DOUBLE SUMUDU-KAMAL TRANSFORM WITH APPLICATIONS

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Abstract.

In this paper, we introduce a new operator integral transform called double Sumudu-Kamal transform, some valuable properties for the transform are present. Furthermore, we use this transform for solving some linear partial differential equations and find some functions.

Keywords: Double Sumudu-Kamal transform, Sumudu transform, Kamal transform, linear partial differential equations.

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1. Introduction

Many phenomenon and process of engineering, science and real life can be expressed mathematically and solved by using integral transforms. There are many types of integral transforms such as Laplace transform, Sumudu transform, Fourier transform, Hunaiber transform, Kamal transform, Shehu transform, Aboodh transform, and so on which have many applications in numerous fields of mathematical sciences and engineering such as physics, chemistry, acoustics, etc.,. The integral transform was applied to partial differential equations with non-homogenous forcing term and having singular variable data. Watugula derived a new integral transform and named Sumudu transform to it. Also, he discussed its basic properties and then used it for finding the solutions of differential equations and control engineering problems [14]. The Sumudu transform was proposed by A. Kiliman and H. Eltayeb. Eltayeb and Kilicman, applied double Laplace transform to solve wave, Laplaces and heat equations with convolution terms, general linear telegraph and partial integro-differential equations [7].

In 2016, Abdelilah Kamal introduced Kamal transform. It's very useful to solve intricate problem in applied science and engineering mathematic [11],[12]. The solutions of initial and boundary value problems are given by numerous integral transforms methods. In previous years, numerous notice has been given to deal with the double integral transform, for instance and so on which have many applications in various fields of mathematical sciences and engineering such as acoustics, physics, chemistry, etc., [9]. We applied new double Sumudu-Kamal transform to solve Poisson, Wave and Heat equations. The main of this paper is to present a new method for solving some linear partial differential equations with initial and boundary conditions called double SumuduKamal transform, the definition of double Sumudu-Kamal transform and its inverse. We also discuss some theorems, properties, elementary functions about Key words and phrases. Double Sumudu-Kamal transform, Sumudu transform, Kamal transform, linear partial differential equations.

the double Sumudu-Kamal transform. Also, we implement the double Sumudu-Kamal transform method to some examples.

The Sumudu transform of the continuous function $\phi(\gamma)$ for all $\gamma \geq 0$ is defined by

$$\mathcal{S}[\phi(\gamma)] = \frac{1}{\mu}\int_0^\infty e^{-\frac{\gamma}{\mu}}\phi(\gamma)d\gamma = \Phi(\mu), \ \kappa_1 \leq \mu \leq \kappa_2, \tag{1.1}$$

where $\mathcal{S}$ is the Sumudu operator[6]. Provided that the integral exists. If the integral is convergent for some value of γ, then the Sumudu transformation of $\phi(\gamma)$ exists otherwise not[16].

The inverse Sumudu transform of the function $\Phi(\mu)$ is defined by

$$\phi(\gamma) = \mathcal{S}^{-1}[\Phi(\mu)] = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{\mu}e^{\frac{\gamma}{\mu}}\Phi(\mu)d\mu, \tag{1.2}$$

where α is a real constant and S^{-1} is an operator and it is called inverse Sumudu transform operator [5].

The Kamal transform of the function $\phi(\tau)$ denoted by the operator $K(.)$, defined by

$$K[\phi(\tau)] = \Phi(\vartheta) = \int_0^\infty e^{-\frac{\tau}{\vartheta}}\phi(\tau)d\tau, \ \lambda_1 \leq \tau \leq \lambda_2, \tag{1.3}$$

Here, the variable ϑ is used to factor the variable τ in the argument of the function ϕ [11].

The function $\phi(\tau)$ is called the inverse Kamal transform of the function $\Phi(\vartheta)$, it can be defined as

$$\phi(\tau) = K^{-1}[\Phi(\vartheta)] = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}e^{\frac{\tau}{\vartheta}}\Phi(\frac{1}{\vartheta})d\vartheta, \tag{1.4}$$

where β is a real constant and the operator K^{-1} is called inverse Kamal transform [15].

1.1. **Definition.** The double Sumudu-Kamal transform of the function $\phi(\gamma,\tau)$ for all variables $\gamma,\tau > 0$ is denoted by $\mathcal{S}_\gamma K_\tau[\phi(\gamma,\tau)] = \Phi(\mu,\vartheta)$ and defined by

$$\mathcal{S}_\gamma K_\tau[\phi(\gamma,\tau)] = \Phi(\mu,\vartheta) = \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\phi(\gamma,\tau)d\gamma d\tau, \tag{1.5}$$

provided the integral exists.

1.2. **Definition.** The inverse double Sumudu-Kamal transform of the function $\Phi(\mu,\vartheta)$ is defined by

$$\phi(\gamma,\tau) = \mathcal{S}_\gamma^{-1}K_\tau^{-1}[\Phi(\mu,\vartheta)] = \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\frac{\gamma}{\mu}}\Big(\int_{\beta-i\infty}^{\beta+i\infty}e^{\frac{\tau}{\vartheta}}\Phi(\mu,\frac{1}{\vartheta})d\vartheta\Big)d\mu, \tag{1.6}$$

where α and β are real constants.

2. Existence and Uniqueness of The Double Sumudu-Kamal Transform

2.1. **Definition.** A function $\phi(\gamma,\tau)$ is said to be of exponential orders $\lambda,\eta>0$, on $0\leq\gamma,\tau\leq\infty$, if there exists positive constants P,M and N such that

$$|\phi(\gamma,\tau)|\leq Pe^{(\lambda\gamma+\eta\tau)},\ for\ all\ \gamma>M,\ \tau>N,$$

and we write

$$\phi(\gamma,\tau)=o(e^{\lambda\gamma+\eta\tau})\ as\ \gamma,\tau\to\infty.$$

Or, equivalently,

$$\lim_{\gamma\to\infty,\tau\to\infty}e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}|\phi(\gamma,\tau)|\leq P\lim_{\gamma\to\infty,\tau\to\infty}e^{-(\frac{1-\lambda\mu}{\mu})\gamma}e^{-(\frac{1-\eta\vartheta}{\vartheta})\tau}=0,\ \mu>\lambda,\ \vartheta>\eta.$$

2.2. **Theorem.** Let $\phi(\gamma,\tau)$ be a continuous function in every finite intervals $(0,M)$ and $(0,N)$ and of exponential order $e^{(\lambda\gamma+\eta\tau)}$, then the double Sumudu-Kamal transform of function $\phi(\gamma,\tau)$ exists for all $\mu>\lambda$ and $\vartheta>\eta$.

Proof.

Let $\phi(\gamma,\tau)$ be of exponential order $e^{(\lambda\gamma+\eta\tau)}$, such that

$$|\phi(\gamma,\tau)|\leq Pe^{(\lambda\gamma+\eta\tau)},\ for\ all\ \gamma>M,\ \tau>N$$

. Then, we have

$$\begin{aligned}\Big|\Phi(\mu,\vartheta)\Big| &= \Big|\frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\phi(\gamma,\tau)d\gamma d\tau\Big|\\ &\leq \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}|\phi(\gamma,\tau)|d\gamma d\tau\\ &\leq \frac{P}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}e^{(\lambda\gamma+\eta\tau)}d\gamma d\tau\\ &= \frac{P}{\mu}\int_0^\infty e^{-(\frac{1-\lambda\mu}{\mu})\gamma}d\gamma\int_0^\infty e^{-(\frac{1-\eta\vartheta}{\vartheta})\tau}d\tau\\ &= \frac{P\vartheta}{\Big(1-\lambda\mu\Big)\Big(1-\eta\vartheta\Big)}.\end{aligned}$$

Thus, the proof is complete.

2.3. **Theorem.** Let $\Phi_1(\mu,\vartheta)$ and $\Phi_1(\mu,\vartheta)$ be the double Sumudu-Kamal transform of the continuous functions $\phi_1(\gamma,\tau)$ and $\phi_2(\gamma,\tau)$ defined for $\gamma,\tau\geq 0$ respectively. If $\Phi_1(\mu,\vartheta)=\Phi_2(\mu,\vartheta)$, then $\phi_1(\gamma,\tau)=\phi_2(\gamma,\tau)$.

Proof.

Assume that α and β are adequately large, since

$$\phi(\gamma,\tau)=\mathcal{S}_\gamma^{-1}K_\tau^{-1}[\Phi(\mu,\vartheta)]=\frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\frac{\gamma}{\mu}}\Big(\int_{\beta-i\infty}^{\beta+i\infty}e^{\frac{\tau}{\vartheta}}\Phi(\mu,\frac{1}{\vartheta})d\vartheta\Big)d\mu,$$

we deduce that

$$\begin{aligned}\phi_1(\gamma,\tau) &= \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\frac{\gamma}{\mu}}\Big(\int_{\beta-i\infty}^{\beta+i\infty} e^{\frac{\tau}{\vartheta}}\Phi_1(\mu,\frac{1}{\vartheta})d\vartheta\Big)d\mu\\ &= \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\frac{\gamma}{\mu}}\Big(\int_{\beta-i\infty}^{\beta+i\infty} e^{\frac{\tau}{\vartheta}}\Phi_2(\mu,\frac{1}{\vartheta})d\vartheta\Big)d\mu\\ &= \phi_2(\gamma,\tau).\end{aligned}$$

This proves the uniqueness of the double Sumudu-Kamal transform.

3. Some Useful Properties of Sumudu-Kamal Transform

3.1. **Linearity property.** If the double Sumudu-Kamal transform of functions $\phi_1(\gamma,\tau)$ and $\phi_2(\gamma,\tau)$ are $\Phi_1(\mu,\vartheta)$ and $\Phi_2(\mu,\vartheta)$ respectively, then double Sumudu-Kamal transform of $\alpha\phi_1(\gamma,\tau)+\beta\phi_2(\gamma,\tau)$ is given by $\alpha\Phi_1(\mu,\vartheta)+\beta\Phi_2(\mu,\vartheta)$, where α and β are arbitrary constants.
Proof.

$$\begin{aligned}\mathcal{S}_\gamma K_\tau[\alpha\phi_1(\gamma,\tau)+\beta\phi_2(\gamma,\tau)] &= \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\Big(\alpha\phi_1(\gamma,\tau)+\beta\phi_2(\gamma,\tau)\Big)d\gamma d\tau\\ &= \frac{\alpha}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\phi_1(\gamma,\tau)d\gamma d\tau\\ &+ \frac{\beta}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\phi_2(\gamma,\tau)d\gamma d\tau\\ &= \alpha\Phi_1(\mu,\vartheta)+\beta\Phi_2(\mu,\vartheta). \qquad (3.1)\end{aligned}$$

3.2. **Shifting property.** If the double Sumudu-Kamal transform of function $\phi(\gamma,\tau)$ is a function $\Phi(\mu,\vartheta)$, then for any pair of real constants $\alpha,\beta>0$

$$\mathcal{S}_\gamma K_\tau[e^{(\alpha\gamma+\beta\tau)}\phi(\gamma,\tau)] = \frac{1}{1-\alpha\mu}\,\Phi\Big(\frac{\mu}{1-\alpha\mu}\,,\,\frac{\vartheta}{1-\beta\vartheta}\Big). \qquad (3.2)$$

Proof. Using the definition of double Sumudu-Kamal transform, we get

$$\begin{aligned}\mathcal{S}_\gamma K_\tau[e^{(\alpha\gamma+\beta\tau)}\phi(\gamma,\tau)] &= \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}e^{(\alpha\gamma+\beta\tau)}\phi(\gamma,\tau)d\gamma d\tau\\ &= \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-[(\frac{1-\alpha\mu}{\mu})\gamma+(\frac{1-\beta\vartheta}{\vartheta})\tau]}\phi(\gamma,\tau)d\gamma d\tau\\ &= \frac{\mu}{\mu(1-\alpha\mu)}\int_0^\infty\int_0^\infty \frac{1-\alpha\mu}{\mu}\,e^{-[(\frac{1-\alpha\mu}{\mu})\gamma+(\frac{1-\beta\vartheta}{\vartheta})\tau]}\phi(\gamma,\tau)d\gamma d\tau\\ &= \frac{1}{1-\alpha\mu}\Phi\Big(\frac{\mu}{1-\alpha\mu}\,,\,\frac{\vartheta}{1-\beta\vartheta}\Big).\end{aligned}$$

3.3. **Change of scale property.** Let $\Phi(\mu,\vartheta)$ be the double Sumudu-Kamal transform of function $\phi(\gamma,\tau)$, then for α and β are positive constants, we have

$$\mathcal{S}_\gamma K_\tau[\phi(\alpha\gamma,\beta\tau)] = \frac{1}{\beta}\Phi(\mu,\beta\vartheta). \qquad (3.3)$$

Proof. Using the definition of double Sumudu-Kamal transform, we get

$$\mathcal{S}_\gamma K_\tau[\phi(\alpha\gamma,\beta\tau)] = \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\phi(\alpha\gamma,\beta\tau)d\gamma d\tau.$$

Set $\xi=\alpha\gamma,\ \eta=\beta\tau$, then

$$\begin{aligned}\mathcal{S}_\gamma K_\tau[\phi(\alpha\gamma,\beta\tau)] &= \frac{1}{\alpha\beta\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\xi}{\alpha\mu}+\frac{\eta}{\beta\vartheta})}\phi(\xi,\eta)d\xi d\eta\\ &= \frac{1}{\beta}\Phi(\alpha\mu,\beta\vartheta).\end{aligned}$$

3.4. **Derivatives properties.** If $\mathcal{S}_\gamma K_\tau[\phi(\gamma,\tau)]=\Phi(\mu,\vartheta)$, then

$$(1).\ \mathcal{S}_\gamma K_\tau\Big[\frac{\partial\phi(\gamma,\tau)}{\partial\gamma}\Big]=\frac{1}{\mu}\ \Phi(\mu,\vartheta)-\frac{1}{\mu}K[\phi(0,\tau)]. \tag{3.4}$$

Proof.

$$\begin{aligned}\mathcal{S}_\gamma K_\tau\Big[\frac{\partial\phi(\gamma,\tau)}{\partial\gamma}\Big] &= \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\ \frac{\partial\phi(\gamma,\tau)}{\partial\gamma}d\gamma d\tau\\ &= \frac{1}{\mu}\int_0^\infty e^{-\frac{\tau}{\vartheta}}d\tau\Big(\int_0^\infty e^{-\frac{\gamma}{\mu}}\phi_\gamma(\gamma,\tau)d\gamma\Big).\end{aligned}$$

Using integration by parts, let $\xi=e^{-\frac{\gamma}{\mu}},\ d\eta=\phi_\gamma(\gamma,\tau)d\gamma$, then we get

$$\begin{aligned}\mathcal{S}_\gamma K_\tau\Big[\frac{\partial\phi(\gamma,\tau)}{\partial\gamma}\Big] &= \frac{1}{\mu}\int_0^\infty e^{-\frac{\tau}{\vartheta}}d\tau\Big(-\phi(0,\tau)+\frac{1}{\mu}\int_0^\infty e^{-\frac{\gamma}{\mu}}\phi(\gamma,\tau)d\gamma\Big)\\ &= \frac{1}{\mu}\ \Phi(\mu,\vartheta)-\frac{1}{\mu}K[\phi(0,\tau)].\end{aligned}$$

$$(2).\ \mathcal{S}_\gamma K_\tau\Big[\frac{\partial\phi(\gamma,\tau)}{\partial\tau}\Big]=\frac{1}{\vartheta}\Phi(\mu,\vartheta)-\mathcal{S}[\phi(\gamma,0)]. \tag{3.5}$$

Proof.

$$\begin{aligned}\mathcal{S}_\gamma K_\tau\Big[\frac{\partial\phi(\gamma,\tau)}{\partial\tau}\Big] &= \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\ \frac{\partial\phi(\gamma,\tau)}{\partial\tau}d\gamma d\tau\\ &= \frac{1}{\mu}\int_0^\infty e^{-\frac{\gamma}{\mu}}d\gamma\Big(\int_0^\infty e^{-\frac{\tau}{\vartheta}}\phi_\tau(\gamma,\tau)d\tau\Big).\end{aligned}$$

Using integration by parts, let $\xi=e^{-\frac{\tau}{\vartheta}},\ d\eta=\phi_\tau(\gamma,\tau)d\tau$, then we obtain

$$\begin{aligned}\mathcal{S}_\gamma K_\tau\Big[\frac{\partial\phi(\gamma,\tau)}{\partial\tau}\Big] &= \frac{1}{\mu}\int_0^\infty e^{-\frac{\gamma}{\mu}}d\gamma\Big(-\phi(\gamma,0)+\frac{1}{\vartheta}\int_0^\infty e^{-\frac{\tau}{\vartheta}}\ \phi(\gamma,\tau)d\tau\Big)\\ &= \frac{1}{\vartheta}\Phi(\mu,\vartheta)-\mathcal{S}[\phi(\gamma,0)].\end{aligned}$$

$$(3).\ \mathcal{S}_\gamma K_\tau\Big[\frac{\partial^2\phi(\gamma,\tau)}{\partial\gamma^2}\Big]=\frac{1}{\mu^2}\Big[\Phi(\mu,\vartheta)-K[\phi(0,\tau)]-\mu K[\phi_\gamma(0,\tau)]\Big]. \tag{3.6}$$

Proof.

$$\begin{aligned}\mathcal{S}_\gamma K_\tau\Big[\frac{\partial^2\phi(\gamma,\tau)}{\partial\gamma^2}\Big] &= \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\,\frac{\partial^2\phi(\gamma,\tau)}{\partial\gamma^2}d\gamma d\tau\\ &= \frac{1}{\mu}\int_0^\infty e^{-\frac{\tau}{\vartheta}}d\tau\Big(\int_0^\infty e^{-\frac{\gamma}{\mu}}\phi_{\gamma\gamma}(\gamma,\tau)d\gamma\Big).\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}\mathcal{S}_\gamma K_\tau\Big[\frac{\partial^2\phi(\gamma,\tau)}{\partial\gamma^2}\Big] &= \frac{1}{\mu}\int_0^\infty e^{-\frac{\tau}{\vartheta}}d\tau\Big(-\phi_\gamma(0,\tau)+\frac{1}{\mu}\Big\{-\phi(0,\tau)+\frac{1}{\mu}\int_0^\infty e^{-\frac{\gamma}{\mu}}\phi(\gamma,\tau)d\gamma\Big\}\Big)\\ &= \frac{1}{\mu^2}\Big[\Phi(\mu,\vartheta)-K[\phi(0,\tau)]-\mu K[\phi_\gamma(0,\tau)]\Big].\end{aligned}$$

Similarly, we can prove that:

$$(4).\ \mathcal{S}_\gamma K_\tau\Big[\frac{\partial^2\phi(\gamma,\tau)}{\partial\tau^2}\Big]=\frac{1}{\vartheta^2}\Phi(\mu,\vartheta)-\frac{1}{\vartheta}\mathcal{S}[\phi(\gamma,0)]-\mathcal{S}[\phi_\tau(\gamma,0)]. \qquad (3.7)$$

4. The Double Sumudu-Kamal Transform of Some Elementary Functions

(1). If the function $\phi(\gamma,\tau)=1$, then

$$\mathcal{S}_\gamma K_\tau[\phi(\gamma,\tau)]=\frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}d\gamma d\tau=\vartheta. \qquad (4.1)$$

(2). If the function $\phi(\gamma,\tau)=\gamma\tau$, then

$$\mathcal{S}_\gamma K_\tau[\phi(\gamma,\tau)]=\frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\gamma\tau d\gamma d\tau=\mu\vartheta^2. \qquad (4.2)$$

(3). If the function $\phi(\gamma,\tau)=\gamma^n\tau^m,\ n,m\in\mathbb{N}$, then

$$\mathcal{S}_\gamma K_\tau[\phi(\gamma,\tau)]=\frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\gamma^n\tau^m d\gamma d\tau \;=\; n!m!\ \mu^n\ \vartheta^{m+1}. \qquad (4.3)$$

(4). If the function $\phi(\gamma,\tau)=\gamma^n\tau^m,\ n,m>-1$, then

$$\begin{aligned}\mathcal{S}_\gamma K_\tau[\phi(\gamma,\tau)] &= \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\gamma^n\tau^m d\gamma d\tau\\ &= \Gamma(n+1)\mu^n\Gamma(m+1)\vartheta^{m+1}. \qquad (4.4)\end{aligned}$$

(5). If the function $\phi(\gamma,\tau)=e^{\alpha\gamma+\beta\tau}$, then

$$\begin{aligned}\mathcal{S}_\gamma K_\tau[\phi(\gamma,\tau)] &= \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}e^{n\gamma+m\tau}\ d\gamma d\tau\\ &= \frac{\vartheta}{(1-\alpha\mu)(1-\beta\vartheta)}. \qquad (4.5)\end{aligned}$$

(6). If the function $\phi(\gamma,\tau)=\sin(\alpha\gamma+\beta\tau)$, then

$$\begin{aligned}\mathcal{S}_\gamma K_\tau[\phi(\gamma,\tau)] &= \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\sin(\alpha\gamma+\beta\tau)d\gamma d\tau\\ &= \frac{\vartheta(\alpha\mu+\beta\vartheta)}{(1+\alpha^2\mu^2)(1+\beta^2\vartheta^2)}. \qquad (4.6)\end{aligned}$$

(7). If the function $\phi(\gamma,\tau)=\cos(\alpha\gamma+\beta\tau)$, then

$$\begin{aligned}\mathcal{S}_\gamma K_\tau[\phi(\gamma,\tau)] &= \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\cos(\alpha\gamma+\beta\tau)d\gamma d\tau \\ &= \frac{\vartheta-\alpha\beta\mu\vartheta^2}{(1+\alpha^2\mu^2)(1+\beta^2\vartheta^2)}. \qquad (4.7)\end{aligned}$$

(8). If the function $\phi(\gamma,\tau)=\sinh(\alpha\gamma+\beta\tau)$, then

$$\begin{aligned}\mathcal{S}_\gamma K_\tau[\phi(\gamma,\tau)] &= \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\sinh(\alpha\gamma+\beta\tau)d\gamma d\tau \\ &= \frac{\vartheta(\alpha\mu+\beta\vartheta)}{(1-\alpha^2\mu^2)(1-\beta^2\vartheta^2)}. \qquad (4.8)\end{aligned}$$

(9). If the function $\phi(\gamma,\tau)=cosh(\alpha\gamma+\beta\tau)$, then

$$\begin{aligned}\mathcal{S}_\gamma K_\tau[\phi(\gamma,\tau)] &= \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta})}\cosh(\alpha\gamma+\beta\tau)d\gamma d\tau \\ &= \frac{\vartheta+\alpha\beta\mu\vartheta^2}{(1-\alpha^2\mu^2)(1-\beta^2\vartheta^2)}. \qquad (4.9)\end{aligned}$$

5. Applications

In this section, to establish the efficiency of the suggestion method we consider second- order linear partial differential equations with initial and boundary value problems. Let the second-order nonhomogeneous linear partial differential equation in two independent variables (γ,τ) be in the form:

$$A\phi_{\gamma\gamma}(\gamma,\tau)+B\phi_{\tau\tau}(\gamma,\tau)+C\phi_\gamma(\gamma,\tau)+D\phi_\tau(\gamma,\tau)+E\phi(\gamma,\tau)=f(\gamma,\tau),\ (\gamma,\tau)\in\mathbb{R}^2_+,\qquad (5.1)$$

with the initial conditions:

$$\phi(\gamma,0)=\hbar_1(\gamma),\ \phi_\tau(\gamma,0)=\hbar_2(\gamma), \qquad (5.2)$$

and the boundary conditions:

$$\phi(0,\tau)=\hbar_3(\tau),\ \phi_\gamma(0,\tau)=\hbar_4(\tau), \qquad (5.3)$$

where A,B,C,D and E are constants and $f(\gamma,\tau)$ is the source term.
Using the property of partial derivative of the double Sumudu-Kamal transform for Eq.(5.1), single Sumudu transform for Eq.(5.2) and single Kamal transform for equation Eq.(5.3) and simplifying, we get:

$$\Phi(\mu,\vartheta)=\frac{(\frac{1}{\vartheta}B+D)\hbar_1(\mu)+B\ \hbar_2(\mu)+(\frac{1}{\mu^2}A+\frac{1}{\mu}C)\hbar_3(\vartheta)+\frac{1}{\mu}A\ \hbar_4(\vartheta)+F(\mu,\vartheta)}{\frac{1}{\mu^2}A+\frac{1}{\vartheta^2}B+\frac{1}{\mu}C+\frac{1}{\vartheta}D+E} \qquad (5.4)$$

where $F(\mu,\vartheta)=\mathcal{S}_\gamma K_\tau[f(\gamma,\tau)]$.
Solving this algebraic equation in $\Phi(\mu,\vartheta)$ and taking the inverse double Sumudu-Kamal transform on both sides of Eq.(5.4), yields

$$\phi(\gamma,\tau)=\mathcal{S}_\gamma^{-1}K_\tau^{-1}\Big[\frac{(\frac{1}{\vartheta}B+D)\hbar_1(\mu)+B\ \hbar_2(\mu)+(\frac{1}{\mu^2}A+\frac{1}{\mu}C)\hbar_3(\vartheta)+\frac{1}{\mu}A\ \hbar_4(\vartheta)+F(\mu,\vartheta)}{\frac{1}{\mu^2}A+\frac{1}{\vartheta^2}B+\frac{1}{\mu}C+\frac{1}{\vartheta}D+E}\Big] \qquad (5.5)$$

which represent the general formula for the solution of Eq.(5.1) by double Sumudu-Kamal transform method.

Example 5.1. *Consider the following boundary Poisson equation*

$$\phi_{\gamma\gamma}(\gamma,\tau)+\phi_{\tau\tau}(\gamma,\tau)=\gamma^2+\tau^2, \qquad (\gamma,\tau)\in\mathbb{R}^2_+, \tag{5.6}$$

with the conditions:

$$\begin{cases} \phi(\gamma,0)=0, & \phi_\tau(\gamma,0)=0, \\ \phi(0,\tau)=0, & \phi_\gamma(0,\tau)=0. \end{cases}$$

Solution:
Substituting $F(\mu,\vartheta)=2\mu^2\vartheta+2\vartheta^3$ in Eq.(5.4) and simplifying, we obtain

$$\Phi(\mu,\vartheta)=\frac{2\mu^2\vartheta 2(\mu^2\vartheta+\vartheta^3)}{\mu^2+\vartheta 2}=2\mu^2\vartheta^3 \tag{5.7}$$

Taking the inverse double Sumudu-Kamal transform of Eq.(5.7), we get a solution of Eq.(5.6)

$$\phi(\gamma,\tau)=\mathcal{S}_\gamma^{-1}K_\tau^{-1}[2\mu^2\vartheta^3]=\frac{\gamma^2\tau^2}{2} \tag{5.8}$$

Example 5.2. *Consider the following boundary wave equation*

$$\phi_{\gamma\gamma}(\gamma,\tau)=\phi_{\tau\tau}(\gamma,\tau), \qquad (\gamma,\tau)\in\mathbb{R}^2_+, \tag{5.9}$$

with the conditions:

$$\begin{cases} \phi(\gamma,0)=\sin\gamma=\hbar_1(\mu), & \phi_\tau(\gamma,0)=2=\hbar_2(\mu), \\ \phi(0,\tau)=2\tau=\hbar_3(\vartheta), & \phi_\gamma(0,\tau)=\cos\tau=\hbar_4(\vartheta). \end{cases}$$

Solution:
Substituting

$$\begin{cases} \hbar_1(\mu)=\frac{\mu}{1+\mu^2}, & \hbar_2(\mu)=2, \\ \hbar_3(\vartheta)=2\vartheta^2, & \hbar_4(\vartheta)=\frac{\vartheta}{1+\vartheta^2}, \\ F(\mu,\vartheta)=0, \end{cases}$$

in Eq. (5.4) and simplifying, we obtain

$$\begin{aligned}\Phi(\mu,\vartheta)&=\frac{\mu^2\vartheta^2}{\vartheta^2-\mu^2}\Big[\frac{\vartheta^2-\mu^2}{\mu\vartheta(1+\mu^2)(1+\vartheta 2)}+\frac{2\vartheta^2-2\mu^2}{\mu^2}\Big]\\ &=\frac{\mu\vartheta}{(1+\mu^2)(1+\vartheta 2)}+2\vartheta^2 \end{aligned}\tag{5.10}$$

Taking the inverse double Sumudu-Kamal transform of Eq.(5.10), we get a solution of Eq.(5.9)

$$\phi(\gamma,\tau)=\sin\gamma\cos\tau+2\tau. \tag{5.11}$$

Example 5.3. *Consider the following boundary heat equation*

$$\phi_\tau(\gamma,\tau)=\phi_{\gamma\gamma}(\gamma,\tau)-6\gamma, \qquad (\gamma,\tau)\in\mathbb{R}^2_+, \tag{5.12}$$

with the conditions:

$$\begin{cases} \phi(\gamma,0)=\gamma^3+\sin\gamma=\hbar_1(\mu), & \phi_\tau(\gamma,0)=-\sin\gamma=\hbar_2(\mu), \\ \phi(0,\tau)=0=\hbar_3(\vartheta), & \phi_\gamma(0,\tau)=e^{-\tau}=\hbar_4(\vartheta). \end{cases}$$

Solution:
Substituting

$$\begin{cases} \hbar_1(\mu) = 6\gamma^3 + \frac{\mu}{1+\mu^2}, & \hbar_2(\mu) = -\frac{\mu}{1+\mu^2}, \\ \hbar_3(\vartheta) = 0, & \hbar_4(\vartheta) = \frac{\vartheta}{1+\vartheta}, \\ F(\mu,\vartheta) = 6\mu\vartheta, & \end{cases}$$

in Eq. (5.4) and simplifying, we obtain

$$\begin{aligned} \Phi(\mu,\vartheta) &= \frac{\mu^2\vartheta}{\vartheta-\mu^2}\Big[6\mu^3 - \frac{\mu}{1+\mu^2} + \frac{\vartheta}{\mu(1+\vartheta)} + 6\mu\vartheta\Big] \\ &= 6\mu^3\vartheta + \frac{\mu\vartheta}{(1+\mu^2)(1+\vartheta)} \end{aligned} \tag{5.13}$$

Taking the inverse double Sumudu-Kamal transform of Eq.(5.13), we get a solution of Eq.(5.12)

$$\phi(\gamma,\tau) = \gamma^3 + e^{-\tau}\sin\gamma. \tag{5.14}$$

conclusion

In conclusion, double Sumudu-Kamal transform is an influential transform among all the integral transforms of exponential sort kernels, the double Sumudu-Kamal transform method for solving partial differential equations is studied. We showed the popular properties and theorems for double Sumudu-Kamal transform and equipped some examples.alled the double Sumudu-Kamal. Moreover, some properties, theorems, examples of this transform. The author's also showed the Sumudu-Kamal transforms of some functions.

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