Solving the Heat and Wave Equations Using the Double Aboodh-Summed Transform

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Abstract

We combine the Aboodh transform and Sumudu transform to give another double transform which is called the double Aboodh-Sumudu trans-form. This interesting transform reduces a linear partial differential equation with unknown function of two independent variables to an algebraic equation, which can then be solved by the formal rules of algebra. Also the original differential equation can then be solved by applying the inverse double Aboodh-Sumudu transform.

Key words and phrases. *Double Aboodh-Sumudu, Aboodh transform, Sumudu transform*

1. Introduction

The topic of partial differential equations is one of the most important subjects in mathematics and other sciences. Therefore it is very important to know methods to solve such partial differential equations. One of the most popular method for solving partial differential equations is the integral transform method. In the literature, several different transforms are introduced and applied to find the solution of partial differential equations such as Laplace transform [1], Sumudu transform [7], Aboodh transform [5], Shehu transform [3], and so on. Aboodh transform has deeper connection with Laplace and Sumudu transforms
[15], and the Sumudu transform is a simple variant of the Laplace transform. In recent years, great attention has been given to deal with double integral transforms, see for example [4, 6, 14, 16] and others. Aboodh [12] in 2013 introduced a new integral transform called Aboodh transform, which is derived from the Fourier integral and similar to Laplace transform, and applied it to solve ordinary differential equations. After that, he introduced the double Aboodh transform and used it to solve integral differential equation and partial differential equation [11]. Recently, in 2020, the authors in [2] introduced a new double integral transform called Laplace-Sumudu transform and applied it to solve partial differential equations. The aim of this work is to study a new operator integral transform called double Aboodh-Sumudu Transform with its main properties. In order to illustrate the applicability and efficiency of the double Aboodh-Sumudu transform, we apply this interesting transform to solve the heat and wave equations. Recall that the heat and the wave equations investigate the evolution of temperature and displacement respectively.

1.1. **Definition.** [12] The Aboodh transform of the real function $h(x)$ of exponential order is defined over the set of functions,

$$\mathcal{M} = \Big\{h(x) : \exists K, \tau_1, \tau_2 > 0, |h(x)| < Ke^{|x|\tau_i},\ x \in (-1)^i \times [0, \infty),\ i = 1, 2\Big\},$$

by the following integral

$$A[h(x)] = H(q) = \frac{1}{q}\int_0^\infty e^{-qx}h(x)dx,\ \ \tau_1 \le q \le \tau_2. \tag{1.1}$$

And the inverse Aboodh transform is

$$A^{-1}[H(q)] = h(x) = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} qe^{qx}H(q)dq,\ \ \omega \ge 0. \tag{1.2}$$

1.2. **Definition.** [7] The Sumudu transform of the function $h(t)$ is defined over the set of functions,

$$\mathcal{N} = \Big\{h(t) : \exists M, \rho_1, \rho_2 > 0, |h(t)| < Me^{\frac{|t|}{\rho_j}},\ t \in (-1)^j \times [0, \infty),\ j = 1, 2\Big\},$$

by

$$S[h(t)] = H(r) = \frac{1}{r}\int_0^\infty e^{-\frac{t}{r}}h(t)dt. \tag{1.3}$$

Moreover, the inverse of Sumudu transform is

$$S^{-1}[H(r)] = h(t) = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{1}{r}e^{\frac{t}{r}}H(r)dr,\ \ \omega \ge 0. \tag{1.4}$$

Now, we introduce the definition of our main work.

1.3. **Definition.** The double Aboodh-Sumudu transform of the continuous function $h(y,t)$, $\ y,t > 0$ is denoted by the operator $A_yS_t[h(y,t)] = H(q,r)$ and defined by

$$\begin{aligned} A_xS_t[h(x,t)] = H(q,r) &= \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}h(x,t)dxdt \\ &= \frac{1}{qr}\lim_{\alpha\to\infty,\beta\to\infty}\int_0^\alpha\int_0^\beta e^{-(qx+\frac{t}{r})}h(x,t)dxdt. \end{aligned} \tag{1.5}$$

It converges if the limit of the integral exists, and diverges if not.
And the inverse double Aboodh-Sumudu transform is defined by

$$h(x,t) = A_y^{-1}S_t^{-1}[H(q,r)] = \frac{1}{2\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty} qe^{qx}dq\Big\{\frac{1}{2\pi i}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty}\frac{1}{r}e^{\frac{t}{r}}H(q,r)dr\Big\}, \tag{1.6}$$

where γ_1 and γ_2 are real constants.

2. Existence and uniqueness of double Aboodh-Sumudu transform

In this section, we prove the existence and uniqueness of double Aboodh-Sumudu transform.

2.1. **Definition.** A function $h(x,t)$ is said to be of exponential order $e^{(ax+bt)}$, $a,b>0$ on $[0,\infty)$, if there are positive constants K, X and T such that

$$|h(x,t)| \leq Ke^{(ax+bt)}, \qquad for\ \ all\ x>X,\ t>T,$$

and, we write

$$h(x,t)=o(e^{(ax+bt)}) \qquad as\ (x,t\to\infty).$$

Or, equivalently,

$$\sup_{x,t>0}\Big(\frac{|h(x,t)|}{e^{(ax+bt)}}\Big)<\infty.$$

2.2. **Theorem.** [8] Let $h(x,t)$ be a continuous function in every finite intervals $(0,X)$ and $(0,T)$, and of exponential order $e^{(ax+bt)}$, then the double Aboodh-Sumudu transform of $h(x,t)$ exists for all $q>a$ and $\frac{1}{r}>b$.
Proof. Let $h(x,t)$ be of exponential order $e^{(ax+bt)}$ such that

$$\big|h(x,t)\big|\leq Ke^{(ax+bt)},\ \forall\ x>X,\ t>T.$$

Then, from the definition of double Aboodh-Sumudu transform, we have

$$\begin{aligned}
\big|H(q,r)\big| &= \Big|\frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}h(x,t)dydt\Big|\\
&\leq \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}|h(x,t)|dxdt\\
&\leq \frac{K}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}e^{(ax+bt)}dxdt\\
&= \frac{K}{qr}\int_0^\infty e^{-(q-a)x}dx\int_0^\infty e^{-(\frac{1}{r}-b)t}dt\\
&= \frac{K}{q(q-a)(1-br)}
\end{aligned}$$

Thus, the proof is complete.

2.3. **Theorem.** Let $h_1(x,t)$ and $h_2(x,t)$ be continuous functions defined for $x,t\geq 0$ and having the double Aboodh-Sumudu transform $H_1(q,r)$ and $H_2(q,r)$ respectively. If $H_1(q,r)=H_2(q,r)$, then $h_1(x,t)=h_2(x,t)$.

Proof. Assume γ_1 and γ_2 to be sufficiently large, then since

$$h(x,t)=A_x^{-1}S_t^{-1}[H(q,r)]=\frac{1}{2\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}qe^{qx}\Big\{\frac{1}{2\pi i}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty}\frac{1}{r}e^{\frac{t}{r}}H(q,r)dr\Big\}dq.$$

We deduce that

$$\begin{aligned}
h_1(x,t) &= \frac{1}{2\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}qe^{qx}\Big\{\frac{1}{2\pi i}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty}\frac{1}{r}e^{\frac{t}{r}}H_1(q,r)dr\Big\}dq\\
&= \frac{1}{2\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}qe^{qx}\Big\{\frac{1}{2\pi i}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty}\frac{1}{r}e^{\frac{t}{r}}H_2(q,r)dr\Big\}dq\\
&= h_2(x,t).
\end{aligned}$$

This proves the uniqueness of the double Aboodh-Sumudu Transform.

3. Some Properties of Double Aboodh-Sumudu Transform

3.1. **Linearity property.** If $h(x,t)$ and $g(x,t)$ be two functions such that

$$A_xS_t[h(x,t)] = H(q,r),$$
$$A_xS_t[g(x,t)] = G(q,r).$$

Then for any constants α and β, we have

$$A_xS_t[\alpha h(x,t) + \beta g(x,t)] = \alpha A_xS_t[h(x,t)] + \beta A_xS_t[g(x,t)]. \tag{3.1}$$

Proof. Using the definition of double Aboodh-Sumudu transform, we obtain

$$\begin{aligned} A_xS_t[\alpha h(x,t) + \beta g(x,t)] &= \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}\Big(\alpha h(x,t) + \beta g(x,t)\Big)dxdt \\ &= \frac{\alpha}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}h(x,t)dxdt \\ &+ \frac{\beta}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}g(x,t)dxdt \\ &= \alpha A_xS_t[h(x,t)] + \beta A_xS_t[g(x,t)]. \end{aligned}$$

3.2. **Shifting property.** If $A_xS_t[h(x,t)] = H(q,r)$, then for any pair of real constants $a, b > 0$

$$A_xS_t[e^{(ax+bt)}h(x,t)] = \frac{q-a}{q(1-br)}H\big(q-a, \frac{r}{1-br}\big). \tag{3.2}$$

Proof. Using the definition of double Aboodh-Sumudu transform, we get

$$\begin{aligned} A_xS_t\big[e^{(ax+bt)}h(x,t)\big] &= \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}e^{(ax+bt)}h(x,t)dxdt \\ &= \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-\big((q-a)x+(\frac{1}{r}-b)t\big)}h(x,t)dxdt \end{aligned}$$

Put $z = \frac{r}{1-br}$, then

$$\begin{aligned} A_xS_t\big[e^{(ax+bt)}h(x,t)\big] &= \frac{1}{qz(1-br)}\int_0^\infty\int_0^\infty e^{-\big((q-a)x+\frac{t}{z}\big)}h(x,t)dxdt \\ &= \frac{q-a}{q(1-br)}\frac{1}{(q-a)z}\int_0^\infty\int_0^\infty e^{-\big((q-a)x+\frac{t}{z}\big)}h(x,t)dxdt \\ &= \frac{q-a}{q(1-br)}H\big(q-a, z\big) = \frac{q-a}{q(1-br)}H\big(q-a, \frac{r}{1-br}\big). \end{aligned}$$

3.3. **Changing of scale property.** Let $h(x,t)$ be a function such that

$$A_xS_t[h(x,t)] = H(q,r).$$

Then for a and b are positive constants, we have

$$A_xS_t[h(ax,bt)] = \frac{1}{a^2}H\big(\frac{q}{a}, br\big). \tag{3.3}$$

Proof. We have

$$A_xS_t[h(ax,bt)] = \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}h(ax,bt)dxdt.$$

Let $\tau = ax,\ \upsilon = bt$, then

$$\begin{aligned} A_xS_t[h(\tau,\upsilon)] &= \frac{1}{abqr}\int_0^\infty\int_0^\infty e^{-(\frac{q}{a}\tau+\frac{\upsilon}{br})}h(\tau,\upsilon)d\tau d\upsilon \\ &= \frac{1}{a^2}\frac{1}{\frac{q}{a}br}\int_0^\infty\int_0^\infty e^{-(\frac{q}{a}\tau+\frac{\upsilon}{br})}h(\tau,\upsilon)d\tau d\upsilon \\ &= \frac{1}{a^2}H\big(\frac{q}{a},br\big). \end{aligned}$$

3.4. **Derivatives properties.** If $A_xS_t[h(x,t)] = H(q,r)$, then

$$(1).\ A_xS_t\Big[\frac{\partial h(x,t)}{\partial x}\Big] = qH(q,r) - \frac{1}{q}S[h(0,t)]. \tag{3.4}$$

Proof.

$$\begin{aligned} A_xS_t\Big[\frac{\partial h(x,t)}{\partial x}\Big] &= \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}\frac{\partial h(x,t)}{\partial x}dxdt \\ &= \frac{1}{qr}\int_0^\infty e^{-\frac{t}{r}}dt\Big\{\int_0^\infty e^{-qx}\frac{\partial h(x,t)}{\partial x}dx\Big\}. \end{aligned}$$

Using integration by parts, let $u = e^{-qx},\ dv = \frac{\partial h(x,t)}{\partial x}dx$, then we obtain

$$\begin{aligned} A_xS_t\Big[\frac{\partial h(x,t)}{\partial x}\Big] &= \frac{1}{qr}\int_0^\infty e^{-\frac{t}{r}}dt\Big\{-h(0,t) + q\int_0^\infty e^{-qx}h(x,t)dx\Big\} \\ &= qH(q,r) - \frac{1}{q}S[h(0,t)] \end{aligned}$$

$$(2).\ A_xS_t\Big[\frac{\partial h(x,t)}{\partial t}\Big] = \frac{1}{r}H(q,r) - \frac{1}{r}A[h(x,0)]. \tag{3.5}$$

Proof.

$$\begin{aligned} A_xS_t\Big[\frac{\partial h(x,t)}{\partial t}\Big] &= \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}\frac{\partial h(x,t)}{\partial t}dxdt \\ &= \frac{1}{qr}\int_0^\infty e^{-qx}dy\Big\{\int_0^\infty e^{-\frac{t}{r}}\frac{\partial h(x,t)}{\partial t}dt\Big\}. \end{aligned}$$

Using integration by parts, let $u = e^{-\frac{t}{r}},\ dv = \frac{\partial h(x,t)}{\partial t}dt$, then we obtain

$$\begin{aligned} A_xS_t\Big[\frac{\partial h(x,t)}{\partial t}\Big] &= \frac{1}{qr}\int_0^\infty e^{-qx}dx\Big\{-h(x,0) + \frac{1}{r}\int_0^\infty e^{-\frac{t}{r}}h(x,t)dt\Big\} \\ &= \frac{1}{r}H(q,r) - \frac{1}{r}A[h(x,0)]. \end{aligned}$$

Similarly, we can prove

$$A_yS_t\Big[\frac{\partial^2 h(x,t)}{\partial x^2}\Big]= q^2H(q,r)-S[h(0,t)]-\frac{1}{q}S[h_x(0,t)],$$
$$A_yS_t\Big[\frac{\partial^2 h(x,t)}{\partial t^2}\Big]= \frac{1}{r^2}H(q,r)-\frac{1}{r^2}A[h(x,0)]-\frac{1}{r}A[h_t(x,0)],$$
$$A_yS_t\Big[\frac{\partial^2 h(x,t)}{\partial x\partial t}\Big]= \frac{q}{r}H(q,r)-\frac{q}{r}A[h(x,0)]-\frac{1}{q}S[h_t(0,t)].$$

4. Double Aboodh-Sumudu Transform of some Functions

(1). Let $h(x,t)=1$, then

$$A_xS_t[h(x,t)]=\frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}dxdt=\frac{1}{q^2}. \tag{4.1}$$

(2). Let $h(x,t)=xt$, then

$$A_xS_t[h(x,t)]=\frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}xtdxdt=\frac{r}{q^3}. \tag{4.2}$$

(3). Let $h(x,t)=x^mt^k,\ m,k=0,1,2,\ \dots$, then

$$A_xS_t[h(x,t)]=\frac{1}{q}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}x^mt^kdxdt=\frac{m!\ k!\ r^k}{q^{m+2}}. \tag{4.3}$$

(4). Let $h(x,t)=x^\nu t^\rho,\ \ \nu\geq -1,\ \ \rho\geq -1$, then

$$A_xS_t[h(x,t)]=\frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}x^\nu t^\rho dxdt \ = \ \int_0^\infty\frac{1}{q}e^{-qx}x^\nu dx\int_0^\infty\frac{1}{r}e^{-\frac{t}{r}}t^\rho dt,$$

let $\theta=qx$ and $\varphi=\frac{t}{r}$

$$\begin{aligned} A_xS_t[h(x,t)] &= \frac{1}{q^{\nu+2}}\int_0^\infty e^{-\theta}\theta^\nu d\theta\ \Big\{r^\rho\int_0^\infty e^{-\varphi}\varphi^\rho d\varphi\Big\}\\ &= \frac{\Gamma(\nu+1)}{q^{\nu+2}}\ \Gamma(\rho+1)r^\rho, \end{aligned} \tag{4.4}$$

where, $\Gamma(.)$ is the Euler gamma function.
(5). Let $h(x,t)=e^{(mx+kt)},\ m,k=0,1,2,\ \dots$, then

$$A_xS_t[h(x,t)]=\frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}e^{(mx+kt)}dxdt=\frac{1}{q(q-m)(1-kr)}. \tag{4.5}$$

Similarly,

$$\begin{aligned} A_xS_t[e^{i(mx+kt)}] &= \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}e^{i(mx+kt)}dxdt=\frac{1}{q(q-im)}\frac{1}{(1-ikr)}\\ &= \frac{(q-mkr)+i(m+kqr)}{q(q^2+m^2)(1+k^2r^2)}. \end{aligned} \tag{4.6}$$

Consequently,

$$A_xS_t[\cos(mx+kt)] = \frac{q-mkr}{q(q^2+m^2)(1+k^2r^2)},$$
$$A_xS_t[\sin(mx+kt)] = \frac{m+kqr}{q(q^2+m^2)(1+k^2r^2)}.$$

(6). If $h(x,t)=\sinh(mx+kt)$ or $\cosh(mx+kt)$, $n,m=0,1,2,\ \dots$.
Recall that

$$\sinh(mx+kt) = \frac{e^{(mx+kt)}-e^{-(mx+kt)}}{2},\ \cosh(mx+kt) = \frac{e^{(mx+kt)}+e^{-(mx+kt)}}{2}.$$

Therefore,

$$A_xS_t[\cosh(mx+kt)] = \frac{q+mkr}{q(q^2-n^2)(1-m^2r^2)},$$
$$A_xS_t[\sinh(mx+kt)] = \frac{kqr+m}{q(q^2-m^2)(1-k^2r^2)}.$$

(7). If $h(x,t)=h_1(x)h_2(t)$, then

$$\begin{aligned} A_xS_t[h(x,t)] &= \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qx+\frac{t}{r})}\Big\{h_1(x)h_2(t)\Big\}dxdt \\ &= \frac{1}{q}\int_0^\infty e^{-qx}f_1(x)dx\ \left\{\frac{1}{r}\int_0^\infty e^{-\frac{t}{r}}f_2(t)dt\right\} \\ &= A_x[h_1(x)]\ S_t[h_2(t)]. \end{aligned} \tag{4.7}$$

Therefore,

$$A_xS_t[\sin(ax)\sin(bt)] = \frac{a}{q(q^2+a^2)}\frac{br}{(1+b^2r^2)},$$
$$A_xS_t[\cos(ax)\cos(bt)] = \frac{1}{(q^2+a^2)}\frac{1}{(1+b^2r^2)}.$$

5. Applications

In this section, we apply double Aboodh-Sumudu transform method to linear partial differential equations. Let the second-order nonhomogeneous linear partial differential equation in two independent variables (x,t) be in the form:

$$aU_{xx}(x,t)+bU_{tt}(x,t)+cU_x(x,t)+dU_t(x,t)+eU(x,t) = f(x,t),\ (x,t)\in\mathbb{R}^2_+, \tag{5.1}$$

with the initial conditions:

$$U(x,0)=\mathcal{G}_1(x),\ U_t(x,0)=\mathcal{G}_2(x), \tag{5.2}$$

and the boundary conditions:

$$U(0,t)=\mathcal{G}_3(t),\ U_x(0,t)=\mathcal{G}_4(t), \tag{5.3}$$

where a,b,c,d and e are constants and $f(x,t)$ is the source term. In (5.1), the dependent variable $U=U(x,t)$ depends on the position x and on the time variable

t. Using the property of partial derivative of the double Aboodh-Sumudu transform for equation (5.1), single Aboodh transform for equation (5.2) and single Sumudu transform for equation (5.3) and simplifying, we obtain that:

$$H(q,r) = \left(\frac{(\frac{b}{r^2}+\frac{d}{r})\mathcal{G}_1(q)+\frac{b}{r}\mathcal{G}_2(q)+(a+\frac{c}{q})\mathcal{G}_3(r)+\frac{a}{q}\mathcal{G}_4(r)+F(q,r)}{(aq^2+\frac{b}{r^2}+cq+\frac{d}{r}+e)}\right), \quad (5.4)$$

where $F(q,r) = A_xS_t[f(x,t)]$.

Finally, Solving this algebraic equation in $H(q,r)$ and taking the inverse double Aboodh-Sumudu transform on both sides of equation (5.4), yields:

$$U(x,t) = A_x^{-1}S_t^{-1}\Big[\frac{(\frac{b}{r^2}+\frac{d}{r})\mathcal{G}_1(q)+\frac{b}{r}\mathcal{G}_2(q)+(a+\frac{c}{q})\mathcal{G}_3(r)+\frac{a}{q}\mathcal{G}_4(r)+F(q,r)}{(aq^2+\frac{b}{r^2}+cq+\frac{d}{r}+e)}\Big]. \quad (5.5)$$

Which is the general format for the solution of equation (5.1) by double Aboodh-Sumudu transform method.

Example 5.1. *Consider the following homogeneous Heat equation*

$$U_t(x,t) = U_{xx}(x,t) - U(x,t), \qquad (x,t) \in \mathbb{R}_+^2, \quad (5.6)$$

subject to the initial and boundary conditions

$$U(x,0) = \sin x = \mathcal{G}_1(x),\ \ U(0,t) = 0 = \mathcal{G}_3(t),\ \ U_x(0,t) = e^{-2t} = \mathcal{G}_4(t).$$

Solution. Substituting

$$\mathcal{G}_1(q) = \frac{1}{q(q^2+1)},\ \ \mathcal{G}_3(r) = 0,\ \ \mathcal{G}_4(r) = \frac{1}{1+2r},$$

in (5.5) and simplifying, we get a solution of (5.6)

$$U(x,t) = A_x^{-1}S_t^{-1}\Big[\frac{1}{q(q^2+1)(1+2r)}\Big] = e^{-2t}\sin x. \quad (5.7)$$

Example 5.2. *Consider the following nonhomogeneous Heat equation*

$$U_t(x,t) = U_{xx}(x,t) - 2U(x,t) + 2e^{-x+t} + 4e^t\cos x, \qquad (x,t) \in \mathbb{R}_+^2, \quad (5.8)$$

subject to the initial and boundary conditions

$$U(x,0) = e^{-x} + \cos x = \mathcal{G}_1(x),\ \ U(0,t) = 2e^t = \mathcal{G}_3(t),\ \ U_x(0,t) = -e^t = \mathcal{G}_4(t).$$

Solution. Taking the double Aboodh-Sumudu transform to both sides of equation (5.8) and rearranging the terms, we get

$$H(q,r) = \frac{r\Big(S[U(0,t)]+\frac{1}{q}S[U_x(0,t)]-\frac{1}{r}A[U(x,0)]-F(q,r)\Big)}{(q^2r-2r-1)}. \quad (5.9)$$

Substituting

$$\begin{cases} \mathcal{G}_1(q) = A[U(x,0)] = \frac{1}{q(q+1)} + \frac{1}{(q^2+1)}, & \mathcal{G}_3(r) = S[U(0,t)] = \frac{2}{1-r}, \\ \mathcal{G}_4(r) = S[U_x(0,t)] = \frac{-1}{1-r}, & F(q,r) = \frac{2}{q(q+1)(1-r)} + \frac{4}{(q^2+1)(1-r)}, \end{cases}$$

in (5.9) and simplifying, we get

$$H(q,r) = \frac{1}{q(q+1)(1-r)} + \frac{1}{(q^2+1)(1-r)}. \quad (5.10)$$

Taking the inverse double Aboodh-Sumudu transform, we get

$$\begin{aligned} U(x,t) &= A_x^{-1}S_t^{-1}\Big[\frac{1}{q(q+1)(1-r)}+\frac{1}{(q^2+1)(1-r)}\Big] \\ &= e^{-x+t}+e^t\cos x. \end{aligned} \tag{5.11}$$

Which is the required solution of the considered nonhomogeneous Heat equation.

Example 5.3. *Consider the following homogeneous Wave equation*

$$U_{tt}(x,t)=U_{xx}(x,t), \qquad (x,t)\in\mathbb{R}_+^2, \tag{5.12}$$

subject to the initial and boundary conditions

$$\begin{cases} U(x,0)=1+x=\mathcal{G}_1(x), & U_t(x,0)=\sin x=\mathcal{G}_2(x), \\ U(0,t)=1=\mathcal{G}_3(t), & U_x(0,t)=1+\sin t=\mathcal{G}_4(t). \end{cases} \tag{5.13}$$

Solution. Substituting

$$\begin{cases} \mathcal{G}_1(q)=\frac{1}{q^2}+\frac{1}{q^3}, & \mathcal{G}_2(q)=\frac{1}{q(q^2+1)}, \\ \mathcal{G}_3(r)=1, & \mathcal{G}_4(r)=1+\frac{r}{1+r^2}, \end{cases}$$

in (5.5) and simplifying, we get a solution of (5.12)

$$\begin{aligned} U(x,t) &= A_x^{-1}S_t^{-1}\Big[\frac{1}{q^2}+\frac{1}{q^3}+\frac{1}{q(q^2+1)}.\frac{r}{(1+r^2)}\Big] \\ &= 1+x+\sin x\sin t. \end{aligned} \tag{5.14}$$

Example 5.4. *Consider the following nonhomogeneous Wave equation*

$$2U_{tt}(x,t)=U_{xx}(x,t)-U(x,t)+2x^2-4, \qquad (x,t)\in\mathbb{R}_+^2, \tag{5.15}$$

subject to the initial and boundary conditions

$$U(x,0)=2x^2+\sin x=\mathcal{G}_1(x),\ \ U_t(x,0)=0=\mathcal{G}_2(x),\ \ U(0,t)=0=\mathcal{G}_3(t),\ \ U_x(0,t)=\cos t=\mathcal{G}_4(t).$$

Solution. Applying the double Aboodh-Sumudu transform on both sides of equation (5.15), we have

$$A_xS_t[2U_{tt}(x,t)]=A_xS_t[U_{xx}(x,t)-U(x,t)+2x^2-4].$$

By linearity property and partial derivatives properties of double Aboodh-Sumudu transform and rearranging the terms, we get

$$H(q,r)=\frac{r^2\Big(S[U(0,t)]+\frac{1}{q}S[U_x(0,t)]-\frac{2}{r^2}A[U(x,0)]-\frac{2}{r}A[U_t(x,0)]-F(q,r)\Big)}{(q^2r^2-r^2-2)} \tag{5.16}$$

Substituting

$$\begin{cases} \mathcal{G}_1(q)=A[U(x,0)]=\frac{4}{q^4}+\frac{1}{q(q^2+1)}, & \mathcal{G}_4(r)=S[U_x(0,t)]=\frac{1}{1+r^2}, \\ F(q,r)=\frac{4}{q^4}-\frac{4}{q^2}, & \end{cases}$$

in (5.16) and simplifying, we get

$$H(q,r)=\frac{4}{q^4}+\frac{1}{q(q^2+1)(1+r^2)}. \tag{5.17}$$

Taking the inverse double Aboodh-Sumudu transform of equation (5.17), we get

$$\begin{aligned} U(x,t) &= A_x^{-1} S_t^{-1}\Big[\frac{4}{q^4} + \frac{1}{q(q^2+1)(1+r^2)}\Big] \\ &= 2x^2 + \sin x \cos t. \end{aligned} \tag{5.18}$$

Which is the required solution of the considered nonhomogeneous Wave equation.

conclusion

In this paper, double Aboodh-Sumudu transform is introduced and its properties are proved. To see the efficiency of Aboodh-Sumudu transform, we applied this new transform on four different examples that are related to the Euclidean dimension and the time dimension and the results show that the double Aboodh-Sumudu transform method is an appropriate method for solving the second-order linear partial differential equation in two independent variables.

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