Correspondence between two bicolored plane trees And generalized Tchebytchev polynomial

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DOI: https://doi.org/10.56807/buj.v1i2.23

Abstract

In this paper, we give some theoretical results, for the Tchebytchev polynomials of a tree. A Tchebytchev polynomial, any polynomial has at most two critical values. we consider two bicolored plane trees (two neighboring vertices are always indifferent colorings).We will introduce the composition and decomposition of two bicolored plane trees A and B denoted by (A o B) and we are interested in the relation between two bicolored plane trees and generalized Tchebytchev polynomials.

الملخص

سوف نقدم في هذا البحث بعض النتائج عن العلاقة بين two bicolored plane trees مع generalized Tchebytchev polynomial

ونعني بذلك كثيرة الحدود Tchebytchev التي لها قيمتان حرجتان وحساب هذه القيم وكيفية تركيب وتحليل القمم المتجاورة في كثيرة الحدود بصورة عامة .

Keywords: *Graph, Tree, Plane Trees, Distance , bicolored, Composition Tchebytchev polynomials, Decomposition*
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Introduction

In this paper we study the Correspondence between two bicolored plane trees and generalized Tchebytchev polynomial also composition and decomposition for generalized Tchebytchev polynomial. It is well known that a tree is a connected graph without circuits. Each tree can be drawn on the plane in a number of ways. We are going to introduce an additional structure on a tree to fix the type of picture of this tree. A graph is a diagram consist of a set (which supposes we are finished here) points $s_1, s_2, \dots, s_n$. By using a set of edges we can connect every two points of vertices. Those points by a set edges of the graph are called vertices of the graph. A connected graph is a graph such as that for any pair of two distinct vertices, there exists a chain connecting these two points. It is called Plane trees, all connected graphs except cyclic are drawn in that plane
[2, 6, 7, 16]. A bicolored plane tree is a tree whose vertices are colored consecutively black and white. Generally each plane tree is related to two bicolored plane trees [1, 3, 17]. The distance between two distinct vertices v_i, v_j of the tree T and the path length in accordance (with the number of edges) connecting v_i, v_j is defined and denoted by $d(v_i, v_j)$
[4, 5]. The distance study of a tree has been made objective in several publications [14, 15, 18, 19, 20]. The notion of TChebyshev polynomials is well known. They are useful in various fields of studies. A lot of generalizations of them have been found and investigated. In this paper we shall talk about one of them, which can also be referred to as Shabat polynomials [8, 9]. Here we shall discuss one-to-one correspondence of Shabat polynomials, plane trees and finite extensions of rational numbers.

Definition:

a) A plane tree is a tree with a prescribed cyclic order of edges adjacent to each vertex.

b) A bicolored plane tree is a tree T embedded in a plane (with no edge crossings) such that each vertex of T has one of two colors, and each edge contains one vertex of each color.

Definition: Plane trees are called isomorphic if there exists an isomorphism of trees as graphs which also preserves cyclic permutations from the definition of plane trees[8, 9].

The equivalence class of plane tree is called a combinatorial plane tree.

Every tree has a natural structure of a bipartite graph: its vertices can be colored in two colors. So through fixing one of the two colorings we obtain a bicolored plane tree.

Definition: A Tchebytchev polynomial, any polynomial has at most two critical values.

Definition: Let α ∈ ℂ and A a complex polynomial

1. We say that α is a critical point of A if À(α) = 0.
2. if α is a critical point of A, then the order of α is the smallest integer k when $A^{(k)}$ (α) ≠ 0 where $A^{(k)}$ is the derivative k-eme of A.
3. If exists α ∈ ℂ when À(α) = 0, then A(α) is applied critical value of the polynomial A.

Example: Consider the polynomial

$$A(z) = z^5 - 8z^4 + 24z^3 - 37z^2 + 30z - 8$$

The polynomial is featured in the following form

$$A(z) = (z-1)^2 (z-2)^3.$$

1. The polynomial A has two critical points $\alpha_0 = 1$ and $\alpha_2 = 2$ because À (1) = À (2) = 0 when the order of α_0 equals 1 and the order of α_1 equals 2.
2. A has only one critical value w = 0 because A(1) = A(2) = 0.

Now A is a fixed polynomial whose only critical values are 0 and 1.

In other words, any critical point of A is a root of A or else A – 1. Let

$$\Gamma = A^{-1}([0, 1]).$$

So Γ consists of a collection of segments that connect the roots of A to the roots of A - 1 sounds cut themselves. From where Γ is the drawing of a planar graph whose vertices are alternatively roots of A and of A - 1. That explains well what is a plane tree. In the following, we will call point black (respecting, white) the apices corresponding to the roots of A (respecting, A -1). The question now is, can we build a polynomial A when the plane tree Γ given, such as Γ = A^{-1} ([0, 1]). The rethinking is given in the next theorem:

Theorem 1: For any plane trees Γ, there is a polynomial A admitting for critical values at most 0 and 1, and such as the image reciprocal of the segment [0, 1] by A isomorphic to Γ. If more A is unique to both of the following transformations[8].

i- Switch z in az + b (direct similarity).

ii- Switch A in A - 1 (exchanging summit colors).

Main results

In this paper we studied Correspondence between two bicolored plane trees and generalized Tchebytchev polynomial , and we studied (composition and decomposition) for Tchebytchev polynomial generalized. Also we calculated Tchebytchev polynomials of plane trees.

Calculation of generalized Tchebytchev polynomial

We know that every generalized Tchebytchev polynomial T_n of degree n∈ N has two critical values 1 and -1. If we ask

$$A = \frac{T_n - 1}{2}.$$

We get the polynomial A has two critical values 0 and 1. So A is a polynomial of Tchebytchev, Where to calculate the polynomial T_n, it suffices to calculate the generalized Tchebytchev polynomial A.

Definition: The body of definition of a plane tree Γ is the body of definition of the coefficients of a generalized Tchebtchev polynomial associated to Γ.

Theorem 2: If Γ is a plane tree, then there exists a minimal algebraic body K of definition of the coefficients of a generalized Tchebytchev polynomial associate Γ. The following proposition assures the existence of a plane tree from the data of a generalized Tchebtchev polynomial.

Proposition: Let A a non-constant polynomial admitting for only values
critics 0 and 1, in other words :
À (z) = 0 → A(z) = 0 ; A(z) = 1.
then Γ = A^{-1} ([0,1]) is a plane trees.

Proof: We note that Γ cannot have a circuit because A cannot be zero
on a circuit sounds be zero in the entire region bounded by this circuit. If
not, according to the principle maximum of the holomorphic and bounded functions, A is zero on the region bounded by this circuit. So
A = 0 hence the contradiction with the fact that A is not constant, n ∈ N the degree of A,

$$A(z) = \lambda \prod_{i=1}^{p} (z - a_i)^{\alpha_i}$$

and

$$A(z) - 1 = \lambda \prod_{i=1}^{q} (z - b_i)^{\beta_i}$$

the early decompositions, respectively, of A and A - 1 or λ ∈ $\mathbb{C}^*$. It is clear
that.

By solving the system of equations we can find all the polynomials which correspond to the trees of this family.

Here< is , example , of , trees , and >their polynomials: $\alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \beta_2, \ldots, \beta_q$

$$\sum_{i=1}^{p} \alpha_i = \sum_{i=1}^{q} \beta_i = n$$

It is also clear that if α_i> 1 and β_i> 1, then a_i and b_i are roots of À orders
of multiplicities, respectively, α_i- 1 and β_i – 1,

$$\deg(\grave{A}) = \sum_{i=1}^{p} (\alpha_i - 1) + \sum_{i=1}^{q} (\beta_i - 1) = n - 1$$

Where

$$\sum_{i=1}^{p} \alpha_i - p + \sum_{i=1}^{q} \beta_i - q = n - 1$$

Therefore, 2n - p - q = n – 1. This gives that $p + q = n + 1$.
Also Γ is connected and therefore Γ is a plane tree.
Fix, in the whole suite:

$$A(z) = \lambda \prod_{i=1}^{p} (z - a_i)^{\alpha_i}$$

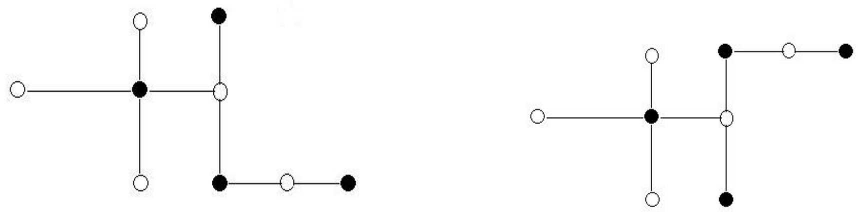
and

$$A(z) - 1 = \lambda \prod_{i=1}^{q} (z - b_i)^{\beta_i}$$

It is quite clear that the plane tree associated with A includes p blacks vertex and q whites vertex. Let $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_p$ and $\beta = \beta_1, \beta_2, \ldots, \beta_q$ orderly in the decreasing sub.

Definition: Set of degrees
Set of degrees of a plane tree associates á A is the couple (α, β). We will also say that the tree is of type

Example: The type tree < 4211; 32111 > is drawn in the complex plane as shown in figure (1).



A_p A_Q

Figure 1: The trees A_p and A_q

Proposition: If p, q ∈ N such as p + q = n + 1, then there is at least a two-colored tree admitting these partition for set of degrees.

Proof:
It is easy to verify that the assumptions imply that for n≥2, we have
To $\alpha_1 > 1$ and $\beta_q = 1$ (or the opposite); by hypothesis of recurrence, there is a type tree $< (\alpha_1 - 1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_{q-1} >$. So just connect the degree vertex $\alpha_1 - 1$, a new blank sheet to get a tree from desired type. □
The calculation of the generalized Tchebtchev polynomial A associated with a type tree

$$< \alpha_1, \alpha_2, \dots, \alpha_p, \beta_1, \beta_2, \dots, \beta_q >$$

returns to finding complex value

$$\lambda, a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q$$

In between values a_i *and* b_i must be all distinct. Let's divide the polynomials A and A - 1 by the scalar λ we can assume that it's losing generality, then

$$\mathrm{A(z)} = \prod_{i=1}^{p} (z - a_i)^{\alpha_i}$$

and

$$\mathrm{B(z)} = \mathrm{A(z)} - 1$$

$$= \prod_{i=1}^{q} (z - b_i)^{\beta_i}$$

It is clear that À = B` this provides a system of linear equations of n - 1
algebraic relations and p + q = n - 1 unknown. We know that if a_i (respecting, b_i) a root of A (respecting, B) of multiplicity order $\alpha_i > 1$ (respecting, $\beta_i > 1$) then $a_i$is a root of À (respecting, B`) of multiplicity order $\alpha_i - 1$,

$$\mathrm{pgcd(A, \grave{A})} = \prod_{i=1}^{p} (z - a_i)^{\alpha_{i-1}}$$

$$\mathrm{pgcd}(\mathrm{B}, B`) = \prod_{i=1}^{q} (z - b_i)^{\beta_{i-1}}$$

Let R = pgcd(A, À) and S = pgcd(B, B`). There are two polynomials T and U when À = RT and $B$` = SU by R Split À and S Split B`. As À = $B$`, RT = SU. So R Split SU and S Split RT. So R and S are first between them (because $a_i \neq b_i \forall i$) R divide U and S Split T by Gauss theorem. Since *deg(R)* = *deg(U)* and *deg(S)* = *deg(T)* and the term coefficient z^{n-1} in the *polynomial* À, *U* = *nR* and *T* = *nS*.

Example: Consider the type tree < 411; 3111 >. It is clear that this
tree admits a black vertex (respecting, White) of order 4 (respecting, 3). So we can

place them arbitrarily in 0 and 1. In other words, 0 is a root of the generalized Tchebyshev polynomial A of order 4 and 1 , is a 3 order root of the polynomial B(z) = A(z) - 1

Where

$$A(z) = z^4(z^2+a_1 z + a_0),$$
$$B(z) = (z-1)^3(z^3+ b_2 z^2+ b_1 z+b_0)$$

this implies

$$\grave{A}(z) = z^3 (6z^2+ 5a_1 z+4 a_0)$$
$$B`(z) = (z-1)^2(6z^3+(-3+5 b_2) z^2+(-2b_2 +4 b_1 z) - b_1 +3 b_0)$$

$\grave{A} = B`$, using the method of comparing gets the following linear equation system:

$4a_0 = 6$
$5a_1 = -12$
$-b1 + 3 b_0 = 0$
$-2b_2 +4 b_1 = 0$
$-3+5 b_2 = 0$

Where

$a_0 = \frac{3}{2}$
$a_1 = \frac{-12}{5}$
$b_0 = \frac{1}{10}$
$b_1 = \frac{3}{10}$
$b_2 = \frac{3}{5}$

Then

$$A(z) = z^4 (z^2 + \frac{-12}{5} z + \frac{3}{2})$$
$$B(z) = (z-1)^3 (z^3+ \frac{3}{5} z^2 + \frac{3}{10} z + \frac{1}{10})$$

Where A(z) - B(z) = $\frac{1}{10}$. That implies that 10×A(z) - 10×B(z) = 1. Then the generalized Tchebyshev polynomial given by

$$A(z) = 10 \times z^4 (z^2 - \frac{-12}{5} z + \frac{3}{2})$$
$$= z^4 (10z^2 - 24z + 15).$$

Composition of the plane tree

It is well-known that a tree is a connected graph without circuits. Each tree can be drawn on the plane in a number of ways. We are going to introduce an additional structure on a tree to fix the type of picture of this tree.

Proposition: Under the preceding assumptions one has

1. if α_i is the degree of a white top (respecting, black) not marked of the tree A_p, then one finds *deg(Q)* white tops (respecting; black) in the tree A_R, and of degree α_i.

2. if α_i is the degree of a black top (respecting, white), marked by □ in the tree A_p, then all tops marked by one □ of the tree A_Q , of degree γ_j(j =1, …, r) find in the tree A_R,with the black color (respecting, white) and of degree $\alpha_i \times \gamma_j$(j =1, …, r).

3. If α_i is the degree of a black top (respecting, white), marked by Δ in the tree A_p, then all them tops marked by one Δ of the tree A_Q, of degree γ_j(j =1, …, s), find themselves in the tree A_R, with the white color (respecting, black), and of degree $\alpha_i \times \gamma_j$(j =1, …, s).

Proof: Let us suppose

$$P(z) = \lambda \prod_{i=1}^{k} (z - a_i)^{\alpha_i}$$

,

$$Q(z) = \delta \prod_{i=1}^{r} (z - c_i)^{\theta_i}$$

,

$$P(z) - 1 = \lambda \prod_{i=1}^{l} (z - b_i)^{\beta_i}$$

,

$$Q(z) - 1 = \delta \prod_{i=1}^{l} (z - d_i)^{\lambda_i}$$

.

Sets of degrees of P and Q respectively are $[(\alpha_1, \alpha_2, \dots, \alpha_k), (\beta_1, \beta_2, \dots, \beta_p)]$ and $[(\theta_1, \theta_2, \dots, \theta_k), (\lambda_1, \lambda_2, \dots, \lambda_p)]$. Let us suppose that one of black tops of A_p of α_i is marked by Δ, then of suit the division Euclidean of P by $(z-1)^{\alpha_i}$ a polynomial is obtained P_1 such as

$P(z) = (z-1)^{\alpha_i} P_1(z)$ with $P_1 \neq 0$ and $\alpha_i > 0$. One has

$$R(z) = P \circ Q(z)$$

$$= P(Q(z))$$

$$= (Q(z)-1)^{\alpha_i} \times P_1(Q(a))$$

$$= (\delta \prod_{i=1}^{s} (z - d_i)^{\lambda_i})^{\alpha_i} \times P_1(Q(z))$$

$$= \delta^{\alpha_i} \prod_{i=1}^{s} (z - d_i)^{\lambda_i \alpha_i} P_1(Q(z))$$

Thus for all s ≤ i≤1, dj is a root of R of order of multiplicity $\lambda_i \times \alpha_i$.

To finish the demonstration, it is enough to take in consideration the other case.

Composition: geometrical description

The presentation which follows is adapted to the trees and for this reason one does not repeat word for word what we described . All is when self based on the principal observation : for the composition trees, the fundamental circuit which must pass by both points □ and Δ a version easy to describe. To know, it is the single way in the tree A_p who connects □ with Δ. One calls "vertebra" the only way which binds the two tops marked of A_p and then the tree is cut out A_p in three parts : "the head ","the body " and "the tail " :

● "The body" is the part of the tree composed by the union of all branches related to "vertebra", except those which are dependent at the tops □ and Δ. While removing "the body", there remain two parts of the tree.

● "The head" : is the union of all the branches connected to the top □, except that connected to "vertebra".

● "The tail": is the union of all the branches connected to the top Δ, except that connected to "vertebra".

On the tree A_Q one proceeds in the following manner :

On each edge of the tree A_Q one inserts the "body " of the tree A_Q,by respecting the orientation of □ at Δ. With each top □ (respecting; Δ) one attaches the "head " (respecting; the "tail") of the tree A_p, as many time as the degree of the if top the degree top □ (respecting; Δ) is higher than 1, one insert the heads (respecting; tails) in the angles between two bodies.

Composition of generalized Tchebytchev polynomial

As already mentioned, the calculation of generalized Tchebytchev polynomial becomes very quickly impractical in general. So to go further we will try to build them by composition, thanks to the following lemma

Lemma: Let P and Q two of generalized Tchebytchev polynomial,
with the values critics fixed 0 and 1. Suppose the polynomial P checked the following condition

P(0) and P(1) $\in \{0,1\}$

$$R'(z) \Leftrightarrow Q'(z).\,P'(Q(z))$$

Then the composed polynomial R(z) = P o Q(z) is also a of generalized Tchebytchev polynomial.

Proof: We have R = P o Q, $R'(z) = Q'(z) .\, P'(Q(z))$ for z $\in \mathbb{C}$. Let's now look at the critical points of R.

$$\Leftrightarrow Q'(z) = 0 \text{ or } P'(Q(z)) = 0$$

To conclude we will distinguish two cases

1- If $P'(Q(z)) = 0$, then Q(z) is a root of P' and as a result P(Q(z)) is a critical value of P. Where R(z) = P(Q(z)) $\in \{0,1\}$, R is a of generalized Tchebytchev polynomial.

2- If $Q'(z) = 0$, then z is a critical point of Q and Q(z) is a value criticism of Q, Q(z) $\in \{0,1\}$, Where R(z) = P(Q(z)) $\in \{P(0), P(1)\}$ because P(0) and P(1) $\in \{0,1\}$ by hypothesis. This shows that R is a of generalized Tchebytchev polynomial.

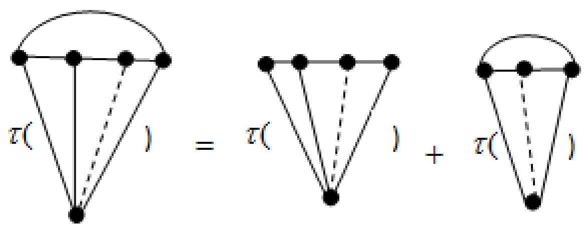


Figure 2: Composition of two polynomials

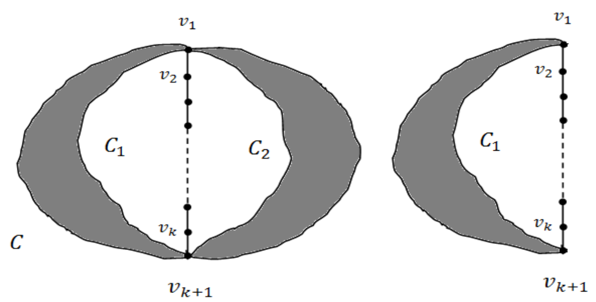


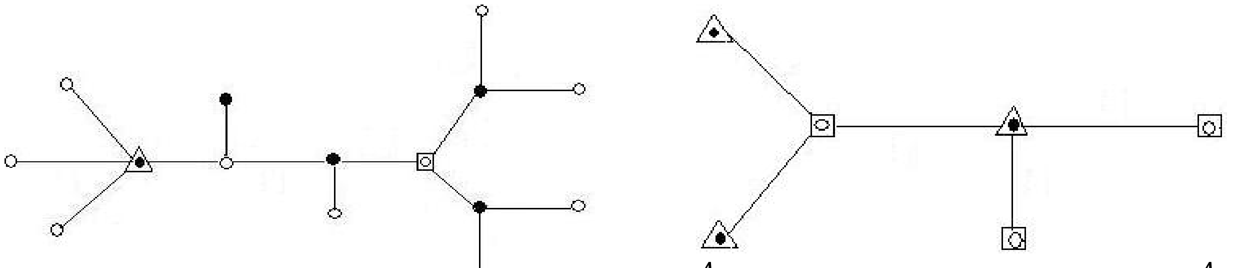
Figure 3: Composition of two Tchebytchev polynomials

The previous lemma gives us the opportunity to introduce the composition of two trees. The condition P(0); P(1) $\in \{0,1\}$,. of lemma (1), is geometrically translated by the fact that two vertex of the tree A_p are distinguished, and placed at the points u = 0 and u = 1. to distinguish them from the other vertices, we mark the one placed at u = 0 by a square □, and we note it by t, we mark the one place to u = 1 by a triangle Δ and we note it q (see the figure 2) the vertices of the tree A_Q which are the images reciprocal points t and q, are marked by squares and triangles instead of black and white colors (see figure 3).

Example: Let A_P and A_Q the two trees in the figure (4) Whose passports are respectively

$$\pi_1 = [1 : 4\Delta 3^3 1,\ 2 : 33\square 11111111]$$

$$\pi_2 = [\square : 311,\ \Delta : 311]$$



A_p A_Q

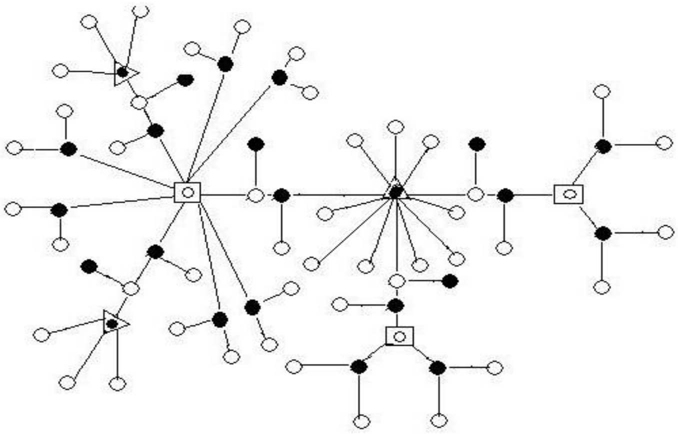
Figure 4: The trees A_p and A_Q

Two vertices of the tree A_p are marked by Δ and □. The black vertices of 4 degree marked by Δ given a degree vertices 3×3 in the tree A_R. The white vertices of degree 3 marked by □ gives a degree vertices 3×3.

The black and white vertices in A_p that are not marked give 3 vertices of the same glue and same degree. So the passport of A_R is given by

$$\pi_R = [1 : ((12)44)\ \Delta 3^{15}\ 1^5,\ 2 : 3^5\ (933)\ \square 140]$$

and in the figure it is drawn as follows:



A_R

Figure 5: The tree A_R is the composition of two trees A_p and A_Q

Composition Combinatorics

Let A and B, respectively, the sets of the edges of trees A_p and A_Q. In the tree A_p , we note, by t the edge which connects the "body" to the "head" of the tree and by q the edge that connects the "body" to the "tail" of the tree. With these ratings we have the following theorem:

Theorem 3: Let A_p and A_Q two bicolored plan trees coded by the permutation pairs $(g_\bullet, g_o)$ and $(g_\square, g_\Delta)$ defined on sets A and B. then the tree A_R and determined by the permutations $(G_\bullet, G_o)$ defines on the set A × B (Cartesian product of two sets) by

$$G_o(a,b) = \begin{cases} (g_o(a), b) & \text{if } a \neq t;\ q, \\ (g_o(a), g_\square(b)) & \text{if } a = t \text{ and the head is white}, \\ (g_o(a), g_\Delta(b)) & \text{if } a = q \text{ and the tail is black}, \end{cases}$$

and we define the same $G_\bullet$ replacing $g_o$by $g_\bullet$ and "white" by "black".

Example: Consider the finite sets A ={ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13} and B = {1, 2, 3, 4} the permutations $g_\bullet = (1,2,3)(5,8,6)(10,12,11)$ and $g_o = (3,4,5)(6,7)(8,9,10)(12,13)$ who encode the tree A_p and operate transitively on A. Let $g_\square = (2,3,4)$ and $g_\Delta = (1,2)$ the two permutations that encode the tree A_Q, (see the figure 6). In the tree A_p the ridge that wears the number 5 connects the head to the body at the tail. Apply the theorem 5 we find the permutations $G_\bullet$ and G_o which encode the tree A_R and which operate on the whole

A × B = {(x, y)/x = 1, 2,… , 13 and y = 1, 2, 3, 4}:

$G_\bullet$ = {(10, 1)(12, 2)(11, 2)(10, 2)(12, 1)(11, 1)}{(10, 3)(12, 3)(11, 3)}
{(10, 4)(12, 4)(11, 4)}{(1, y)(2, y)(3, y)}{(5, y)(8, y)(6, y)}
for y = 1, 2, 3, 4

G_o = {(3, 1)(4, 1)(5, 1)}{(3, 2)(4, 2)(5, 2)(3, 3)(4, 3)(5, 3)(3, 4)(4, 4)(5, 4)}{(6, y)(7, y)}{(8, y)(9, y)(10, y)(12, y)(13, y)}
for y = 1, 2, 3, 4

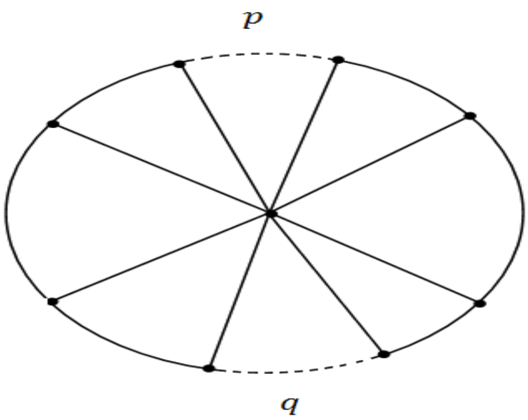


Figure 6: Sets of edges of trees

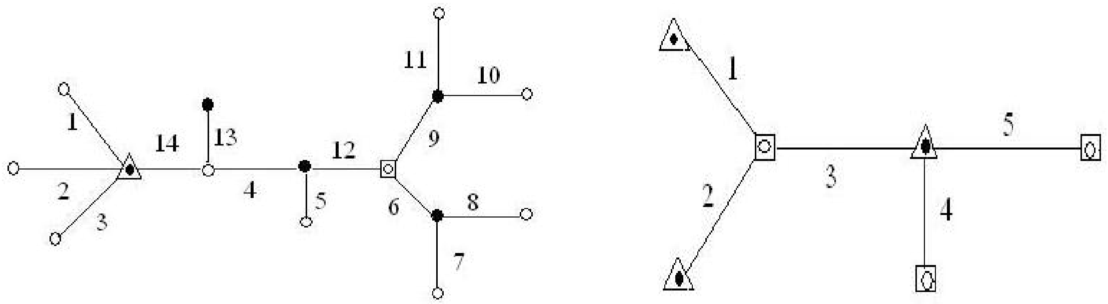
Example: Consider the finite sets
A ={ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14}

and B = {1, 2, 3, 4, 5} The permutations that encode (see figure 7) the tree A_p and operate transitively on A are given by

$$g_{\bullet} = (1, 2, 3, 14)(4, 5, 12)(6, 7, 8)(9, 10, 11)$$

$$g_o = (4, 13, 14)(6, 9, 12)$$

The permutation that encodes the tree A_Q are g_{Δ} (3, 4, 5) and $g_{\square}$ = (1, 2, 3).



A_p A_Q

Figure 7: The trees A_p and A_Q

In the tree A_p the ridge that carries the memo t = 12 connects the head to body at the tail apply the theorem 3 we find permutation $G_{\bullet}$ and G_o which encode the tree A_{PoQ} and which operate on the whole

$$A \times B = \{(x, y)/x = 1, 2, \ldots, 14 \text{ and } y = 1, 2, 3, 4, 5\}$$

as a result

$G_o =$ {(4, 2)(4, 3)(4, 1)(4, 4)(4, 5)}{(6, 2)(6, 3)(6, 1)(6, 4)(6, 6)}
{(4, y)(13, y)(6, y)(9, y)}
for y = 1, 2, 3, 4, 5

$G_{\bullet}$ = {(1, 1)(1, 2)(1, 4)(1, 6)(1, 3)}{(4, 1)(4, 2)(4, 4)(4, 5)(4, 3)}
{(1, y)(2, y)(3, y)(4, y)(5, y)(6, y)(7, y)(8, y)(9, y)(10, y)(11, y)}
for y = 1, 2, 3, 4, 5

Decomposition of polynomials

The starting point is the decomposition of polynomials and rational functions in one variable. First, we will define the basic concepts of this topic.

Definition. If $f = g \circ h$, *f, g, h* ∈ *K(x)*, we call this a decomposition off in K(x) and say that g is a component on the left of *f* and h is a component on the right of *f*.

We call it a decomposition trivial if any of the components is a unit with respect to decomposition. Given two decompositions $f = g_1 \circ h_1 = g_2 \circ h_2$ of a rational function, we call them equivalent if there exists a unit u such as that $h_1 = u \circ h_2$, $g_1 = g_2 \circ u^{-1}$

where the inverse is taken with respect to composition. Given a non-constant f, we say that it is indecomposable, it is not a unit and all its decompositions are trivial [10; 11; 12; 13]. Ritt studied the functional decomposition of a un variety complex polynomial f into prime (indecomposable) polynomials, f = u_1 o u_2 …o u_r. His main achievement was a procedure for obtaining any decomposition of f from any other by repeatedly applying certain transformations. However, Ritt's results provide no control on the number of times one must apply the basic transformations, which makes his procedure unsuitable for many theoretical and algorithmic applications. We solve this problem by giving a new description of the collection of all decompositions of a polynomial. One consequence is as follows: if f has degree, n > 1 but f is not conjugate by a linear polynomial to either X^n or T_n (with T_n the Tchebychev polynomial), and if the composition a o b of polynomials a, b is the K^{th} iterate of f for some $k > log_2$(n+2), then either $a = f$ o c or $b = c$ o f for some polynomial c. This result has been used by Ghioca, Tucker and Zieve to describe the polynomials f, g having orbits with infinite intersection; our results have also been used by Medevedev and Scanlon to describe the affine varieties invariant under a coordinate wise polynomial action. Ritt also proved that the sequence (deg(u_1),…, deg(u_r)) is uniquely determined by f, up to permutation.

Theorem 4: A coating $h : X \to Z$ is decomposable, $h = g$ o f, $f : X \to Y$, $g : Y \to$Z, if and only if the H monodrama group of the h coating is imprimatur.

Decomposition of generalized Tchebytchev polynomial

The problem comes next. If a constellation [h_1,…, h_i], $h_j \in s_{mn}$ which corresponds to a function $h : X \to Z$ degree $deg(h) = mn$. How to decide whether or not this function is decomposition : $h = g$ o f, and if yes how to find the corresponding constellations f and g. It is more natural to discuss all the problems that arise on an example. But before presenting this example we will give an example of decomposition (analytic) of a given polynomial

Example: Consider the polynomial $R(z) = -z^4 + 2z^4$ and ask $P(x) = -x^2 + 2x$ and $Q(z) = z^2$. We note that $R(z) = PoQ(z)$. This shows that the R polynomial is decomposable in two polynomials P and Q. Let us first show that R is a generalized Tchebytchev polynomial. Indeed, let's look at the critical points of R

$$R'(z) \Leftrightarrow -z^4 + 2z^4 = 0$$
$$\Leftrightarrow 4z(-z^2+1) = 0$$
$$\Leftrightarrow 4z(1-z)(1+z) = 0$$
$$\Leftrightarrow z = o, z = 1 \; or \; z = -1$$

We have $R(0) = 0$ and $R(1) = R(-1) = 1$. Which leads to R has two critical values 0 and 1, R is a generalized Tchebytchev polynomial.

$$\begin{aligned} R'(z) &= -z^4 + 2z^2 \\ &= z^2(-z^2+2) \\ &= z^2(\sqrt{2}-z)(\sqrt{2}+z) \end{aligned}$$

and

$$\begin{aligned} R'(z) - 1 &= -z^4 + 2z^2 - 1 \\ &= -(z^4 - 2z^2 + 1) \\ &= -((z^2)^2 + 2z^2 + 1^2) \end{aligned}$$

$$= -(z^2 - 1)^2$$
$$= -(z^2 - 1)^2(z + 1)^2$$

So the plane tree A_R associated at R is type < 211; 22 > (see the figure 8).

Figure 8: The tree A_R

for P we have

$$P(z) = -2z^2 + 2z$$
$$= -z(z - 2)$$

seek the critical values of P. We have

$$P^{\square}(z) \Leftrightarrow -2z + 2 = 0$$
$$\Leftrightarrow -2(z - 1) = 0$$
$$\Leftrightarrow z = 1$$

Where P(1) = −1 + 2 = 1. Therefore P has only one point critical w = 1. We have

$$P(z) - 1 = -z^2 + 2z - 1$$
$$= -(z^2 - 2z + 1)$$
$$= -(z - 1)^2$$

where plane tree A_P associated to P is of type < 11, 2 > (see figure 9)
for Q we have Q(z) = z^2, $Q^{\square}$ = 0 if and only if $Z = 0$.
So Q has the point 0.

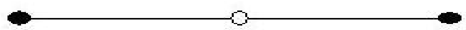


Figure 9: The tree A_P

Only one critical value. We also have Q(z) − 1 = $z^2 - 1 = (z - 1)(z + 1)$
Where A_Q is of type < 2, 11 > (see the figure 10)
We notice that the plane tree A_R decomposes in two trees AP and A_Q.

Figure 10: The tree A_Q

Conclusion

We see in [21] the correspondence between DZ-pairs and weighted bicolored plane trees. When such a tree is uniquely determined by the set of black and white degrees of its vertices, it is called unitree, and the corresponding DZ- pair is defined over Q, also in [22] it studied pairs of polynomials with a given factorization pattern and such that the degree of their difference attains its minimum. In this paper we studied Correspondence between two bicolored plane trees and generalized Tchebytchev polynomial, and we studied (composition and decomposition) for generalized Tchebytchev polynomial. Also we calculated Tchebyshev polynomials of plane trees

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