Big| d_{p,q}t \\ &\leq \frac{M}{(r_n(x))^n \gamma_{p,q}(n)} \int_0^\infty q^{\frac{n(n-1)}{2}} t^{n-1} e_{p,q}\Big(-\frac{pt}{r_n(x)} \Big) \\ &\quad \times \frac{\Big| \frac{q^n t}{[n]_{p,q}} - x \Big|}{\sqrt{\frac{q^n t}{[n]_{p,q}} + x}} d_{p,q}t. \end{aligned}$$

Using $\sqrt{x} < \sqrt{\frac{q^n t}{[n]_{p,q}} + x}$ and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |G^*_{n,p,q}(f;x) - f(x)| &\leq \frac{M}{\sqrt{x}(r_n(x))^n \gamma_{p,q}(n)} \int_0^\infty q^{\frac{n(n-1)}{2}} t^{n-1} e_{p,q}\Big(-\frac{pt}{r_n(x)} \Big) \\ &\quad \times \Big| \frac{q^n t}{[n]_{p,q}} - x \Big| d_{p,q}t \\ &= \frac{M}{\sqrt{x}} G^*_{n,p,q}((t-x)^2;x) \leq M \left(\frac{\alpha_{n,p,q}(x)}{x} \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore the result is true for $\beta = 1$. we prove that the result is true for $0 < \beta \leq 1$, applying Holder's inequality with $p = \frac{2}{\beta}, q = \frac{1}{2-\beta}$,

$$
\begin{aligned}
|G^*_{n,p,q}(f;x)-f(x)| &\leq \frac{1}{(r_n(x))^n\gamma_{p,q}(n)}\int_0^\infty q^{\frac{n(n-1)}{2}}t^{n-1}e_{p,q}\Big(-\frac{pt}{r_n(x)}\Big)\\
&\quad\times\Big|f(\frac{q^nt}{[n]_{p,q}})-f(x)\Big|d_{p,q}t\\
&\leq \Bigg\{\Bigg(\frac{1}{(r_n(x))^n\gamma_{p,q}(n)}\int_0^\infty q^{\frac{n(n-1)}{2}}t^{n-1}e_{p,q}\Big(-\frac{pt}{r_n(x)}\Big)\\
&\quad\times\Big|f(\frac{q^nt}{[n]_{p,q}})-f(x)\Big|d_{p,q}t\Bigg)^{\frac{2}{\beta}}\Bigg\}^{\frac{\beta}{2}}\\
&\quad\times\Bigg\{\frac{1}{(r_n(x))^n\gamma_{p,q}(n)}\int_0^\infty q^{\frac{n(n-1)}{2}}t^{n-1}\\
&\quad\times e_{p,q}\Big(-\frac{pt}{r_n(x)}\Big)d_{p,q}t\Bigg\}^{\frac{2-\beta}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Bigg\{\frac{1}{(r_n(x))^n\gamma_{p,q}(n)}\int_0^\infty q^{\frac{n(n-1)}{2}}t^{n-1}e_{p,q}\Big(-\frac{pt}{r_n(x)}\Big)\\
&\quad\times\Big|f(\frac{q^nt}{[n]_{p,q}})-f(x)\Big|^{\frac{2}{\beta}}d_{p,q}t\Bigg\}^{\frac{\beta}{2}}.
\end{aligned}
$$

Since $f\in lip_M(\beta)$, we have

$$
\begin{aligned}
|G^*_{n,p,q}(f;x)-f(x)| &\leq \frac{M}{x^{\frac{\beta}{2}}}\Bigg\{\frac{1}{(r_n(x))^n\gamma_{p,q}(n)}\int_0^\infty q^{\frac{n(n-1)}{2}}t^{n-1}e_{p,q}\Big(-\frac{pt}{r_n(x)}\Big)\\
&\quad\times\Big(\frac{q^nt}{[n]_{p,q}}-x\Big)^2d_{p,q}t\Bigg\}^{\frac{\gamma}{2}}\\
&= \frac{M}{x^{\frac{\beta}{2}}}\Big(G^*_{n,p,q}((t-x)^2;x)\Big)^{\frac{\beta}{2}}\leq M\Big(\frac{\alpha_{n,p,q}(x)}{x}\Big)^{\frac{\beta}{2}}.
\end{aligned}
$$

Hence, the desired result is obtained. □

Conclusion

In this paper we have constructed a new King type modification of Gamma operators based on (p,q)-integers and investigated their approximation properties. We have obtained weighted approximation and some other results of these operators are studied by means of modulus of continuity and Peetre K -functional.

References

1. Agratini, O. (2006). *Linear operators that preserve some test functions.* Int. J. Math. Math. Sci. 11, Art. ID 94136.
2. Agratini, O. (2006). *On the iterates of a class of summation-type linear positive operators.* Comput. Math. Appl. 55, 1178-1180.
3. CAI, Q. B. 4(2014). *Properties of Convergence for a Class of Generalized q-Gamma Operators.* J. Math. Study. 47, , 388-395.
4. Cheng, W. Zhang, W. Shu, A. (2019). *Approximation properties of* (p, q)*-Gamma operators.* J. Math. Inequal. 13(3), 879-890.
5. Devore, R. A. Lorentz, G. G. (1993). *Constructive Approximation, Springer.* Berlin.
6. Duman, O. Özarslan, M. A. (2007). *Szász-Mirakjan type operators providing a better error estimation.* Appl. Math. Lett. 20, 1184-1188.
7. Gadzhiev, A. D. (1974). *The convergence problem for a sequence of linear positive operators on unbounded sets and theorems analogous to that P. P. Korovkin.* Sov. Math. Dokl. 15(5), 1433-1436.
8. Lenze, B. (1988). *On Lipschitz-type maximal functions and their smoothness spaces.* Nederl. Akad. Wetensch. Indag. Math. 50(1), 53-63.
9. King, J. P. (2003). *Positive linear operators which preserve* x^2. Acta. Math. Hungar. 99, 203-208.
10. Mursaleen, M. Ansari, K. J. (2017). *Approximation of q-Stancu-Beta Operators Which Preserve* x^2. Bulletin of the Malaysian Mathematical Sciences Society. Oct 1;40(4):1479-91.
11. Mursaleen, M. Nasiruzzaman, MD. Al-Abied, A. A. H. (2017). *Dunkl generalization of q-parametric Szász-Mirakjan operators.* International Journal of Analysis and Applications, ISSN 2291-8639, Volume 13, Number 2, 206-215.
12. Mursaleen, M. Khan, A. (2013). *Statistical approximation properties of modified q-Stancu-Beta operators.* Bull. Malays. Math. Sci. Soc. (2) 36(3), 683690.
13. Mursaleen, M. Khan, A. 7(2013). *Generalized q-Bernstein-Schurer operators and some approximation theorems.* J. Funct. Spaces Appl. (719834).
14. Mursaleen, M. Al-Abied, A. A. H. Alotaibi, A. (2017)(1). *On* (p, q)*-Szász-Mirakyan operators and their approximation properties.* Journal of inequalities and applications 2017 Dec 1:196.
15. Mursaleen, M. Al-Abied, A. Nasiruzzaman, M. (2016). *Modified (p, q)-Bernstein-Schurer operators and their approximation properties.* Cogent Mathematics. 31;3(1):1236534.
16. Mursaleen, M. Alotaibi, A. Ansari, K. J. (2016). *On a Kantorovich variant of* (p, q)*-Szász-Mirakjan operators.* Jour. Func. Spaces, Volume, Article ID 1035253, 9 pages.
17. Özarslan, M. A. Duman, O. (2007). *MKZ type operators providing a better estimation on* $[1/2, 1]$. Canad. Math. Bull. 50,434-439.
18. ÖZGE, D. ÖRKCÜ, M. 4(2016). *On convergence properties of Gamma-Stancu operators based on q-integers.* Math. Inequal. Appl. 19, 1335-1348.
19. Rempulska, L. Tomczak, K. (2009). *Approximation by certain linear operators preserving* x^2. Turkish J. Math. 33, 273-281.
20. SADJANG, P. N. 9(2017). *On two (p,q) -analogues of the Laplace transform.* J. Differ. Equ. Appl., 23. 1562-1583.
21. ZENG, X. M. (2005). *Approximation properties of Gamma operators.* J. Math. Anal. Appl. 311, 389-401.