NEW INTEGRAL TRANSFORM: OUIDEEN TRANSFORM FOR SOLVING BOTH ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

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DOI: https://doi.org/10.56807/buj.v4i03.330

Abstract
In this paper, we introduce a new-type integral transform called the Ouideen transform which is a generalization of the Aboodh and the Sumudu integral transforms for solving linear ordinary differential equations with constant coefficients and linear partial differential equations with two variables. The proposed integral transform is successfully derived from the classical Fourier integral transform and is applied to both ordinary and partial differential equations to show its simplicity, efficiency, and high accuracy.

1. Introduction

The differential equation plays a central role in Applied Mathematics, Physics and in Engineering. Linear and nonlinear partial differential equations are generally difficult to be solved and their exact solutions are difficult to be obtained. The exact solution and numerical solutions of this kind of equations play an important role in physical science and in engineering fields. Therefore, there have been attempts to develop new techniques for obtaining analytical solutions which reasonably approximate the exact solutions [7]. One of most popular and rather method for solving both ordinary and partial differential equations is the integral transform method. The origin of the integral transforms can be traced back to the work of P.S. Laplace in 1780s [2] and Joseph Fourier in 1822. The Laplace transform and Fourier integral transforms are the most commonly used in the literature. In recent years, differential and integral equations have been solved using many integral transforms such as Aboodh transform [1], Sumudu transform [10], Shehu transform [8], Elzaki transform [9], Yang transform [7], and so on. In 2016, Atangana and Alkaltani introduced a new double integral equation and their properties based on the Laplace transform and decomposition method. The double integral transform was successfully applied to second order partial differential equation with singularity called the two dimensional Mboctara equation [3]. Recently, Eltayeb applied double Laplace decomposition method to nonlinear partial differential equations [5]. In 2017, Belgacem el at. extended the applications of the natural and the Sumudu transforms to fractional diffusion and Stokes fluid flow realms [4]. In this paper, we proposed a new integral transform called Ouideen transform for solving both ordinary and partial differential equations.

The new integral transform converges to Aboodh transform when $q = 1$, and

***Key** words and phrases. New integral transform, Ouideen transform, ordinary differential equations, Partial differential equations.*

to Sumudu transform when $p = 1$. The proposed integral transform is successfully applied to both ordinary and partial differential equations. Throughout this paper, the Ouideen transform is denoted by an operator $\mathbb{O}[.]$.

2. Main result

Definition 2.1. The Ouideen transform of the function $f(t)$ is defined over the set of functions

$$Y = \{f(t) : \exists M, \rho_1, \rho_2, \tau_1, \tau_2 > 0, | f(t) |< Me^{\frac{\tau_i |t|}{\rho_i}}, x \in (-1)^i \times [0, \infty), i = 1, 2\}, \tag{2.1}$$

by the following integral

$$\mathbb{O}[f(t)] = T(p, q) = \frac{1}{pq} \int_0^\infty e^{-\frac{pt}{q}} f(t)dt \tag{2.2}$$

$$= \frac{1}{pq} \lim_{\beta \to \infty} \int_0^\beta e^{-\frac{pt}{q}} f(t)dt, \qquad \rho_1 < \rho_2, \ \tau_1 < \tau_2 \tag{2.3}$$

It converges if the limit of the integral exists, and diverges if not.
or

$$\mathbb{O}[f(t)] = T(p, q) = \frac{1}{p} \int_0^\infty e^{-pt} f(qt)dt \tag{2.4}$$

where $\rho_1 < p < \rho_2, \ \tau_1 < q < \tau_2, \ t \geq 0$ and M is constant.
The inverse Ouideen transform is given by

$$\mathbb{O}^{-1}[T(p, q)] = f(t), \ \ for \ t \geq 0. \tag{2.5}$$

Equivalently

$$f(t) = \mathbb{O}^{-1}[T(p, q)] = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} pe^{\frac{pt}{q}} T(p, q)dp, \tag{2.6}$$

where p and q are the Ouideen transform variables, and β is areal constant.

Theorem 2.2. ***The sufficient condition for the existence of Ouideen transform.*** *If the function $f(t)$ is piecewise continues in every finite interval $0 \leq t \leq \alpha$ and of exponential order β for $t > \alpha$. Then its Ouideen transform $T(p, q)$ exists.*

Proof. For any positive number α, we algebraically deduce

$$\frac{1}{pq} \int_0^\infty e^{-\frac{pt}{q}} f(t)dt = \frac{1}{pq} \int_0^\alpha e^{-\frac{pt}{q}} f(t)dt + \frac{1}{pq} \int_\alpha^\infty e^{-\frac{pt}{q}} f(t)dt \tag{2.7}$$

Since the function $f(t)$ is piecewise continues in every finite interval $0 \leq t \leq \alpha$. Then the first integral on the right hand side exists. Besides, the second integral on the right hand side exists, since the function $f(t)$ is of exponential order β for

$t > \alpha$. To verify this claim, we consider the following case

$$\begin{aligned}
\mid \frac{1}{pq}\int_{\alpha}^{\infty} e^{-\frac{pt}{q}} f(t)dt \mid &\leq \frac{1}{pq}\int_{0}^{\infty} \mid e^{-\frac{pt}{q}} f(t) \mid dt \\
&\leq \frac{1}{pq}\int_{0}^{\infty} e^{-\frac{pt}{q}} \mid f(t) \mid dt \\
&\leq \frac{1}{pq}\int_{\alpha}^{\infty} e^{-\frac{pt}{q}} N e^{\beta t} dt \\
&= \frac{N}{pq}\int_{\alpha}^{\infty} e^{-\frac{(p-\beta q)t}{q}} dt \\
&= -\frac{N}{p(p-\beta q)} \lim_{\delta\to\infty}\left[e^{-\frac{(p-\beta q)t}{q}}\right]_0^{\delta} \\
&= \frac{N}{p(p-\beta q)}
\end{aligned}$$

The proof is complete. □

Theorem 2.3. ***Derivative of Ouideen transform****. If the function $f^{(n)}(t)$ is the nth derivative of the function $f(t) \in Y$ with respect to 't', then its Oueedin transform is defined by*

$$\mathbb{O}\left[f^{(n)}(t)\right] = \frac{p^n}{q^n}T(p,q) - \sum_{k=0}^{n-1}(\frac{1}{p})^{2-n+k}(\frac{1}{q})^{n-k} f^k(0). \tag{2.8}$$

When $n = 1,\ 2,$ and 3 in Equ. (2.8) above, we obtain the following derivatives with respect to t.

$$\mathbb{O}[f'(t)] = \frac{p}{q}T(p,q) - \frac{1}{pq}f(0) \tag{2.9}$$

$$\mathbb{O}\left[f''(t)\right] = \frac{p^2}{q^2}T(p,q) - \frac{1}{q^2}f(0) - \frac{1}{pq}f'(0) \tag{2.10}$$

$$\mathbb{O}\left[f'''(t)\right] = \frac{p^3}{q^3}T(p,q) - \frac{1}{p^{-1}q^3}f(0) - \frac{1}{q^2}f'(0) - \frac{1}{pq}f''(0) \tag{2.11}$$

Proof. Now suppose *Equ.* (2.8) is true for $n = k$. Then using *Equ.* (2.9) and the induction hypothesis, we deduce

$$\begin{aligned}
\mathbb{O}\left[(f^{(k)}(t))'\right] &= \frac{p}{q}\left[f^{(k)}(t)\right] - \frac{1}{pq}f^k(0) \\
&= \frac{p}{q}\left[\frac{p^k}{q^k}T(p,q) - \sum_{j=0}^{k-1}(\frac{1}{p})^{2-k+j}(\frac{1}{q})^{k-j}f^j(0)\right] - f^k(0) \\
&= \left(\frac{p}{q}\right)^{k+1}\mathbb{O}[f(t)] - \sum_{j=0}^{k}(\frac{1}{p})^{1-k+j}(\frac{1}{q})^{k+1-j}f^j(0),
\end{aligned}$$

which implies that *Equ.* (2.9) holds for $n = k + 1$. By induction hypothesis the proof is complete. □

The following important properties are obtained using the Leibniz's rule

$$\mathbb{O}\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}}\frac{\partial f(x,t)}{\partial x}dt = \frac{\partial}{\partial x}\left[\frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}}f(x,t)dt\right]$$

$$= \frac{\partial}{\partial x}[T(x,p,q)] \Rightarrow \mathbb{O}\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{d}{dx}[T(x,p,q)],$$

$$\mathbb{O}\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}}\frac{\partial^2 f(x,t)}{\partial x^2}dt = \frac{\partial^2}{\partial x^2}\left[\frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}}f(x,t)dt\right]$$

$$= \frac{\partial^2}{\partial x^2}[T(x,p,q)] \Rightarrow \mathbb{O}\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2}[T(x,p,q)],$$

and

$$\mathbb{O}\left[\frac{\partial^n f(x,t)}{\partial x^n}\right] = \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}}\frac{\partial^n f(x,t)}{\partial x^n}dt = \frac{\partial^n}{\partial x^n}\left[\frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}}f(x,t)dt\right]$$

$$= \frac{\partial^n}{\partial x^n}[T(x,p,q)] \Rightarrow \mathbb{O}\left[\frac{\partial^n f(x,t)}{\partial x^n}\right] = \frac{d^n}{dx^n}[T(x,p,q)].$$

3. Some useful results of Ouideen transform

3.1. Property. First translation or shifting property of Ouideen transform. *Let $e^{at}f(t) \in Y$, where a is constant. Then*

$$\mathbb{O}[e^{at}f(t)] = \frac{p}{p-aq}T\left(p, \frac{pq}{p-aq}\right) \tag{3.1}$$

Proof. By Definition 2.1 of Ouideen transform, we have

$$\mathbb{O}[e^{at}f(t)] = \frac{1}{p}\int_0^\infty e^{-(p-aq)}f(qt)dt. \tag{3.2}$$

Let $\beta p = (p-aq)t$ which implies $t = \frac{p\beta}{p-aq}$ and $dt = \frac{p}{p-aq}d\beta$ in *Equ.*3.2, we deduce

$$\begin{aligned}\mathbb{O}[e^{at}f(t)] =& \frac{1}{p} \times \frac{p}{p-aq}\int_0^\infty e^{-p\beta}f\left(\frac{pq\beta}{p-aq}\right)d\beta \\ =& \frac{1}{p-aq}\int_0^\infty e^{-pt}f\left(\frac{pqt}{p-aq}\right)dt \\ =& \frac{p}{p-aq}T\left(p, \frac{pq}{p-aq}\right).\end{aligned}$$

This ends the proof. □

Based on variables transformation in property 3.1, we deduce

$$\mathbb{O}[e^{at}f(t)] = \begin{cases} \frac{1}{1-aq}T\left(1, \frac{q}{1-aq}\right), & p = 1 \ (Sumudu\ transform), \\ \frac{p}{p-a}T\left(p, \frac{p}{p-a}\right), & q = 1 \ (Aboodh\ transform). \end{cases} \tag{3.3}$$

3.2. **Property.** ***Linearity property of Ouideen transform.*** *Let the functions $af_1(t)$ and $bf_2(t)$ be in a set Y, then $(af_1(t)+bf_2(t)) \in Y$, where a and b are nonzero arbitrary constants, and*

$$\mathbb{O}[af_1(t)+bf_2(t)] = a\mathbb{O}[f_1(t)] + b\mathbb{O}[f_2(t)] \tag{3.4}$$

Proof. Using the Definition 2.1 of Ouideen transform, we get

$$\begin{aligned}
\mathbb{O}[af_1(t)+bf_2(t)] &= \frac{1}{pq}\int_0^{\infty} e^{-\frac{pt}{q}}(af_1(t)+bf_2(t))dt \\
&= \frac{1}{pq}\int_0^{\infty} e^{-\frac{pt}{q}}(af_1(t))dt + \frac{1}{pq}\int_0^{\infty} e^{-\frac{pt}{q}}(bf_2(t))dt \\
&= \frac{a}{pq}\int_0^{\infty} e^{-\frac{pt}{q}}f_1(t)dt + \frac{b}{pq}\int_0^{\infty} e^{-\frac{pt}{q}}f_2(t)dt \\
&= a[\frac{1}{pq}\int_0^{\infty} e^{-\frac{pt}{q}}f_1(t)dt] + b[\frac{1}{pq}\int_0^{\infty} e^{-\frac{pt}{q}}f_2(t)dt] \\
&= a\mathbb{O}[f_1(t)] + b\mathbb{O}[f_2(t)].
\end{aligned}$$

The proof is complete. □

Generalization: If $f_1(x), f_2(x), ..., f_n(x)$ have Ouideen transform, then

$$\mathbb{O}[b_1f_1(x)+b_2f_2(x)+...+b_nf_n(x)] = b_1\mathbb{O}[f_1(x)] + b_2\mathbb{O}[f_2(x)] + ... + b_n\mathbb{O}[f_n(x)], \tag{3.5}$$

where $b_1, b_2, ..., b_n$ are constants and $f_1(x), f_2(x), ..., f_n(x)$ are defined function.

3.3. **Property.** ***Change of scale property of Ouideen transform.*** *Let the function $f(bt)$ be in a set Y, where b is an arbitrary constant. Then*

$$\mathbb{O}[f(bt)] = \frac{1}{b^2}T\big(\frac{p}{b}, q\big). \tag{3.6}$$

Proof. Using the Definition 2.1 of Ouideen transform, we deduce

$$\mathbb{O}[f(bt)] = \frac{1}{pq}\int_0^{\infty} e^{-\frac{pt}{q}}f(bt)dt \tag{3.7}$$

Substituting $\varphi = bt$ which implies $t = \frac{\varphi}{b}$ and $dt = \frac{d\varphi}{b}$ in $Equ.$(3.7) yields

$$\begin{aligned}
\mathbb{O}[f(bt)] &= \frac{1}{pqb}\int_0^{\infty} e^{-\frac{p\varphi}{qb}}f(\varphi)d\varphi \\
&= \frac{1}{pqb}\int_0^{\infty} e^{-\frac{pt}{qb}}f(t)dt \\
&= \frac{1}{pb}\int_0^{\infty} e^{-\frac{pt}{b}}f(qt)dt \\
&= \frac{1}{b^2}[\frac{b}{p}\int_0^{\infty} e^{-\frac{pt}{b}}f(qt)dt] \\
&= \frac{1}{b^2}T\big(\frac{p}{b}, q\big).
\end{aligned}$$

The proof is complete. □

3.4. **Property.** *Let the function* $f(t) = 1$ *be in a set* Y. *Then its Ouideen transform is given by*

$$\mathbb{O}[1] = \frac{1}{p^2}. \tag{3.8}$$

Proof. Using $Equ.$(2.2), we deduce

$$\mathbb{O}[1] = \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}}dt = \frac{1}{pq}\Big(-\frac{q}{p}\lim_{\delta\to\infty}\big[e^{-\frac{pt}{q}}\big]_0^\delta\Big) = \frac{1}{p^2}.$$

This ends the proof. □

3.5. **Property.** *Let the function* $f(t) = t$ *be in a set* Y. *Then its Ouideen transform is given by*

$$\mathbb{O}[t] = \frac{q}{p^3}. \tag{3.9}$$

Proof. Using the Definition 2.1 of the Ouideen transform and integration by parts, we get

$$\mathbb{O}[t] = \frac{1}{pq}\int_0^\infty te^{-\frac{pt}{q}}dt = \frac{1}{pq}\left(\frac{q}{p}\lim_{\delta\to\infty}\left[te^{-\frac{pt}{q}}\right]_0^\delta + \frac{q}{p}\int_0^\infty e^{-\frac{pt}{q}}dt\right)$$
$$= -\frac{q}{p^3}\lim_{\delta\to\infty}\left[e^{-\frac{pt}{q}}\right]_0^\delta = \frac{q}{p^3}.$$

Thus the proof ends. □

3.6. **Property.** *Let the function* $f(t) = t^n$ $n = 0, 1, 2...$ *be in a set* Y. *Then its Ouideen transform is given by*

$$\mathbb{O}[t^n] = n!\frac{q^n}{p^{n+2}}. \tag{3.10}$$

—

$\frac{t^n}{n!}$ ——

we deduce

$$\begin{aligned}
\mathbb{O}[t^n] &= \frac{1}{pq}\int_0^\infty t^n e^{-\frac{pt}{q}}dt = \frac{1}{p^2}n\int_0^\infty t^{n-1}e^{-\frac{pt}{q}}dt \\
&= \frac{q}{p^3}n(n-1)\int_0^\infty t^{n-2}e^{-\frac{pt}{q}}dt \\
&= \frac{q^2}{p^4}n(n-1)(n-2)\int_0^\infty t^{n-3}e^{-\frac{pt}{q}}dt \\
&= \frac{q^3}{p^5}n(n-1)(n-2)(n-3)\int_0^\infty t^{n-4}e^{-\frac{pt}{q}}dt \\
&= \frac{q^4}{p^6}n(n-1)(n-2)(n-3)(n-4)\int_0^\infty t^{n-5}e^{-\frac{pt}{q}}dt = ... = n!\frac{q^n}{p^{n+2}}.
\end{aligned}$$

The proof is completed. □

3.7. **Property.** *Let the function* $f(t) = \frac{t^n}{\Gamma(n+1)}$ $n = 0, 1, 2, ...$ *be in the a set* Y. *Then its Ouideen transform is given by*

$$\mathbb{O}\left[\frac{t^n}{\Gamma(n+1)}\right] = \frac{q^n}{p^{n+2}}. \tag{3.12}$$

Proof. The proof of property 6 follows immediately from the previous property 3.5. □

3.8. **Property.** *Let the function* $f(t) = e^{at}$ *be in* Y. *Then its Ouideen transform is given by*

$$\mathbb{O}[e^{at}] = \frac{1}{p(p-aq)}. \tag{3.13}$$

Proof. Using $Equ.$(2.2) we get

$$\begin{aligned}\mathbb{O}[e^{at}] &= \frac{1}{pq}\int_0^\infty e^{-\frac{(p-aq)t}{q}}dt \\ &= -\frac{1}{p(p-aq)}\lim_{\delta\to\infty}\left[e^{-\frac{(p-aq)t}{q}}\right]_0^\delta = \frac{1}{p(p-aq)}.\end{aligned}$$

The proof is complete. □

3.9. **Property.** *Let the function* $f(t) = te^{at}$ *be in a set* Y. *Then its Ouideen transform is given by*

$$\mathbb{O}[te^{at}] = \frac{q}{p(p-aq)^2}. \tag{3.14}$$

Proof. Using the Definition 2.1 of the Ouideen transform and integration by parts, we get

$$\begin{aligned}\mathbb{O}[te^{at}] &= \frac{1}{pq}\int_0^\infty te^{-\frac{(p-aq)t}{q}}dt \\ &= -\frac{1}{p(p-aq)}\lim_{\delta\to\infty}\left[te^{-\frac{(p-aq)t}{q}}\right]_0^\delta \\ &+ \frac{1}{p(p-aq)}\int_0^\infty e^{-\frac{(p-aq)t}{q}}dt \\ &= -\frac{q}{p(p-aq)^2}\lim_{\delta\to\infty}\left[e^{-\frac{(p-aq)t}{q}}\right]_0^\delta = \frac{q}{p(p-aq)^2}.\end{aligned}$$

The proof is complete. □

3.10. **Property.** *Let the function* $f(t) = t^n e^{at}$ $n = 0, 1, 2, ...$ *be in a set* Y. *Then its Ouideen transform is given by*

$$\mathbb{O}[t^n e^{at}] = \frac{n!q^n}{p(p-aq)^{n+1}}. \tag{3.15}$$

Equivalently, if the function $f(t) = \frac{t^n e^{at}}{n!}$. *Then its Ouideen transform is given by*

$$\mathbb{O}\left[\frac{t^n e^{at}}{n!}\right] = \frac{q^n}{p(p-aq)^{n+1}}. \tag{3.16}$$

Proof. Using the Definition 2.1 of the Ouideen transform and integration by parts, we deduse

$$\begin{aligned}
\mathbb{O}[t^n e^{at}] &= \frac{1}{pq}\int_0^\infty t^n e^{-\frac{(p-aq)t}{q}} dt \\
&= \frac{n}{p(p-aq)}\int_0^\infty t^{n-1} e^{-\frac{(p-aq)t}{q}} dt \\
&= \frac{qn(n-1)}{p(p-aq)^2}\int_0^\infty t^{n-2} e^{-\frac{(p-aq)t}{q}} dt = ... = \frac{n! q^n}{p(p-aq)^{n+1}}.
\end{aligned}$$

Thus the proof is complete. □

3.11. **Property.** *Let the function* $f(t) = \frac{t^n}{\Gamma(n+1)} e^{at}$ $n = 0, 1, 2, ...$ *be in a set* Y. *Then its Ouideen transform is given by*

$$\mathbb{O}\left[\frac{t^n e^{at}}{\Gamma(n+1)}\right] = \frac{q^n}{p(p-aq)^{n+1}}. \tag{3.17}$$

Proof. The proof of Property 3.10 follows as a direct consequence of Property 3.9. □

3.12. **Property.** *Let the function* $f(t) = \sin(at)$ *be in a set* Y. *Then its Ouideen transform is given by*

$$\mathbb{O}[\sin(at)] = \frac{aq}{p(p^2 + a^2q^2)}. \tag{3.18}$$

Proof. Using the Definition 2.1 of the Ouideen transform and integration by parts, we get

$$\begin{aligned}
\mathbb{O}[\sin(at)] &= \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}} \sin(at)dt \\
&= \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}} \left(\frac{e^{iat} - e^{-iat}}{2i}\right) dt \\
&= \frac{1}{2ipq}(\int_0^\infty e^{(ia-\frac{p}{q})t} - e^{(-ia-\frac{p}{q})t})dt = \frac{1}{2ipq}(\int_0^\infty e^{(ia-\frac{p}{q})t} - e^{-(\frac{p}{q}+ia)t})dt \\
&= \frac{1}{2ipq}\left[\frac{1}{\frac{p}{q} - ia} - \frac{1}{\frac{p}{q} + ia}\right] = \frac{1}{2ipq}\left[\frac{1}{\frac{p-iaq}{q}} - \frac{1}{\frac{p+iaq}{q}}\right] \\
&= \frac{1}{2ipq}\left[\frac{q}{p - iaq} - \frac{q}{p + iaq}\right] = \frac{1}{2ipq}\left[\frac{q(p + iaq - p + iaq)}{p^2 + a^2q^2}\right] \\
&= \frac{aq}{p(p^2 + a^2q^2)}.
\end{aligned}$$

Thus the proof is complete. □

3.13. **Property.** *Let the function* $f(t) = \cos(at)$ *be in a set* Y*. Then its Ouideen transform is given by*

$$\mathbb{O}[\cos(at)] = \frac{1}{p^2 + a^2q^2}. \tag{3.19}$$

Proof. Using the Definition 2.1 of the Ouideen transform and integration by parts, we deduce

$$\begin{aligned}
\mathbb{O}[\sin(at)] &= \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}} \sin(at)dt \\
&= \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}} \left(\frac{e^{iat} + e^{-iat}}{2}\right) dt \\
&= \frac{1}{2pq}(\int_0^\infty e^{(ia-\frac{p}{q})t} + e^{(-ia-\frac{p}{q})t})dt = \frac{1}{2pq}(\int_0^\infty e^{(ia-\frac{p}{q})t} + e^{-(\frac{p}{q}+ia)t})dt \\
&= \frac{1}{2pq}\left[\frac{1}{\frac{p}{q} - ia} + \frac{1}{\frac{p}{q} + ia}\right] = \frac{1}{2pq}\left[\frac{1}{\frac{p-iaq}{q}} + \frac{1}{\frac{p+iaq}{q}}\right] \\
&= \frac{1}{2pq}\left[\frac{q}{p - iaq} + \frac{q}{p + iaq}\right] = \frac{1}{2pq}\left[\frac{q(p + iaq + p - iaq)}{p^2 + a^2q^2}\right] \\
&= \frac{1}{p^2 + a^2q^2}.
\end{aligned}$$

Thus the proof is complete. $\square$

3.14. **Property.** *Let the function* $f(t) = \sinh(at)$ *be in a set* Y*. Then its Ouideen transform is given by*

$$\mathbb{O}[\sinh(at)] = \frac{aq}{p(p^2 + a^2q^2)}. \tag{3.20}$$

Equivalently, if the function $f(t) = \left[\frac{\sinh(at)}{a}\right]$ *be in a set* Y*. Then its Ouideen transform is given by*

$$\mathbb{O}\left[\frac{\sinh(at)}{a}\right] = \frac{q}{p(p^2 + a^2q^2)}. \tag{3.21}$$

Proof. From the Definition 2.1 of the Ouideen transform and integration by parts, we get

$$\begin{aligned}
\mathbb{O}[\sinh(at)] &= \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}}\sinh(at)dt \\
&= \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}}\left(\frac{e^{at}-e^{-at}}{2}\right)dt \\
&= \frac{1}{2pq}(\int_0^\infty e^{(a-\frac{p}{q})t}-e^{(-a-\frac{p}{q})t})dt = \frac{1}{2pq}(\int_0^\infty e^{(a-\frac{p}{q})t}-e^{-(\frac{p}{q}+a)t})dt \\
&= \frac{1}{2pq}\left[\frac{1}{\frac{p}{q}-a}-\frac{1}{\frac{p}{q}+a}\right] = \frac{1}{2pq}\left[\frac{1}{\frac{p-aq}{q}}-\frac{1}{\frac{p+aq}{q}}\right] \\
&= \frac{1}{2pq}\left[\frac{q}{p-aq}-\frac{q}{p+aq}\right] = \frac{1}{2pq}\left[\frac{q(p+aq-p+aq)}{p^2+a^2q^2}\right] \\
&= \frac{aq}{p(p^2+a^2q^2)}.
\end{aligned}$$

Thus the proof is complete. □

3.15. **Property.** *Let the function* $f(t) = \cosh(at)$ *be in a set* Y. *Then its Ouideen transform is given by*

$$\mathbb{O}[\cosh(at)] = \frac{1}{p^2-a^2q^2}. \tag{3.22}$$

Proof. Applying the Definition 2.1 of the Ouideen transform and integration by parts, we get

$$\begin{aligned}
\mathbb{O}[\sinh(at)] &= \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}}\cosh(at)dt \\
&= \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}}\left(\frac{e^{at}+e^{-at}}{2}\right)dt \\
&= \frac{1}{2pq}(\int_0^\infty e^{(a-\frac{p}{q})t}+e^{(-a-\frac{p}{q})t})dt = \frac{1}{2pq}(\int_0^\infty e^{(a-\frac{p}{q})t}+e^{-(\frac{p}{q}+a)t})dt \\
&= \frac{1}{2pq}\left[\frac{1}{\frac{p}{q}-a}+\frac{1}{\frac{p}{q}+a}\right] = \frac{1}{2pq}\left[\frac{1}{\frac{p-aq}{q}}+\frac{1}{\frac{p+aq}{q}}\right] \\
&= \frac{1}{2pq}\left[\frac{q}{p-aq}+\frac{q}{p+aq}\right] = \frac{1}{2pq}\left[\frac{q(p+aq+p-aq)}{p^2-a^2q^2}\right] \\
&= \frac{1}{p^2-a^2q^2}.
\end{aligned}$$

Thus the proof is complete. □

3.16. **Property.** *Let the function* $f(t) = e^{bt}\sin(at)$ *be in a set* Y. *Then its Ouideen transform is given by*

$$\mathbb{O}[e^{bt}\sin(at)] = \frac{aq}{p[(p-bq)^2+a^2q^2]}. \tag{3.23}$$

Equivalently, if the function $f(t) = \dfrac{e^{bt}\sin(at)}{a}$ *be in a set* Y. *Then its Ouideen transform is given by*

$$\mathbb{O}\left[\frac{e^{bt}\sin(at)}{a}\right] = \frac{q}{p[(p-bq)^2 + a^2q^2]}. \tag{3.24}$$

Proof. Applying the Definition 2.1 of the Ouideen transform and integration by parts, we get

$$\begin{aligned}
\mathbb{O}[e^{bt}\sin(at)] &= \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}} e^{bt}\sin(at)dt \\
&= \frac{1}{pq}\int_0^\infty e^{\frac{(bq-p)t}{q}} \left(\frac{e^{iat} - e^{-iat}}{2i}\right) dt \\
&= \frac{1}{2ipq}\int_0^\infty (e^{\frac{(bq-p+iaq)t}{q}} - e^{\frac{(bq-p-iaq)t}{q}})dt \\
&= \frac{1}{2ipq}\left[\frac{q}{(p-pq) - iaq} - \frac{q}{(p-pq) + iaq}\right] \\
&= \frac{aq}{p[(p-bq)^2 + a^2q^2]}.
\end{aligned}$$

Thus the proof is complete. □

3.17. **Property.** *Let the function* $f(t) = e^{bt}\cos(at)$ *be in a set* Y. *Then its Ouideen transform is given by*

$$\mathbb{O}[f(t) = e^{bt}\cos(at)] = \frac{(p-bq)}{p[(p-bq)^2 + a^2q^2]}. \tag{3.25}$$

Proof. Applying the Definition 2.1 of the Ouideen transform and integration by parts, we get

$$\begin{aligned}
\mathbb{O}[e^{bt}\sin(at)] &= \frac{1}{pq}\int_0^\infty e^{-\frac{pt}{q}} e^{bt}\cos(at)dt \\
&= \frac{1}{pq}\int_0^\infty e^{\frac{(bq-p)t}{q}} \left(\frac{e^{iat} + e^{-iat}}{2}\right) dt \\
&= \frac{1}{2pq}\int_0^\infty (e^{\frac{(bq-p+iaq)t}{q}} + e^{\frac{(bq-p-iaq)t}{q}})dt \\
&= \frac{1}{2pq}\left[\frac{q}{(p-bq) - iaq} + \frac{q}{(p-bq) + iaq}\right] \\
&= \frac{(p-bq)}{p[(p-bq)^2 + a^2q^2]}.
\end{aligned}$$

Thus the proof is complete. □

4. Applications

In this section, the applications of the proposed transform are presented. The simplicity, efficiency and high accuracy of the Ouideen transform are clearly illustrated.

Example 4.1. *consider the following first order ordinary differential equation*

$$\frac{df(t)}{dt} + f(t) = 0, \tag{4.1}$$

subject to initial conditions

$$f(0) = 1. \tag{4.2}$$

Applying the Ouideen transform on both sides of Equ.(4.1), we get

$$\frac{p}{q}T(p,q) - \frac{1}{pq}f(0) + T(p,q) = 0. \tag{4.3}$$

Substituting the given initial condition and simplifying, we deduce

$$T(p,q)\left(\frac{p}{q} + 1\right) = \frac{1}{pq},$$

$$T(p,q)\left(\frac{p+q}{q}\right) = \frac{1}{pq},$$

$$T(p,q) = \frac{1}{p(p+q)}. \tag{4.4}$$

Taking the inverse Ouideen transform of Equ.(4.4), yields

$$f(t) = e^{-t}. \tag{4.5}$$

Example 4.2. *consider the following second ordinary differential equation*

$$\frac{d^2f(t)}{dt^2} - 3\frac{df(t)}{dt} + 2f(t) = 0 \tag{4.6}$$

subject to initial conditions

$$f(0) = 1,\ f'(0) = 4. \tag{4.7}$$

Applying the Ouideen transform on both sides of Equ.(4.9), we get

$$\frac{p^2}{q^2}T(p,q) - \frac{1}{q^2}f(0) - \frac{1}{pq}f'(0) - \frac{3p}{q}T(p,q) + \frac{3}{pq}f(0) + 2T(p,q) = 0 \tag{4.8}$$

Substituting the given initial condition and simplifying, we deduce

$$\frac{p^2}{q^2}T(p,q) - \frac{1}{q^2} - \frac{4}{pq} - \frac{3p}{q}T(p,q) + \frac{3}{pq} + 2T(p,q) = 0,$$

$$T(p,q)\left[\frac{p^2}{q^2} - \frac{3p}{q} + 2\right] - \left[\frac{1}{q^2} + \frac{4}{pq} - \frac{3}{pq}\right] = 0,$$

$$T(p,q) = \frac{1}{(p-q)(p-2q)} + \frac{q}{p(p-q)(p-2q)}. \tag{4.9}$$

Taking the inverse Ouideen transform of Equ.(4.9), yields

$$T(p,q) = \left[2e^{2t} - e^{t}\right] + \left[e^{2t} - e^{t}\right]$$

$$T(p,q) = 3e^{2t} - 2e^{t}$$

Example 4.3. *consider the following ordinary differential equation*

$$\frac{d^2f(t)}{dt^2} - 3\frac{df(t)}{dt} + 2f(t) = 4e^{3t} \tag{4.10}$$

subject to initial conditions

$$f(0) = -3, \;\; f'(0) = 5. \tag{4.11}$$

Applying the Ouideen transform on both sides of Equ.(4.10), we get

$$\frac{p^2}{q^2}T(p,q) - \frac{1}{q^2}f(0) - \frac{1}{pq}f'(0) - \frac{3p}{q}T(p,q) + \frac{3}{pq}f(0) + 2T(p,q) = 4\frac{1}{p(p-3q)}. \tag{4.12}$$

Substituting the given initial condition and simplifying, we deduce

$$T(p,q)\left[\frac{p^2-3pq+2q^2}{q^2}\right] + \left[\frac{3}{q^2} - \frac{14}{pq}\right] = \frac{4}{p(p-3q)},$$

$$T(p,q) = \left[\frac{4q^2}{p(p-3q)(p-q)(p-2q)}\right] + \left[\frac{14q-3p}{p(p-q)(p-2q)}\right],$$

$$T(p,q) = \frac{1}{p}\left[\frac{2}{p-3q} - \frac{4}{p-2q} + \frac{2}{p-q}\right] + \frac{1}{p}\left[\frac{8}{p-2q} - \frac{11}{p-q}\right],$$

$$T(p,q) = \frac{2}{p(p-3q)} - \frac{4}{p(p-2q)} + \frac{2}{p(p-q)} + \frac{8}{p(p-2q)} - \frac{11}{p(p-q)}. \tag{4.13}$$

Taking the inverse Ouideen transform of Equ.(4.13), yields

$$f(t) = 2e^{3t} + 4e^{2t} - 9e^{t}. \tag{4.14}$$

Example 4.4. *Consider the following homogeneous partial differential equation (heat equation)*

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2} \tag{4.15}$$

subject to the boundary and initial conditions

$$f(0,t) = 0 \;\; f(1,t) = 0, \;\; f(x,0) = 3\sin(2\pi x). \tag{4.16}$$

Applying the Ouideen transform on both sides of Equ.(4.15), we get

$$\frac{p}{q}T(x,p,q) - \frac{1}{pq}f(x,0) = \frac{\partial^2 T(x,p,q)}{\partial x^2}. \tag{4.17}$$

Substituting the given initial condition and simplifying, we deduce

$$\frac{\partial^2 T(x,p,q)}{\partial x^2} - \frac{p}{q}T(x,p,q) = \frac{-3}{pq}\sin(2\pi x). \tag{4.18}$$

The general solution of Equ.(4.18) can be written as

$$T(x,p,q) = T_h(x,p,q) + T_s(x,p,q), \tag{4.19}$$

where $T_h(x,p,q)$ is the solution of the homogeneous part which is given by

$$T_h(x,p,q) = a_1 e^{\sqrt{\frac{p}{q}}x} + a_2 e^{-\sqrt{\frac{p}{q}}x}, \tag{4.20}$$

applying the boundary conditions on Equ.(4.20), we get

$$a_1 + a_2 = 0 \text{ and } a_1 e^{\sqrt{\frac{p}{q}}} + a_2 e^{-\sqrt{\frac{p}{q}}} = 0 \Rightarrow T_h(x,p,q) = 0, \tag{4.21}$$

since $a_1 = a_2 = 0$.
And $T_s(x,p,q)$ *is the solution of the nonhomogeneous part. Using the method of undetermined coefficients on the nonhomogeneous part, we get*

$$T_s(x,p,q) = \frac{3}{p(p+4\pi^2 q)} \sin(2\pi x). \tag{4.22}$$

Then Equ.(4.18) will become

$$T(x,p,q) = \frac{3}{p(p+4\pi^2 q)} \sin(2\pi x). \tag{4.23}$$

Taking the inverse Ouideen transform of Equ.(4.23), we get

$$f(x,t) = 3e^{-4\pi^2 t} \sin(2\pi x). \tag{4.24}$$

Example 4.5. *Consider the following nonhomogeneous partial differential equation*

$$\frac{\partial^2 f(x,t)}{\partial t^2} = b^2 \frac{\partial^2 f(x,t)}{\partial x^2} + \sin(\pi x) \tag{4.25}$$

subject to the boundary and initial conditions

$$f(0,t) = 0,\ f(1,t) = 0,\ f(x,0) = 0,\ \frac{\partial f(x,0)}{\partial t} = 0,\ b^2 = 1. \tag{4.26}$$

Applying the Ouideen transform on both sides of Equ.(4.25), we get

$$\frac{p^2}{q^2} T(x,p,q) - \frac{1}{q^2} f(x,0) - \frac{1}{pq} f'(x,0) = \frac{d^2 T(x,p,q)}{dx^2} + \frac{1}{p^2} \sin(\pi x). \tag{4.27}$$

Substituting the given initial condition and simplifying, we deduce

$$\frac{d^2 T(x,p,q)}{dx^2} - \frac{p^2}{q^2} T(x,p,q) = -\frac{1}{p^2} \sin(\pi x). \tag{4.28}$$

The general solution of Equ.(4.28) can be written as

$$T(x,p,q) = T_h(x,p,q) + T_s(x,p,q), \tag{4.29}$$

where $T_h(x,p,q)$ *is the solution of the homogeneous part which is given by*

$$T_h(x,p,q) = c_1 e^{\frac{p}{q}x} + c_2 e^{-\frac{p}{q}x}, \tag{4.30}$$

applying the boundary conditions on Equ.(4.30), we get

$$c_1 + c_2 = 0 \text{ and } c_1 e^{\frac{p}{q}} + c_2 e^{-\frac{p}{q}} = 0 \Rightarrow T_h(x,p,q) = 0, \tag{4.31}$$

since $c_1 = c_2 = 0$.
And $T_s(x,p,q)$ *is the solution of the nonhomogeneous part. Using the method of undetermined coefficients on the nonhomogeneous part, we get*

$$T_s(x,p,q) = \frac{q^2}{p^2(p^2+\pi^2 q^2)} \sin(\pi x) = \frac{1}{\pi^2}\left(\frac{1}{p^2} - \frac{1}{p^2+\pi^2 q^2}\right) \sin(\pi x). \tag{4.32}$$

Then Equ.(4.29) will become

$$T(x,p,q) = \frac{1}{\pi^2}\left(\frac{1}{p^2} - \frac{1}{p^2+\pi^2 q^2}\right) \sin(\pi x). \tag{4.33}$$

Taking the inverse Ouideen transform of Equ.(5.4), we get

$$f(x,t) = \frac{1}{\pi^2}\,(1-\cos(\pi t))\sin(\pi x). \tag{4.34}$$

5. Solving the Laplace, Poisson and Telegraph equations using the Ouideen transform

(1) Laplace's formula

Consider Laplace equations

$$\Delta f = \sum_{i=1}^{n} f_{x_i x_i} = 0,\ x \in R^n.$$

If $n = 2$, then Laplace equations in two dimension has the form:

$$f_{tt}(x,t) + f_{xx}(x,t) = 0, \tag{5.1}$$

with the initial and boundary conditions

$$f(0,t) = 0,\ f(1,t) = 0,\ f(x,0) = h_1(x),\ \frac{\partial f(x,0)}{\partial t} = h_2(x) \tag{5.2}$$

By applying the Ouideen transform to both sides

$$\frac{p^2}{q^2}T(x,p,q) - \frac{1}{q^2}f(x,0) - \frac{1}{pq}f'(x,0) + \frac{d^2T(x,p,q)}{dx^2} = 0. \tag{5.3}$$

After substituting the initial condition we get

$$\frac{p^2}{q^2}T(x,p,q) - \frac{1}{q^2}h_1(x) - \frac{1}{pq}h_2(x) + \frac{d^2T(x,p,q)}{dx^2} = 0$$

$$\frac{d^2T(x,p,q)}{dx^2} + \frac{p^2}{q^2}T(x,p,q) = \frac{1}{q^2}h_1(x) + \frac{1}{pq}h_2(x). \tag{5.4}$$

The general solution of $Equ.$(5.4) can be written as

$$T(x,p,q) = T_h(x,p,q) + T_s(x,p,q), \tag{5.5}$$

where $T_h(x,p,q)$ is the solution of the homogeneous part which is given by

$$T_h(x,p,q) = A\cos(\frac{p}{q})x + B\sin(\frac{p}{q})x. \tag{5.6}$$

applying the boundary conditions on $Equ.$(5.6), we get

$$A + B = 0 \ and\ A\cos(\frac{p}{q}) + B\sin(\frac{p}{q}) = 0 \Rightarrow T_h(x,p,q) = 0, \tag{5.7}$$

since $A = B = 0$.
And $T_s(x,p,q)$ is the solution of the nonhomogeneous part can be obtained by variation of parameters

$$T_s(x,p,q) = A(x)\cos(\frac{p}{q})x + B(x)\sin(\frac{p}{q})x. \tag{5.8}$$

Since

$$A'(x)\cos(\frac{p}{q})x + B'(x)\sin(\frac{p}{q})x = 0 \tag{5.9}$$

$$-\frac{p}{q}A'(x)\sin(\frac{p}{q})x+\frac{p}{q}B'(x)\cos(\frac{p}{q})x=\frac{1}{q^2}h_1(x)+\frac{1}{pq}h_2(x), \tag{5.10}$$

where

$$\Delta=\begin{vmatrix}\cos(\frac{p}{q})x & \sin(\frac{p}{q})x\\ -\frac{p}{q}\sin(\frac{p}{q})x & \frac{p}{q}\cos(\frac{p}{q})x\end{vmatrix}=\frac{p}{q}.$$

Then

$$A'(x)=\frac{1}{\Delta}\begin{vmatrix}0 & \sin(\frac{p}{q})x\\ \frac{1}{q^2}h_1(x)+\frac{1}{pq}h_2(x) & \frac{p}{q}\cos(\frac{p}{q})x\end{vmatrix}=\frac{-\left(\frac{1}{q^2}h_1(x)+\frac{1}{pq}h_2(x)\right)\sin(\frac{p}{q})x}{\frac{p}{q}},$$

$$\Rightarrow A(x)=-\frac{q}{p}\int\left(\frac{1}{q^2}h_1(x)+\frac{1}{pq}h_2(x)\right)\sin(\frac{p}{q})xdx.$$

In similar way

$$B'(x)=\frac{\left(\frac{1}{q^2}h_1(x)+\frac{1}{pq}h_2(x)\right)\cos(\frac{p}{q})x}{\frac{p}{q}},$$

$$\therefore B(x)=\frac{q}{p}\int\left(\frac{1}{q^2}h_1(x)+\frac{1}{pq}h_2(x)\right)\cos(\frac{p}{q})xdx.$$

So the $Equ.$(5.8) become

$$\begin{aligned}T_s(x,p,q)=&\left[-\frac{q}{p}\int\left(\frac{1}{q^2}h_1(x)+\frac{1}{pq}h_2(x)\right)\sin(\frac{p}{q})xdx\right]\cos(\frac{p}{q})x\\&+\left[\frac{q}{p}\int\left(\frac{1}{q^2}h_1(x)+\frac{1}{pq}h_2(x)\right)\cos(\frac{p}{q})xdx\right]\sin(\frac{p}{q})x.\end{aligned} \tag{5.11}$$

Then, the general solution of equation (5.5) is

$$\begin{aligned}T(x,p,q)=&\left[-\frac{q}{p}\int\left(\frac{1}{q^2}h_1(x)+\frac{1}{pq}h_2(x)\right)\sin(\frac{p}{q})xdx\right]\cos(\frac{p}{q})x\\&+\left[\frac{q}{p}\int\left(\frac{1}{q^2}h_1(x)+\frac{1}{pq}h_2(x)\right)\cos(\frac{p}{q})xdx\right]\sin(\frac{p}{q})x.\end{aligned} \tag{5.12}$$

After taking the inverse of both sides we get the general solution of equation (5.1)

$$\begin{aligned}f(x,t)=&\mathbb{O}^{-1}[\left(-\frac{q}{p}\int\left(\frac{1}{q^2}h_1(x)+\frac{1}{pq}h_2(x)\right)\sin(\frac{p}{q})xdx\right)\cos(\frac{p}{q})x\\&+\left(\frac{q}{p}\int\left(\frac{1}{q^2}h_1(x)+\frac{1}{pq}h_2(x)\right)\cos(\frac{p}{q})xdx\right)\sin(\frac{p}{q})x].\end{aligned} \tag{5.13}$$

This is formula of Laplace $Equ.$(5.1).

Example 5.1. *To solve Laplace equation*

$$f_{tt}(x,t)+f_{xx}(x,t)=0, \tag{5.14}$$

with the conditions

$$f(0,t)=0,\ f(1,t)=0,\ f(x,0)=0,\ \frac{\partial f(x,0)}{\partial t}=x.$$

Sol:

Using the for formula (5.13), then

$$f(x,t)=\mathbb{O}^{-1}[\left(-\frac{q}{p}\int\left(\frac{1}{q^2}(0)+\frac{1}{pq}(x)\right)\sin(\frac{p}{q})xdx\right)\cos(\frac{p}{q})x$$
$$+\left(\frac{q}{p}\int\left(\frac{1}{q^2}(0)+\frac{1}{pq}(x)\right)\cos(\frac{p}{q})xdx\right)\sin(\frac{p}{q})x].$$

$$f(x,t)=\mathbb{O}^{-1}\left[\frac{-1}{p^2}\int\left(x^2\sin(\frac{p}{q})dx\right)\cos(\frac{p}{q})x+\frac{1}{p^2}\int\left(x^2\cos(\frac{p}{q})dx\right)\sin(\frac{p}{q})x\right].$$
$$f(x,t)=\mathbb{O}^{-1}\left[\frac{q}{p^3}\cos^2(\frac{p}{q^2})x-\frac{q^2}{p^4}\sin(\frac{p}{q})x\cos(\frac{p}{q})x+\frac{q}{p^3}\sin^2(\frac{p}{q})x+\frac{q^2}{p^4}\sin(\frac{p}{q})x\cos(\frac{p}{q})x\right]$$
$$f(x,t)=\mathbb{O}^{-1}\left[\frac{q}{p^3}x\right]$$
$$f(x,t)=tx.$$

(2) Poisson's formula

Consider Poisson equations

$$\Delta f(t,x)=h(x,t) \tag{5.15}$$

with the initial and boundary conditions

$$f(0,t)=0,\ f(1,t)=0,\ f(x,0)=c_1(x),\ \textit{and}\ \frac{\partial f(x,0)}{\partial t}=c_2(x)$$

By applying the Ouideen transform to both sides

$$\frac{p^2}{q^2}T(x,p,q)-\frac{1}{q^2}f(x,0)-\frac{1}{pq}f'(x,0)+\frac{d^2T(x,p,q)}{dx^2}=\mathbb{O}\left[h(x,t)\right]. \tag{5.16}$$

After substitute the initial condition we get

$$\frac{p^2}{q^2}T(x,p,q)-\frac{1}{q^2}c_1(x)-\frac{1}{pq}c_2(x)+\frac{d^2T(x,p,q)}{dx^2}=\mathbb{O}\left[h(x,t)\right]$$

$$\frac{d^2T(x,p,q)}{dx^2}+\frac{p^2}{q^2}T(x,p,q)=\mathbb{O}\left[h(x,t)\right]+\frac{1}{q^2}c_1(x)+\frac{1}{pq}c_2(x). \tag{5.17}$$

The general solution of Equ.(5.4) can be written as

$$T(x,p,q)=T_h(x,p,q)+T_s(x,p,q), \tag{5.18}$$

where $T_h(x,p,q)$ is the solution of the homogeneous part which is given by

$$T_h(x,p,q)=A\cos(\frac{p}{q})x+B\sin(\frac{p}{q})x. \tag{5.19}$$

applying the boundary conditions on Equ.(5.19), we get

$$A + B = 0 \text{ and } A\cos(\frac{p}{q}) + B\sin(\frac{p}{q}) = 0 \Rightarrow T_h(x,p,q) = 0, \tag{5.20}$$

since $A = B = 0$.
And $T_s(x,p,q)$ *is the solution of the nonhomogeneous part can be obtained by variation of parameters*

$$T_s(x,p,q) = A(x)\cos(\frac{p}{q})x + B(x)\sin(\frac{p}{q})x. \tag{5.21}$$

Since

$$A'(x)\cos(\frac{p}{q})x + B'(x)\sin(\frac{p}{q})x = 0 \tag{5.22}$$

$$-\frac{p}{q}A'(x)\sin(\frac{p}{q})x + \frac{p}{q}B'(x)\cos(\frac{p}{q})x = \mathbb{O}\left[h(x,t)\right] + \frac{1}{q^2}c_1(x) + \frac{1}{pq}c_2(x), \tag{5.23}$$

where

$$\Delta = \begin{vmatrix} \cos(\frac{p}{q})x & \sin(\frac{p}{q})x \\ -\frac{p}{q}\sin(\frac{p}{q})x & \frac{p}{q}\cos(\frac{p}{q})x \end{vmatrix} = \frac{p}{q}.$$

Then

$$A'(x) = \frac{1}{\Delta}\begin{vmatrix} 0 & \sin(\frac{p}{q})x \\ \mathbb{O}\left[h(x,t)\right] + \frac{1}{q^2}c_1(x) + \frac{1}{pq}c_2(x) & \frac{p}{q}\cos(\frac{p}{q})x \end{vmatrix}$$

$$= \frac{-\left(\mathbb{O}\left[h(x,t)\right] + \frac{1}{q^2}c_1(x) + \frac{1}{pq}c_2(x)\right)\sin(\frac{p}{q})x}{\frac{p}{q}},$$

$$\Rightarrow A(x) = -\frac{q}{p}\int\left(\mathbb{O}\left[h(x,t)\right] + \frac{1}{q^2}c_1(x) + \frac{1}{pq}c_2(x)\right)\sin(\frac{p}{q})x\,dx.$$

In similar way

$$B'(x) = \frac{\left(\mathbb{O}\left[h(x,t)\right] + \frac{1}{q^2}c_1(x) + \frac{1}{pq}c_2(x)\right)\cos(\frac{p}{q})x}{\frac{p}{q}},$$

$$\therefore B(x) = \frac{q}{p}\int\left(\mathbb{O}\left[h(x,t)\right] + \frac{1}{q^2}c_1(x) + \frac{1}{pq}c_2(x)\right)\cos(\frac{p}{q})x\,dx.$$

So the Equ.(5.21) become

$$\begin{aligned} T_s(x,p,q) = &\left[-\frac{q}{p}\int\left(\mathbb{O}\left[h(x,t)\right] + \frac{1}{q^2}c_1(x) + \frac{1}{pq}c_2(x)\right)\sin(\frac{p}{q})x\,dx\right]\cos(\frac{p}{q})x \\ &+ \left[\frac{q}{p}\int\left(\mathbb{O}\left[h(x,t)\right] + \frac{1}{q^2}c_1(x) + \frac{1}{pq}c_2(x)\right)\cos(\frac{p}{q})x\,dx\right]\sin(\frac{p}{q})x. \end{aligned} \tag{5.24}$$

Then, the general solution of equation (5.18) is

$$T(x,p,q)=\left[-\frac{q}{p}\int\left(\mathbb{O}\left[h(x,t)\right]+\frac{1}{q^2}c_1(x)+\frac{1}{pq}c_2(x)\right)\sin(\frac{p}{q})xdx\right]\cos(\frac{p}{q})x$$
$$+\left[\frac{q}{p}\int\left(\mathbb{O}\left[h(x,t)\right]+\frac{1}{q^2}c_1(x)+\frac{1}{pq}c_2(x)\right)\cos(\frac{p}{q})xdx\right]\sin(\frac{p}{q})x. \tag{5.25}$$

After taking the inverse of both sides we get the general solution of equation (5.15)

$$f(x,t)=\mathbb{O}^{-1}[\left(-\frac{q}{p}\int\left(\mathbb{O}\left[h(x,t)\right]+\frac{1}{q^2}c_1(x)+\frac{1}{pq}c_2(x)\right)\sin(\frac{p}{q})xdx\right)\cos(\frac{p}{q})x$$
$$+\left(\frac{q}{p}\int\left(\mathbb{O}\left[h(x,t)\right]+\frac{1}{q^2}c_1(x)+\frac{1}{pq}c_2(x)\right)\cos(\frac{p}{q})xdx\right)\sin(\frac{p}{q})x]. \tag{5.26}$$

This is formula of Poisson Equ.(5.15).

Example 5.2. *To solve Poisson equation*

$$f_{tt}(x,t)+f_{xx}(x,t)=xe^t \tag{5.27}$$

with the conditions

$$f(0,t)=0,\ \ f(1,t)=0,\ \ ,f(x,0)=0\ and\ \frac{\partial f(x,0)}{\partial t}=x.$$

Sol:
Using equation (5.26) and after substitute the conditions

$$f(x,t)=\mathbb{O}^{-1}[\left(-\frac{q}{p}\int\left(x\left[\frac{1}{p(p-q)}\right]+\frac{x}{pq}\right)\sin(\frac{p}{q})xdx\right)\cos(\frac{p}{q})x$$
$$+\left(\frac{q}{p}\int\left(x\left[\frac{1}{p(p-q)}\right]+\frac{x}{pq}\right)\cos(\frac{p}{q})xdx\right)\sin(\frac{p}{q})x].$$

$$f(x,t)=\mathbb{O}^{-1}[\left(\frac{q}{p^3-p^2q}x\cos(\frac{p}{q})x-\frac{q^2}{p^4-q^2p^2}\sin(\frac{p}{q})x\right)\cos(\frac{p}{q})x$$
$$+\left(\frac{q}{p^3-p^2q}x\sin(\frac{p}{q})x+\frac{q^2}{p^4-q^2p^2}\cos(\frac{p}{q})x\right)\sin(\frac{p}{q})x]$$

$$f(x,t) = \mathbb{O}^{-1}[\frac{q}{p^3-p^2q}x\cos^2(\frac{p}{q})x - \frac{q^2}{p^4-q^2p^2}\sin(\frac{p}{q})x\cos(\frac{p}{q})x$$
$$+\frac{q}{p^3-p^2q}x\sin^2(\frac{p}{q})x + \frac{q^2}{p^4-q^2p^2}\cos(\frac{p}{q})x\sin(\frac{p}{q})x]$$
$$f(x,t) = \left[\frac{q}{p^3-p^2q}x\right]$$
$$f(x,t) = \left[\frac{1}{p(p-q)}x - \frac{1}{p^2}x\right]$$
$$f(x,t) = e^t x - x.$$

(3) Telegraph's formula

The Telegraph equation has the form

$$f_{tt}(x,t) + 2Hf_t(x,t) + \beta^2 f(x,t) = f_{xx}(x,t) + h(x,t), \tag{5.28}$$

under the conditions

$$f(0,t) = 0,\ f(1,t) = 0,\ f(x,0) = c_1(x) \text{ and } \frac{\partial f(x,0)}{\partial t} = c_2(x).$$

By taking the Ouideen transform to both sides, yields

$$\frac{p^2}{q^2}T(x,p,q) - \frac{1}{q^2}f(x,0) - \frac{1}{pq}f'(x,0) + 2H\frac{p}{q}T(x,p,q) - 2Hf(x,0) + \beta^2 T(x,p,q) = \frac{d^2T(x,p,q)}{dx^2} + \mathbb{O}[h(x,t)]$$

After substituting initial conditions

$$\frac{p^2}{q^2}T(x,p,q) - \frac{1}{q^2}c_1(x) - \frac{1}{pq}c_2(x) + 2H\frac{p}{q}T(x,p,q) - 2Hc_1(x) + \beta^2 T(x,p,q) = \frac{d^2T(x,p,q)}{dx^2} + \mathbb{O}[h(x,t)],$$

$$\Rightarrow \frac{d^2T(x,p,q)}{dx^2} - \left(\frac{p^2}{q^2} + 2H\frac{p}{q} + \beta^2\right)T(x,p,q) = -\frac{1}{q^2}c_1(x) - 2Hc_1(x) - \frac{1}{pq}c_2(x) - \mathbb{O}[h(x,t)] \tag{5.29}$$

The general solution of Equ.(5.29) can be written as

$$T(x,p,q) = T_h(x,p,q) + T_s(x,p,q), \tag{5.30}$$

where $T_h(x,p,q)$ is the solution of the homogeneous part which is given by

$$T_h(x,p,q) = Ae^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x} + Be^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x}. \tag{5.31}$$

applying the boundary conditions on Equ.(5.31), we get

$$A + B = 0 \text{ and } Ae^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}} + Be^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}} = 0 \Rightarrow T_h(x,p,q) = 0, \tag{5.32}$$

since $A = B = 0$.

And $T_s(x,p,q)$ is the solution of the nonhomogeneous part can be obtained by variation of parameters

$$T_s(x,p,q) = A(x)e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x} + B(x)e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x}. \tag{5.33}$$

Since

$$A'(x)e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x}+B'(x)e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x}=0 \tag{5.34}$$

$$-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{-1}{2}}A'(x)e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x}+\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{-1}{2}}B'(x)e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x}$$

$$=-\frac{1}{q^2}c_1(x)-2Hc_1(x)-\frac{1}{pq}c_2(x)-\mathbb{O}[h(x,t)], \tag{5.35}$$

where

$$\Delta=\begin{vmatrix} e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x} & e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x} \\ -\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{-1}{2}}e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x} & \left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{-1}{2}}e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x} \end{vmatrix}$$

$$=\frac{1}{2}\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}.$$

Then

$$A'(x)=\frac{1}{\Delta}\begin{vmatrix} 0 & e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x} \\ -\frac{1}{q^2}c_1(x)-2Hc_1(x)-\frac{1}{pq}c_2(x)-\mathbb{O}[h(x,t)] & \frac{p}{q}e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x} \end{vmatrix}$$

$$=\frac{-\left(-\frac{1}{q^2}c_1(x)-2Hc_1(x)-\frac{1}{pq}c_2(x)-\mathbb{O}[h(x,t)]\right)e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x}}{2\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}}.$$

Let $\Delta_T==\frac{1}{2}\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{-1}{2}}$.

$$\Rightarrow A(x)=-\Delta_T\int\left(-\frac{1}{q^2}c_1(x)-2Hc_1(x)-\frac{1}{pq}c_2(x)-\mathbb{O}[h(x,t)]\right)e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x}dx.$$

In similar way

$$B'(x)=\frac{\left(-\frac{1}{q^2}c_1(x)-2Hc_1(x)-\frac{1}{pq}c_2(x)-\mathbb{O}[h(x,t)]\right)e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x}}{2\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}},$$

$$\therefore B(x) = \Delta_T \int \left(-\frac{1}{q^2} c_1(x) - 2Hc_1(x) - \frac{1}{pq} c_2(x) - \mathbb{O}[h(x,t)] \right) e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}} x dx.$$

So the Equ.(5.33) become

$$T_s(x,p,q) =$$

$$\left[-\Delta_T \int \left(-\frac{1}{q^2} c_1(x) - 2Hc_1(x) - \frac{1}{pq} c_2(x) - \mathbb{O}[h(x,t)] \right) e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}} x} dx \right] e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}} x}$$

$$+ \left[\Delta_T \int \left(-\frac{1}{q^2} c_1(x) - 2Hc_1(x) - \frac{1}{pq} c_2(x) - \mathbb{O}[h(x,t)] \right) e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}} x} dx \right] e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}} x}.$$

Then, the general solution of equation (5.30) is

$$T(x,p,q) =$$

$$\left[-\Delta_T \int \left(-\frac{1}{q^2} c_1(x) - 2Hc_1(x) - \frac{1}{pq} c_2(x) - \mathbb{O}[h(x,t)] \right) e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}} x} dx \right] e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}} x}$$

$$+ \left[\Delta_T \int \left(-\frac{1}{q^2} c_1(x) - 2Hc_1(x) - \frac{1}{pq} c_2(x) - \mathbb{O}[h(x,t)] \right) e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}} x} dx \right] e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}} x}.$$

After taking the inverse of both sides we get the general solution of equation (5.28)

$$f(x,t) =$$

$$\begin{aligned} &\mathbb{O}^{-1}\Bigg[\left(-\Delta_T \int \left(-\frac{1}{q^2} c_1(x) - 2Hc_1(x) - \frac{1}{pq} c_2(x) - \mathbb{O}[h(x,t)] \right) e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}} x} dx \right) e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}} x} \\ &\quad + \left(\Delta_T \int \left(-\frac{1}{q^2} c_1(x) - 2Hc_1(x) - \frac{1}{pq} c_2(x) - \mathbb{O}[h(x,t)] \right) e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}} x} dx \right) e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}} x} \Bigg]. \end{aligned} \tag{5.36}$$

This is formula of Poisson Equ.(5.28), where $\Delta_T = \frac{1}{2}\left(\frac{p^2}{q^2} + 2H\frac{p}{q} + \beta^2\right)^{\frac{-1}{2}}$.

Moreover, if $h(x,t) = 0$ *then the equation (5.28) has the form*

$$f_{tt}(x,t) + 2Hf_t(x,t) + \beta^2 f(x,t) = f_{xx}(x,t), \tag{5.37}$$

under the conditions

$$f(0,t) = 0, \;\; f(1,t) = 0, \;\; f(x,0) = c_1(x) \text{ and } \frac{\partial f(x,0)}{\partial t} = c_2(x).$$

So, form equation (5.40) the general solution of (5.37), has the form

$$f(x,t) = \mathbb{O}^{-1}\Bigg[\left(-\Delta_T \int \left(-\frac{1}{q^2}c_1(x) - 2Hc_1(x) - \frac{1}{pq}c_2(x)\right) e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x} dx\right) e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x} + \left(\Delta_T \int \left(-\frac{1}{q^2}c_1(x) - 2Hc_1(x) - \frac{1}{pq}c_2(x)\right) e^{\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x} dx\right) e^{-\left(\frac{p^2}{q^2}+2H\frac{p}{q}+\beta^2\right)^{\frac{1}{2}}x}\Bigg]. \tag{5.38}$$

Where $\Delta_T = \frac{1}{2}\left(\frac{p^2}{q^2} + 2H\frac{p}{q} + \beta^2\right)^{\frac{-1}{2}}$.

Example 5.3. *To obtain the solution of Telegraph equation*

$$f_{tt}(x,t) + f_t(x,t) + f(x,t) = f_{xx}(x,t) + 2e^{-2t}\sinh(x), \tag{5.39}$$

under the conditions

$$f(0,t) = 0,\ \ f(1,t) = 0,\ \ f(x,0) = \sinh(x)\ \mathit{and}\ f_t(x,0) = -2\sinh(x).$$

Sol:
From formula (5.40)

$$f(x,t) = \mathbb{O}^{-1}\Bigg[\left(-\Delta_T \int \left(-\frac{p+2q}{q^2p}\sinh(x) - \sinh(x) - \sinh(x)\frac{2}{p(p+2q)}\right) e^{-\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x} dx\right) e^{\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x} + \left(\Delta_T \int \left(-\frac{p+2q}{q^2p}\sinh(x) - \sinh(x) - \sinh(x)\frac{2}{p(p+2q)}\right) e^{\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x} dx\right) e^{-\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x}\Bigg]. \tag{5.40}$$

Where $\Delta_T = \frac{1}{2}\left(\frac{p^2}{q^2} + \frac{p}{q} + 1\right)^{\frac{-1}{2}}$.

$$f(x,t) = \mathbb{O}^{-1}\left[\frac{-\frac{p+2q}{q^2p} - \frac{2}{p(p+2q)}}{4(\frac{p^2}{q^2}+\frac{p}{q}+1)^{\frac{1}{2}}}\left(e^x\left(\frac{2\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}}{1-\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)}\right) - e^{-x}\left(\frac{2\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}}{1-\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)}\right)\right)\right] \tag{5.41}$$

$$f(x,t) = \mathbb{O}^{-1}\left[\frac{-\frac{p+2q}{q^2p} - \frac{2}{p(p+2q)}}{1-(\frac{p^2}{q^2}+\frac{p}{q}+1)}\sinh(x)\right]$$

$$f(x,t) = \mathbb{O}^{-1}\left[\sinh(x)\frac{1}{p(p+2q)}\right]$$

Lastly, the solution of equation (5.42) is

$$f(x.t) = \sinh(x)e^{-2t}$$

Example 5.4. *To obtain the solution of Telegraph equation*

$$f_{tt}(x,t) + f_t(x,t) + f(x,t) = f_{xx}(x,t), \tag{5.42}$$

under the conditions

$$f(0,t) = 0,\ f(1,t) = 0,\ f(x,0) = e^x \text{ and } f_t(x,0) = -e^x.$$

Sol:
From formula (5.38)

$$\begin{aligned} f(x,t) =& \mathbb{O}^{-1}\Bigg[\left(-\Delta_T \int \left(-\frac{1}{q^2}e^x - e^x + \frac{1}{pq}e^x\right) e^{-\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x} dx\right) e^{\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x} \\ &+ \left(\Delta_T \int \left(-\frac{1}{q^2}e^x - e^x + \frac{1}{pq}e^x\right) e^{\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x} dx\right) e^{-\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x}\Bigg]. \end{aligned} \tag{5.43}$$

Where $\Delta_T = \frac{1}{2}\left(\frac{p^2}{q^2} + \frac{p}{q} + 1\right)^{\frac{-1}{2}}$.

$$\begin{aligned} f(x,t) =& \mathbb{O}^{-1}\Bigg[\left(-\Delta_T \int \left(\frac{-p - pq^2 + q^2}{pq^2}e^x\right) e^{-\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x} dx\right) e^{\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x} \\ &+ \left(\Delta_T \int \left(\frac{-p - pq^2 + q^2}{pq^2}e^x\right) e^{\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x} dx\right) e^{-\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x}\Bigg], \end{aligned}$$

$$\begin{aligned} f(x,t) =& \mathbb{O}^{-1}\Bigg[\left(\frac{(-p - pq^2 + q^2)e^{1+\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x}}{2pq^2\left(1 + (\frac{p^2}{q^2} + \frac{p}{q} + 1)^{\frac{1}{2}}\right)(\frac{p^2}{q^2} + \frac{p}{q} + 1)^{\frac{1}{2}}}\right) e^{\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x} \\ &+ \left(\frac{(p + pq^2 - q^2)e^{1+\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x}}{2pq^2\left(1 + (\frac{p^2}{q^2} + \frac{p}{q} + 1)^{\frac{1}{2}}\right)(\frac{p^2}{q^2} + \frac{p}{q} + 1)^{\frac{1}{2}}}\right) e^{-\left(\frac{p^2}{q^2}+\frac{p}{q}+1\right)^{\frac{1}{2}}x}\Bigg]. \end{aligned}$$

After simple calculation

$$f(x,t) = \mathbb{O}^{-1}\left[\frac{(-p - pq^2 + q^2)e^x}{pq^2\left(1 - (\frac{p^2}{q^2} + \frac{p}{q} + 1)\right)}\right]$$

$$f(x,t) = \mathbb{O}^{-1}\left[e^x \frac{1}{p(p+q)}\right]$$

$$f(x,t) = e^{x-t}$$

Conclusion

We introduced a new integral transform called the Ouideen transform for solving both ordinary and partial differential equations. We presented its existence and inverse transform. We presented some useful properties and their applications for solving ordinary and partial differential equations. We presented formulas for Laplace, Poisson and Telegraph equations using Ouideen transform to solve these equations. We provide a comprehensive list of the Ouideen transform in table 1. Finally, based on the mathematical formulations, simplicity and the findings of the proposed integral transform, we conclude that it is highly efficient and it is a generalization of the Aboodh and Sumudu integral transforms.

APPENDIX

Table 1: Here we present a comprehensive list of Ouideen transform of some functions.

$S.NO.$	$f(t)$	$\mathbb{O}[f(t)]$
1	1	$\dfrac{1}{p^2}$
2	t	$\dfrac{q}{p^3}$
3	e^{at}	$\dfrac{1}{p(p-aq)}$
4	$\sin(at)$	$\dfrac{aq}{p(p^2+a^2q^2)}$
5	$\cos(at)$	$\dfrac{1}{p^2+a^2q^2}$
6	$\cosh(at)$	$\dfrac{1}{p^2-a^2q^2}$
7	$\sinh(at)$	$\dfrac{aq}{p(p^2-a^2q^2)}$
8	$t^n \ n=0,1,2...$	$n!\dfrac{q^n}{p^{n+1}}$
9	$\dfrac{t^n}{\Gamma(n+1)} \ n=0,1,2...$	$\dfrac{q^n}{p^{n+1}}$
10	$\cos(t)$	$\dfrac{1}{p^2+q^2}$
11	$\sin(t)$	$\dfrac{q}{p(p^2+q^2)}$
12	$\cosh(t)$	$\dfrac{1}{p^2-q^2}$
13	$e^{bt}\cosh(at)$	$\dfrac{p-bq}{p[(p-bq)^2-a^2q^2]}$
14	$e^{bt}\sinh(at)$	$\dfrac{aq}{p[(p-bq)^2-a^2q^2]}$

15	$t\sin(at)$	$\frac{2apq}{(p^2+a^2q^2)^2}$
16	$t\cos(at)$	$\frac{q(p^2-a^2q^2)^2}{p(p^2+a^2q^2)^2}$
17	$\sin(at)+at\cos(at)$	$\frac{2apq}{(p^2+a^2q^2)^2}$
18	$2\cos(at)-at\sin(at)$	$\frac{2p^2}{p^2+a^2q^2)^2}$
19	$\sin(at)-at\cos(at)$	$\frac{2a^3q^3}{p(p^2+a^2q^2)^2}$
20	$t\sinh(at)+t\cosh(at)$	$\frac{q}{p(p-aq)^2}$
21	$t\sinh(at)$	$\frac{2aq^2}{(p^2-a^2q^2)^2}$
22	$Si(at)(Sineintegral)$	$\frac{1}{p^2}\tan^{-1}(\frac{aq}{p})$
23	$Ci(at)(Cosineintegral)$	$-\frac{1}{2p^2}\log\left(\frac{p^2+a^2}{a^2}\right)$
24	$Ei(at)(Exp.integral)$	$-\frac{1}{p^2}\log\left(\frac{aq-p}{aq}\right)$
25	$(3-a^2t^2)\sin(at)-3at\cos(at)$	$\frac{8a^5q^5}{p(p^2+a^2q^2)^3}$
26	$(3-a^2t^2)\sin(at)+5at\cos(at)$	$\frac{8ap^3q}{(p^2+a^2q^2)^3}$
27	$(3-a^2t^2)\cos(at)-7at\sin(at)$	$\frac{8p^4}{(p^2+a^2q^2)^3}$
28	$t^2\sin(at)$	$\frac{24q^3(3p^2-a^2q^2)}{p(p^2+a^2q^2)^3}$
29	$t^2\cos(at)$	$\frac{2q^2(p^2-3a^2q^2)}{(p^2+a^2q^2)^3}$
30	$t^3\sin(at)$	$\frac{24aq^4(p-aq)^2}{(p^2+a^2q^2)^4}$
31	$(e^{at}-e^{bt})\ a\neq b$	$\frac{(a-b)q}{p(p-aq)(p-bq)}$
32	$(ae^{at}-be^{bt})\ a\neq b$	$\frac{(a-b)}{(p-bq)(p-aq)}$
33	$I_0(at)$	$\frac{1}{p\sqrt{p^2-a^2q^2}}$
34	$\delta(t-a)$	$\frac{1}{p}e^{\frac{-ap}{q}}$
35	$J_0(at)$	$\frac{1}{p\sqrt{p^2-a^2q^2}}$

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