ON DOUBLE ABOODH-SHEHU TRANSFORM AND ITS PROPERTIES WITH APPLICATIONS

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Abstract

We combine the Aboodh transform and Shehu transform to give another double transform which is called the double Aboodh-Shehu transform. This interesting transform reduces a linear partial differential equation with unknown function of two independent variables to an algebraic equation, which can then be solved by the formal rules of algebra, or by applying the double Aboodh-Shehu transform directly to the given equation.

1. Introduction

One of most popular and rather method for solving partial differential equations is the integral transform method. In the literature, several different transforms are introduced and applied to find the solution of partial differential equations such as Laplace transform [10], Sumudu transform [7], Aboodh transform [1], Shehu transform [5], and so on. In 2013, K. Aboodh [1] introduced a new integral transform called Aboodh transform, which is derived from the Fourier integral and similar to Laplace transform, and applied it to solve ordinary differential equations, after that he introduced the double Aboodh transform and used it to solve integral differential equation and partial differential equation [2]. Shehu transform of single variable [12], is a new transform which was recently introduced by M. Shehu and W. Zhao in 2019. Shehu transform is a generalization of Laplace and Sumudu transforms. This transform is used to solve both ordinary and partial differential equations. In recent years, great attention has been given to deal with double integral transforms, see for example [2, 4, 6, 9] and others. The main objective of this paper is to introduce a new method for solving some partial differential equations subject to the initial and boundary conditions called double Aboodh-Shehu transform, the definition of double Aboodh-Shehu transform and its inverse. We also discuss some theorems and popular properties about the double Aboodh-Shehu transform and gave the double Aboodh-Shehu transform of some elementary functions. To show the applicability and efficiency of this interesting transform we apply this transform to some test examples.

Definition 1.1. *[1] The single Aboodh transform of the real function f(x) of exponential order is defined over the set of functions,*

$A = \ f(x) : \exists K, \tau 1, \tau 2 > 0, \mid f(x) \mid < Ke|x|\tau j\ , x \in (-1)j \times [0, \infty), j = 1, 2\ ,$

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by the following integral

$$A[f(x)] = F(r) = \frac{1}{r}\int_0^\infty e^{-rx} f(x)dx,\ \tau_1 \le r \le \tau_2. \tag{1.1}$$

And the inverse Aboodh transform is

$$A^{-1}[F(x)] = f(x) = \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} re^{rx}F(r)dr,\ w \ge 0 \tag{1.2}$$

Definition 1.2. [12] The single Shehu transform of the function $f(t)$ of exponential order is defined over the set of functions,

$$\mathcal{B} = \left\{ f(t) : \exists N, \rho_1, \rho_2 > 0, \mid f(t) \mid < Ne^{\frac{|t|}{\rho_j}}, t \in (-1)^j \times [0,\infty),\ j = 1,2 \right\},$$

by the following integral

$$\mathbb{S}[f(t)] = F(s,u) = \int_0^\infty e^{\frac{-st}{u}} f(t)dt,\ s > 0,\ u > 0. \tag{1.3}$$

The inverse Shehu transform is given by

$$f(t) = \mathbb{S}^{-1}[F(s,u)] = \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} \frac{1}{u}e^{\frac{st}{u}}F(s,u)ds, \tag{1.4}$$

where s and u are the Shehu transform variables, and w is a real constant and the integral in $Eq.$(1.3) is taken along $s = w$ in the complex plane $s = x + iy$. In the next definitions, we introduce the double Aboodh-Shehu transform.

Definition 1.3. The double Aboodh-Shehu transform of the continuous function $f(x,t)$, $x, t > 0$ is denoted by the operator $A_x\mathbb{S}_t[f(x,t)] = F(r,s,u)$ and defined by

$$\begin{aligned} A_x\mathbb{S}_t[f(x,t)] = F(r,s,u) &= \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}f(x,t)dxdt \\ &= \frac{1}{r}\lim_{a\to\infty, b\to\infty}\int_0^a\int_0^b e^{-(rx+\frac{st}{u})}f(x,t)dxdt. \end{aligned} \tag{1.5}$$

It converges if the limit of the integral exists, and diverges if not.
The inverse of double Aboodh-Shehu transform is defined by

$$f(x,t) = A_x^{-1}\mathbb{S}_t^{-1}\left[F(r,s,u)\right] = \frac{1}{(2\pi i)^2}\int_{\rho_1-i\infty}^{\rho_1+i\infty} re^{rx}\left\{\int_{\rho_2-i\infty}^{\rho_2+i\infty}\frac{1}{u}e^{\frac{st}{u}}F(r,s,u)ds\right\}dr, \tag{1.6}$$

where ρ_1 and ρ_2 are real constants.

2. Existence and uniqueness of double Aboodh-Shehu transform

In this section, we prove the existence and uniqueness of double Aboodh-Shehu transform.

Definition 2.1. A function $f(x,t)$ is said to be exponential order $e^{(ax+bt)}$, $a, b > 0$ on $[0,\infty)$, if there exist positive constants L, X and T such that

$$\mid f(x,t) \mid \le Le^{(ax+bt)}, \qquad for\ \ all\ x > X,\ t > T,$$

and, we write

$$f(x,t) = o(e^{(ax+bt)}) \qquad as\ (x \to \infty, t \to \infty).$$

Or, equivalently,

$$\sup_{x,t>0} \left(\frac{|\ f(x,t)\ |}{e^{(ax+bt)}} \right) < \infty.$$

Theorem 2.2. *Let $f(x,t)$ be a continuous function in every finite intervals $(0,X)$ and $(0,T)$, and of exponential order $e^{(ax+bt)}$, then the double Aboodh-Shehu transform of $f(x,t)$ exists for all $r > a$ and $\frac{s}{u} > b$.*

Proof. Let $f(x,t)$ be of exponential order $e^{(ax+bt)}$ such that

$$|\ f(x,t)\ | \leq Le^{(ax+bt)},\ \forall x > X, t > T.$$

Then, from the definition of double Aboodh-Shehu transform, we have

$$\begin{aligned}
\big|F(r,s,u)\big| &= \Big|\frac{1}{r}\int_0^\infty \int_0^\infty e^{-(rx+\frac{s}{u}t)} f(x,t)dxdt\Big| \\
&\leq \frac{1}{r}\int_0^\infty \int_0^\infty e^{-(rx+\frac{s}{u}t)} |f(x,t)| dxdt \\
&\leq \frac{L}{r}\int_0^\infty \int_0^\infty e^{-(rx+\frac{s}{u}t)} e^{(ax+bt)} dxdt \\
&= \frac{L}{r}\int_0^\infty e^{-(r-a)x}dx \int_0^\infty e^{-(\frac{s}{u}-b)t}dt \\
&= \frac{Lu}{r(r-a)(s-bu)}.
\end{aligned}$$

The proof is complete. □

Theorem 2.3. *Let $f_1(x,t)$ and $f_2(x,t)$ be continuous functions defined for $x,t \geq 0$ and having the double Aboodh-Shehu transform $F_1(r,s,u)$ and $F_2(r,s,u)$, respectively. If $F_1(r,s,u) = F_2(r,s,u)$, then, $f_1(x,t) = f_2(x,t)$.*

Proof. Assume β_1 and β_2 are sufficiently large, since

$$f(x,t) = A_x^{-1}\mathbb{S}_t^{-1}[F(r,s,u)] = \frac{1}{(2\pi i)^2}\int_{\beta_1 - i\infty}^{\beta_1 + i\infty} re^{rx} \left\{ \int_{\beta_2 - i\infty}^{\beta_2 + i\infty} \frac{1}{u} e^{\frac{st}{u}} F(r,s,u) ds \right\} dr,$$

we deduce that

$$\begin{aligned}
f_1(x,t) &= \frac{1}{(2\pi i)^2}\int_{\beta_1 - i\infty}^{\beta_1 + i\infty} re^{rx} \left\{ \int_{\beta_2 - i\infty}^{\beta_2 + i\infty} \frac{1}{u} e^{\frac{st}{u}} F_1(r,s,u) ds \right\} dr \\
&= \frac{1}{(2\pi i)^2}\int_{\beta_1 - i\infty}^{\beta_1 + i\infty} re^{rx} \left\{ \int_{\beta_2 - i\infty}^{\beta_2 + i\infty} \frac{1}{u} e^{\frac{st}{u}} F_2(r,s,u) ds \right\} dr \\
&= f_2(x,t).
\end{aligned}$$

Thus, the proof is complete. □

3. Some Properties of Double Aboodh-Shehu Transform

3.1. Linearity property. If $f(x,t)$ and $h(x,t)$ be two functions such that

$$A_x\mathbb{S}_t[f(x,t)] = F(r,s,u),$$
$$A_x\mathbb{S}_t[h(x,t)] = H(r,s,u).$$

Then for any constants a and b, we have

$$A_x\mathbb{S}_t[af(x,t)+bh(x,t)] = aA_x\mathbb{S}_t[f(x,t)] + bA_x\mathbb{S}_t[h(x,t)].$$

Proof. By using the definition of double Aboodh-Shehu transform, we obtain

$$\begin{aligned} A_x\mathbb{S}_t[af(x,t)+bh(x,t)] &= \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}\big(af(x,t)+bh(x,t)\big)dxdt \\ &= \frac{a}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}f(x,t)dxdt \\ &+ \frac{b}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}h(x,t)dxdt \\ &= aA_x\mathbb{S}_t[f(x,t)] + bA_x\mathbb{S}_t[h(x,t)]. \end{aligned}$$

□

3.2. Shifting property. *If $A_x\mathbb{S}_t[f(x,t)] = F(r,s,u)$, then for any pair of real constants $a,b>0$*

$$A_x\mathbb{S}_t[e^{ax+bt}f(x,t)] = \frac{(r-a)}{r}F\left(r-a, s-ub, u\right). \tag{3.1}$$

Proof. Using the definition of double Aboodh-Shehu transform, we obtain

$$\begin{aligned} A_x\mathbb{S}_t[e^{ax+bt}f(x,t)] &= \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}e^{ax+bt}f(x,t)dxdt \\ &= \frac{1}{r}\int_0^\infty\int_0^\infty e^{-\left((r-a)x+(\frac{s-ub}{u})t\right)}f(x,t)dxdt \\ &= \frac{(r-a)}{r(r-a)}\int_0^\infty\int_0^\infty e^{-\left((r-a)x+(\frac{s-ub}{u})t\right)}f(x,t)dxdt \\ &= \frac{r-a}{r}F\left(r-a, s-ub, u\right). \end{aligned}$$

This ends the proof. □

3.3. Changing of scale property. Let $f(x,t)$ be a function such that

$$A_x\mathbb{S}_t[f(x,t)] = F(r,s,u).$$

Then for $\alpha,\beta>0$, we have

$$A_x\mathbb{S}_t[f(\alpha x,\beta t)] = \frac{1}{\alpha^2\beta}F(\frac{r}{\alpha},\frac{s}{\beta},u). \tag{3.2}$$

Proof. Using the definition of double Aboodh-Shehu transform, we deduce

$$A_x\mathbb{S}_t[f(\alpha x,\beta t)] = \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}f(\alpha x,\beta t)dxdt.$$

Let $\phi = \alpha x$ and $\varphi = \beta t$, then

$$\begin{aligned} A_x\mathbb{S}_t[f(\phi,\varphi)] =& \frac{1}{r\alpha\beta}\int_0^\infty\int_0^\infty e^{-(\frac{r\phi}{\alpha}+\frac{s\varphi}{u\beta})}f(\phi,\varphi)d\phi d\varphi \\ &= \frac{1}{\alpha^2\beta}\left\{\frac{\alpha}{r}\int_0^\infty\int_0^\infty e^{-(\frac{r\phi}{\alpha}+\frac{s\varphi}{u\beta})}f(\phi,\varphi)d\phi d\varphi\right\} \\ &= \frac{1}{\alpha^2\beta}F(\frac{r}{\alpha},\frac{s}{\beta},u). \end{aligned}$$

□

3.4. Derivatives properties. If $A_x\mathbb{S}_t[f(x,t)] = F(r,s,u)$, then

(1) $$A_x\mathbb{S}_t[\frac{\partial f(x,t)}{\partial x}] = rF(r,s,u) - \frac{1}{r}\mathbb{S}[f(0,t)]. \quad (3.3)$$

Proof.

$$\begin{aligned} A_x\mathbb{S}_t[\frac{\partial f(x,t)}{\partial x}] &= \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}\frac{\partial f(x,t)}{\partial x}dxdt \\ &= \frac{1}{r}\int_0^\infty e^{-\frac{s}{u}t}dt\left\{\int_0^\infty e^{-rx}f_x(x,t)dx\right\}. \end{aligned}$$

Using integration by parts, let $\mathsf{u} = e^{-rx}$, $d\mathsf{v} = f_x(x,t)dx = \frac{\partial f(x,t)}{\partial x}dx$, then we obtain

$$\begin{aligned} A_x\mathbb{S}_t[\frac{\partial f(x,t)}{\partial x}] &= \frac{1}{r}\int_0^\infty e^{-\frac{s}{u}t}dt\left\{-f(0,t) + r\int_0^\infty e^{-rx}f(x,t)dx\right\} \\ &= rF(r,s,u) - \frac{1}{r}\mathbb{S}[f(0,t)]. \end{aligned}$$

□

(2) $$A_x\mathbb{S}_t[\frac{\partial f(x,t)}{\partial t}] = \frac{s}{u}F(r,s,u) - A[f(x,0)]. \quad (3.4)$$

Proof.

$$\begin{aligned} A_x\mathbb{S}_t[\frac{\partial f(x,t)}{\partial t}] &= \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}\frac{\partial f(x,t)}{\partial t}dxdt \\ &= \frac{1}{r}\int_0^\infty e^{-rx}dx\left\{\int_0^\infty e^{-\frac{s}{u}t}f_t(x,t)dt\right\}. \end{aligned}$$

Using integration by parts, let $\mathsf{u} = e^{\frac{s}{u}t}$ and $d\mathsf{v} = f_t(x,t)dt = \frac{\partial f(x,t)}{\partial t}dt$, then we obtain

$$\begin{aligned} A_x\mathbb{S}_t[\frac{\partial f(x,t)}{\partial t}] &= \frac{1}{r}\int_0^\infty e^{-rx}dx\left\{-f(x,0) + \frac{s}{u}\int_0^\infty e^{-\frac{s}{u}t}f(x,t)dt\right\} \\ &= \frac{s}{u}F(r,s,u) - A[f(x,0)]. \end{aligned}$$

□

Similarly, we can prove that:

$$A_x\mathbb{S}_t[\frac{\partial^2 f(x,t)}{\partial x^2}] = r^2F(r,s,u) - \mathbb{S}[f(0,t)] - \frac{1}{r}\mathbb{S}[f_x(0,t)],$$
$$A_x\mathbb{S}_t[\frac{\partial^2 f(x,t)}{\partial t^2}] = \frac{s^2}{u^2}F(r,s,u) - \frac{s}{u}A[f(x,0)] - A[f_t(x,0)],$$
$$A_x\mathbb{S}_t[\frac{\partial^2 f(x,t)}{\partial x\partial t}] = \frac{sr}{u}F(r,s,u) - rA[f(x,0)] - \frac{1}{r}\mathbb{S}[f_t(0,t)].$$

4. The Double Aboodh-Shehu Transform of some Functions

(1). If $f(x,t)=1$, then

$$A_x\mathbb{S}_t[1] = \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}dxdt = \frac{u}{r^2 s}. \tag{4.1}$$

(2). If $f(x,t)=xt$, then

$$A_x\mathbb{S}_t[xt] = \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}xtdxdt = \frac{1}{r^3}\frac{u^2}{s^2}. \tag{4.2}$$

(3). If $f(x,t)=x^m t^k$, $m,k=0,\ 1,\ 2,\ \dots$, then

$$A_x\mathbb{S}_t[x^m t^k] = \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}x^m t^k dxdt = \frac{m!k!}{r^{m+2}}(\frac{u}{s})^{k+1}. \tag{4.3}$$

(4). If $f(x,t)=x^\alpha t^\beta$, $\alpha\ge -1$, $\beta\ge -1$, then we have

$$A_x\mathbb{S}_t[x^\alpha t^\beta] = \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}x^\alpha t^\beta dxdt = \frac{1}{r}\int_0^\infty e^{-rx}x^\alpha dx\int_0^\infty e^{-\frac{s}{u}t}t^\beta dt.$$

Let $\zeta = rx$ and $\eta=\frac{s}{u}t$, then we have

$$\begin{aligned} A_x\mathbb{S}_t[x^\alpha t^\beta] &= \frac{1}{r^{\alpha+2}}\int_0^\infty e^{-\zeta}\zeta^\alpha d\zeta\left\{(\frac{u}{s})^{\alpha+1}\int_0^\infty e^{-\eta}\eta^\beta d\eta\right\} \\ &= \frac{\Gamma(\alpha+1)}{r^{\alpha+2}}\Gamma(\beta+1)(\frac{u}{s})^{\beta+1}. \end{aligned} \tag{4.4}$$

Where $\Gamma(.)$ is the gamma function.
(5). If $f(x,t)=e^{(nx+mt)}$, $n,m=0,\ 1,\ 2,\ \dots$, then

$$A_x\mathbb{S}_t[e^{(nx+mt)}] = \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}e^{(nx+mt)}dxdt = \frac{u}{r(r-n)(s-mu)}. \tag{4.5}$$

Similarly,

$$\begin{aligned} A_x\mathbb{S}_t[e^{i(nx+mt)}] &= \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}e^{i(nx+mt)}dxdt = \frac{u}{r(r-in)(s-imu)} \\ &= \frac{u(rs-nmu)+iu(ns+mru)}{r(r^2+n^2)(s^2+m^2u^2)}. \end{aligned} \tag{4.6}$$

Consequently,

$$A_x\mathbb{S}_t[\cos(nx+mt)] = \frac{u(rs-nmu)}{r(r^2+n^2)(s^2+m^2u^2)}, \tag{4.7}$$

$$A_x\mathbb{S}_t[\sin(nx+mt)] = \frac{u(rmu+ns)}{r(r^2+n^2)(s^2+m^2u^2)}. \tag{4.8}$$

(6). If $f(x,t)=\sinh(nx+mt)$ or $\cosh(nx+mt)$, $n,m=0,1,2,\ \dots$.
Recall that

$$\sinh(nx+mt) = \frac{e^{(nx+mt)}-e^{-(nx+mt)}}{2},\ \cosh(nx+mt) = \frac{e^{(nx+mt)}+e^{-(nx+mt)}}{2}.$$

Therefore,

$$A_x\mathbb{S}_t[\cosh(nx+mt)] = \frac{u(rs+nmu)}{r(r^2-n^2)(s^2-m^2u^2)}, \tag{4.9}$$

$$A_x\mathbb{S}_t[\sinh(nx+mt)] = \frac{u(rmu+ns)}{r(r^2-n^2)(s^2-m^2u^2)}. \tag{4.10}$$

(7). If $f(x,t)=f_1(x)f_2(t)$, then

$$\begin{aligned} A_x\mathbb{S}_t[f_1(x)f_2(t)] &= \frac{1}{r}\int_0^\infty\int_0^\infty e^{-(rx+\frac{st}{u})}f_1(x)f_2(t)dxdt \\ &= \frac{1}{r}\int_0^\infty e^{-rx}f_1(x)dx\int_0^\infty e^{-\frac{s}{u}t}f_2(t)dt \\ &= A_x[f_1(x)]\mathbb{S}_t[f_2(t)]. \end{aligned} \tag{4.11}$$

Therefore,

$$A_x\mathbb{S}_t[\sin(ax)\sin(bt)] = \frac{a}{r(r^2+a^2)}\frac{bu^2}{(s^2+b^2u^2)}, \tag{4.12}$$

$$A_x\mathbb{S}_t[\cos(ax)\cos(bt)] = \frac{1}{(r^2+a^2)}\frac{us}{(s^2+b^2u^2)}. \tag{4.13}$$

5. Application of double Aboodh-Shehu transform method to linear partial differential equations

Let the second-order nonhomogeneous linear partial differential equation in two independent variables (x,t) be in the form:

$$aU_{xx}(x,t)+bU_{tt}(x,t)+cU_x(x,t)+dU_t(x,t)+eU(x,t)=g(x,t), \quad (x,t)\in\mathbb{R}_+^2 \tag{5.1}$$

with the initial conditions:

$$U(x,0)=\mathcal{G}_1(x), \qquad U_t(x,0)=\mathcal{G}_2(x), \tag{5.2}$$

and the boundary conditions:

$$U(0,t)=\mathcal{G}_3(t), \qquad U_x(0,t)=\mathcal{G}_4(t), \tag{5.3}$$

where $a,\ b,\ c,\ d$ and e are constants and $g(x,t)$ is the source term. In (5.1), the dependent variable $U=U(x,t)$ depends on the position x and on the time variable t. Using the property of partial derivative of the double Aboodh-Shehu transform for equation (5.1), single Aboodh transform for equation (5.2) and single Shehu transform for equation (5.3) and simplifying, we obtain that:

$$F(r,s,u) = \left(\frac{(\frac{bs}{u}+d)\mathcal{G}_1(r) + b\mathcal{G}_2(r) + (a+\frac{c}{r})\mathcal{G}_3(s,u) + \frac{a}{r}\mathcal{G}_4(s,u) + H(r,s,u)}{ar^2 + \frac{bs^2}{u^2} + cr + \frac{ds}{u} + e}\right), \tag{5.4}$$

where $H(r,s,u) = A_x\mathbb{S}_t[g(x,t)]$.

Finally, solving this algebraic equation in $F(r,s,u)$ and taking the inverse double Aboodh-Shehu transform on both sides of equation (5.4), yields:

$$U(x,t) = A_x^{-1}\mathbb{S}_t^{-1}\left[\frac{(\frac{bs}{u}+d)\mathcal{G}_1(r) + b\mathcal{G}_2(r) + (a+\frac{c}{r})\mathcal{G}_3(s,u) + \frac{a}{r}\mathcal{G}_4(s,u) + H(r,s,u)}{ar^2 + \frac{bs^2}{u^2} + cr + \frac{ds}{u} + e}\right]. \tag{5.5}$$

Which is the general format for the solution of equation (5.1) by the double Aboodh-Shehu transform method.

Example 5.1. *Consider the following homogeneous heat equation*

$$U_t(x,t) = U_{xx}(x,t) - U(x,t), \qquad x \in \mathbb{R}_+,\ t > 0, \tag{5.6}$$

subject to the initial and boundary conditions

$$U(x,0) = \sin(x) = \mathcal{G}_1(x),\ U(0,t) = 0 = \mathcal{G}_3(t),\ U_x(0,t) = e^{-2t} = \mathcal{G}_4(t).$$

Substituting

$$\mathcal{G}_1(r) = \frac{1}{r(r^2+1)},\ \mathcal{G}_3(s,u) = 0,\ \mathcal{G}_4(s,u) = \frac{u}{s+2u},$$

in (5.5) and simplifying, we get a solution of (5.6)

$$U(x,t) = A_x^{-1}\mathbb{S}_t^{-1}\left[\frac{u}{r(r^2+1)(s+2u)}\right] = e^{-2t}\sin(x). \tag{5.7}$$

Example 5.2. *Consider the following nonhomogeneous heat equation*

$$U_t(x,t) - U_{xx}(x,t) = -6x, \qquad x \in \mathbb{R}_+,\ t > 0, \tag{5.8}$$

with the initial and boundary conditions:

$$U(x,0) = x^3 + \sin(x) = \mathcal{G}_1(x), \qquad U_t(x,0) = -\sin(x) = \mathcal{G}_2(x),$$
$$U(0,t) = 0 = \mathcal{G}_3(t), \qquad U_x(0,t) = e^{-t} = \mathcal{G}_4(t).$$

Applying the double Aboodh-Shehu transform on both sides of Eq. (5.8) and re-arranging the terms, we get

$$F(r,s,u) = \frac{u}{(s-r^2u)}\left\{A[U(x,0)] - \mathbb{S}[U(0,t)] - \frac{1}{r}\mathbb{S}[U_x(0,t)] - \frac{6}{r^3}(\frac{u}{s})\right\}, \tag{5.9}$$

where

$$A_x\mathbb{S}_t[-6x] = -\frac{6}{r^3}(\frac{u}{s}).$$

Substituting

$$\mathcal{G}_1(r) = \frac{6}{r^5} + \frac{1}{r(r^2+1)},\ \mathcal{G}_2(r) = \frac{-1}{r(r^2+1)},\ \mathcal{G}_3(s,u) = 0,\ \mathcal{G}_4(s,u) = \frac{u}{s+u},$$

in (5.9) and simplify to obtain

$$F(r,s,u) = \frac{u}{(s-r^2u)}\left\{\frac{6(r^2+1)(s+u)(s-r^2u)+r^4s(s-r^2u)}{r^5s(r^2+1)(s+u)}\right\}, = \frac{6}{r^5}(\frac{u}{s}) + \frac{u}{r(r^2+1)(s+u)}. \quad (5.10)$$

Taking the inverse double Aboodh-Shehu transform of equation (5.10), we get a solution of (5.8)

$$U(x,t) = A_x^{-1}\mathbb{S}_t^{-1}\left[\frac{6}{r^5}(\frac{u}{s}) + \frac{u}{r(r^2+1)(s+u)}\right] = x^3 + e^{-t}\sin(x).$$

Example 5.3. *Consider the following nonhomogeneous wave equation*

$$U_{tt}(x,t) = U_{xx}(x,t) - 3U(x,t) + 3, \qquad x \in \mathbb{R}_+, \ t > 0, \quad (5.11)$$

with the initial and boundary conditions:

$$U(x,0) = 1 = \mathcal{G}_1(x), \qquad U_t(x,0) = 2\sin(x) = \mathcal{G}_2(x),$$
$$U(0,t) = 1 = \mathcal{G}_3(t), \qquad U_x(0,t) = \sin(2t) = \mathcal{G}_4(t).$$

Substituting

$$\mathcal{G}_1(r) = \frac{1}{r^2}, \ \mathcal{G}_2(r) = \frac{2}{r(r^2+1)}, \ \mathcal{G}_3(s,u) = \frac{u}{s}, \ \mathcal{G}_4(s,u) = \frac{2u^2}{s^2+4u^2}, \ H(r,s,u) = \frac{3u}{r^2s},$$

in (5.5) and simplifying, we get a solution of (5.11)

$$U(x,t) = A_x^{-1}\mathbb{S}_t^{-1}\left[\frac{u}{r^2s} + \frac{1}{r(r^2+1)}\frac{2u^2}{(s^2+4u^2)}\right] = 1 + \sin(x)\sin(2t).$$

Example 5.4. *Consider the following partial differential Telegraph equation*

$$U_{xx}(x,t) - U_{tt}(x,t) - U_t(x,t) - U = e^{2x+t}, \qquad x \in \mathbb{R}_+, \ t > 0, \quad (5.12)$$

with respect to the initial and boundary conditions:

$$U(x,0) = e^{2x} = \mathcal{G}_1(x), \qquad U_t(x,0) = e^{2x} = \mathcal{G}_2(x),$$
$$U(0,t) = e^{t} = \mathcal{G}_3(t), \qquad U_x(0,t) = 2e^{t} = \mathcal{G}_4(t).$$

Substituting

$$\mathcal{G}_1(r) = \frac{1}{r(r-2)}, \mathcal{G}_2(r) = \frac{1}{r(r-2)}, \mathcal{G}_3(s,u) = \frac{u}{s-u}, \mathcal{G}_4(s,u) = \frac{2u}{s-u}, H(r,s,u) = \frac{u}{r(r-2)(s-u)},$$

in (5.4) and simplifying, we get

$$F(r,s,u) = \frac{\frac{r^2u^2-s^2-su-u^2}{ru(r-2)(s-u)}}{\frac{r^2u^2-s^2-su-u^2}{u^2}} = \frac{u}{r(r-2)(s-u)}.$$

Taking the inverse of double Aboodh-Shehu transform, we get

$$U(x,t) = A_x^{-1}\mathbb{S}_t^{-1}\left[\frac{u}{r(r-2)(s-u)}\right] = e^{2x+t}.$$

Which is the required solution of the considered Telegraph equation.

Example 5.5. *Consider the following boundary Laplace Equation*

$$U_{xx}(x,t)+U_{tt}(x,t)=0, \qquad (x,t)\in\mathbb{R}_+^2, \tag{5.13}$$

with the conditions:

$$U(x,0)=\cos(x)=\mathcal{G}_1(x), \qquad U_t(x,0)=0=\mathcal{G}_2(x),$$
$$U(0,t)=\cosh(t)=\mathcal{G}_3(t), \qquad U_x(0,t)=0=\mathcal{G}_4(t).$$

Substituting

$$\mathcal{G}_1(r)=\frac{1}{(r^2+1)},\ \ \mathcal{G}_2(r)=0,\ \ \mathcal{G}_3(s,u)=\frac{su}{(s^2-u^2)},\ \ \mathcal{G}_4(s,u)=0,$$

in (5.5) and simplifying, we get a solution of (5.13)

$$U(x,t)=A_x^{-1}\mathbb{S}_t^{-1}\left[\frac{1}{(r^2+1)}\frac{su}{(s^2-u^2)}\right]=\cos(x)\cosh(t).$$

Example 5.6. *Consider the following boundary Poisson equation*

$$U_{xx}(x,t)+U_{tt}(x,t)=t\sin(x), \qquad (x,t)\in\mathbb{R}_+^2, \tag{5.14}$$

with the conditions:

$$U(x,0)=0=\mathcal{G}_1(x), \qquad U_t(x,0)=-\sin(x)=\mathcal{G}_2(x),$$
$$U(0,t)=0=\mathcal{G}_3(t), \qquad U_x(0,t)=-t=\mathcal{G}_4(t).$$

Substituting

$$\mathcal{G}_1(r)=0,\ \ \mathcal{G}_2(r)=\frac{-1}{r(r^2+1)},\ \ \mathcal{G}_3(s,u)=0,\ \ \mathcal{G}_4(s,u)=\frac{-u^2}{s^2},\ \ H(r,s,u)=\frac{1}{r(r^2+1)}\frac{u^2}{s^2},$$

in (5.5) and simplifying, we get a solution of (5.14)

$$U(x,t)=A_x^{-1}\mathbb{S}_t^{-1}\left[\frac{-u^2}{s^2r(r^2+1)}\right]=-A_x^{-1}\mathbb{S}_t^{1}\left[\frac{1}{r(r^2+1)}\frac{u^2}{s^2}\right]=-t\sin(x).$$

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