New general results on matrix domains of triangles in sequence spaces

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Abstract.

The notion of matrix domains of triangles in sequence spaces has largely been used to define new sequence spaces in terms of old ones. In this research paper, we will use this idea to introduce some new sequence spaces related to bounded and convergent series. Also, some properties of our spaces will derived. Further, we will establish some new inclusion relations between them. Moreover, the Schauder basis of these spaces will be discussed.

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1 Introduction

In this paper, we use w for the linear space of all real or complex sequences, and any sequence $x \in w$ will be simply written as $x = (x_k)$ instead of $x = (x_k)_{k=1}^{\infty}$. Also, we will use the conventions $e = (1,1,1,\ldots)$ and $e_k = (\delta_{nk})_{n=1}^{\infty}$ for each $k \geq 1$, that is e_k is the sequence whose only one non-zero term which is the k-term and is equal to 1, Also, any term with non-positive subscript is equal to zero, i.e. $x_0 = 0$ and $x_{-1} = 0$. Any linear subspace of w is called a sequence space. We will write ℓ_∞ ,c and c_0 for the classical sequence spaces of bounded, convergent and null sequences, respectively. Also, we will write bs, cs and cs_0 for the sequence spaces consisting of sequences associated with bounded, convergent and null series, respectively. That is

$$bs = \{x \in w : \sup_n |\sum_{k=1}^{n} x_k| < \infty\},\ cs = \{x \in w : \lim_{n\to\infty}(\sum_{k=1}^{n} x_k) \text{ exists}\}$$

$$\text{And}\quad cs_0 = \{x \in w : \lim_{n\to\infty}(\sum_{k=1}^{n} x_k) = 0\}.$$

Thus x belongs to bs, cs or cs_0 whenever the series $\sum_{k=1}^{\infty} x_k$ is bounded, convergent or convergent to zero, respectively. Further, for each $1 \leq p < \infty$, the space $l_p = \{x \epsilon w : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$ contains all sequences associted with p-absolutely convergent series and $bv = \{x \in w : \sum |x_k - x_{k-1}| < \infty\}$ is the space of sequences with bounded variation [8]. By a BK-space we mean a Banach sequence space with continuous coordinates. The spaces ℓ ,c and c_0 are BK-spaces ∞ with their natural norm $\|\cdot\|_\infty$ defined by $\|x\|_\infty = \sup_k |x_k|$, where the supremum is taking over all integers $k \geq 1$, and ℓ_p is a BK-space with the p-norm given by $\|x\|_x = sup_n |\sum_{k=1}^{n} x_k| p$. Also, the spaces bs, cs and cs_0 are BK-spaces with their norm $k\|\cdot\|s$ defined by $\|x\|_p = \sum_{k=1}^{\infty} |x_k|^p$[16]. An infinite matrix A whose real or complex entries a_{nk} for all $n,k \geq 1$ will be written as $A = [a_{nk}]$ instead of $A = [a_{nk}]_{n,k=1}^{\infty}$, and the act of A on any sequence $x \in w$ is called the A-transform of x, and is defined to be the sequence $A(x) = (A_n(x))_{n=1}^{\infty}$, where

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk}\, x_k \,, \qquad (n \geq 1)$$

provided that series on the right hand side converges for each n, and we then say that $A(x)$ exists. For two sequence spaces X and Y, we say that an infinite matrix A defines a matrix operator form X to Y, which is a linear operator, and we denote it by $A : X \to Y$, if A acts form X to Y, i.e, if for every sequence $x \in X$; the A-transform of x exists and is in Y. Moreover, we will write $(X : Y)$ for the class of all infinite matrices that map X into Y, i.e, $A \in (X : Y)$ if and only if $A(x)$ exists and $A(x) \in Y$ for every $x \in X$ [8]. Further, the matrix domain of A in a sequence space X is denoted by X_A which is a sequence space defined as $X_A = \{x \in w : A(x) \in X\}$. An infinite matrix A is called a triangle, if $a_{nk} = 0$ for all $k \geq n$ and $a_{nn} \neq 0$ for all n, where $n, k \geq 1$. The matrix domain of a triangle in a sequence space has a special important. For example, if X is a BK-space with its norm $\|\cdot\|$ and A is a triangle, then the matrix domain X_A is also a BK-space with the norm $\|\cdot\|_A$ defined by $\|x\|_A = \|A(x)\|$ for all $x \in X_A$ [16]. We will write σ for the sum-matrix which is a triangle defining the partial summation, that is $\sigma(x) = (\sum_{k=1}^{n} x_k)_{n=1}^{\infty}$ which means that $\sigma_n(x) = \sum_{k=1}^{n} x_k$ for all n. Then, it can be seen that $bs = (\ell_\infty)_\sigma$, $cs = (c)_\sigma$ and $cs_0 = (c_0)_\sigma$. Also, by Δ we mean the band matrix of difference, i.e, $\Delta(x) = (x_n - x_{n-1})_{n=1}^{\infty} = (x_1, x_2 - x_1, x_3 - x_2, \cdots)$ which means that $\Delta(x_k) = x_k - x_{k-1}$ for all k and so the space bv can be defined as $bv = (\ell_1)_\Delta$. The idea of constructing a new sequence space by means of the matrix domain of a particular triangle has largely been used by several authors in different ways. For instance, see [1, 2, 3, 4, 5, 6, 7, 9, 11, 13, 14, 17, 18, 19] and [20].

2 The new λ-sequence spaces bs^λ, cs^λ and cs_0^λ

In this section, we introduce the new λ-sequence spaces bs^λ, cs^λ and cs_0^λ, and show that these spaces are BK-spaces which are isometrically isomorphic to the spaces ℓ_∞, c and c_0, respectively.

Here and in what follows, we assume throughout that $\lambda = (\lambda_j)_{j=1}^{\infty}$ is a strictly increasing sequence of positive reals tending to ∞. That is $0 < \lambda_1 < \lambda_2 < \cdots$ and $\lambda_j \to \infty$ as $j \to \infty$. Also, we define the triangle $\Lambda = [\lambda_{nk}]$ for every $n, k \geq 1$ by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n}\,; & (1 \leq k \leq n), \\ 0\,; & (k > n \geq 1). \end{cases}$$

Then, for any $x \in w$, we have the sequence $\Lambda(x) = (\Lambda_n(x))_{n=1}^{\infty}$, where

$$\Lambda_k(x) = \frac{1}{\lambda_k} \sum_{j=1}^{k} (\lambda_j - \lambda_{j-1})\, x_j\,; \qquad (k \geq 1). \tag{2.1}$$

The λ-sequence spaces c_0^λ, c^λ, ℓ_∞^λ and ℓ_1^λ have been introduced by Mursaleen and Noman [10, 12] as the matrix domains of Λ in the spaces c_0, c, ℓ_∞ and ℓ_1, respectively. That is $c_0^\lambda = \{x \in w : \ \Lambda(x) \in c_0\}$, $c^\lambda = \{x \in w : \ \Lambda(x) \in c\}$, $\ell_\infty^\lambda = \{x \in w : \ \Lambda(x) \in \ell_\infty\}$ and $\ell_1^\lambda = \{x \in w : \ \Lambda(x) \in \ell_1\}$.

As a natural continuation, we follow them to introduce the new spaces bs^λ, cs^λ and cs_0^λ as the matrix domains of Λ in the spaces bs, cs and cs_0, respectively. That is $bs^\lambda = (bs)_\Lambda = \{x \in w : \Lambda(x) \in bs\}$, $cs^\lambda = (cs)_\Lambda = \{x \in w : \Lambda(x) \in cs\}$ and $cs_0^\lambda = (cs_0)_\Lambda = \{x \in w : \Lambda(x) \in cs_0\}$. So that, our contribution is the following new spaces:

$$bs^\lambda = \left\{ x \in w \, : \, \sup_n \left| \sum_{k=1}^{n} \Lambda_k(x) \right| < \infty \right\},$$

$$cs^\lambda = \left\{ x \in w \, : \, \lim_{n\to\infty} \left(\sum_{k=1}^{n} \Lambda_k(x) \right) \text{ exists} \right\},$$

$$cs_0^\lambda = \left\{ x \in w \, : \, \lim_{n\to\infty} \left(\sum_{k=1}^{n} \Lambda_k(x) \right) = 0 \right\}.$$

Besides, we define the triangle $\hat{\Lambda} = [\hat{\lambda}_{nk}]$ for every $n, k \geq 1$ by

$$\hat{\lambda}_{nk} = \begin{cases} (\lambda_k - \lambda_{k-1}) \displaystyle\sum_{j=k}^{n} \frac{1}{\lambda_j}\,; & (1 \leq k \leq n), \\ 0\,; & (k > n \geq 1). \end{cases}$$

Then, for every $x \in w$, we have

$$\hat{\Lambda}_n(x) = \sum_{k=1}^{n} \left(\sum_{j=k}^{n} \frac{1}{\lambda_j} \right) (\lambda_k - \lambda_{k-1}) x_k\,, \qquad (n \geq 1). \tag{2.2}$$

Thus, it can easily be seen that $\hat{\Lambda}(x) = \sigma(\,\Lambda(x)\,)$ for every $x \in w$ which can be written as follows:

$$\hat{\Lambda}_n(x) = \sum_{k=1}^{n} \Lambda_k(x)\,, \qquad (n \geq 1). \tag{2.3}$$

It follows that our bs^λ, cs^λ and cs_0^λ are sequence spaces which can be redefined as the matrix domains of $\hat{\Lambda}$ in the spaces ℓ_∞, c and c_0, respectively. That is

$$bs^\lambda = (\ell_\infty)_{\hat{\Lambda}}\,, \quad cs^\lambda = (c)_{\hat{\Lambda}} \quad \text{and} \quad cs_0^\lambda = (c_0)_{\hat{\Lambda}}. \tag{2.4}$$

Thus, we have $bs^\lambda = \{x \in w : \hat{\Lambda}(x) \in \ell_\infty\}$, $cs^\lambda = \{x \in w : \hat{\Lambda}(x) \in c\}$ and $cs_0^\lambda = \{x \in w : \hat{\Lambda}(x) \in c_0\}$, and we may begin now with the following result which is essential in the text:

Lemma 2.1 *The λ-sequence spaces bs^λ, cs^λ and cs_0^λ are BK-spaces with the norm $\|\cdot\|_\lambda$ defined, for every sequence x in these spaces, by*

$$\|x\|_\lambda = \|\hat{\Lambda}(x)\|_\infty = \sup_n \left| \hat{\Lambda}_n(x) \right| = \sup_n \left| \sum_{k=1}^{n} \Lambda_k(x) \right|.$$

Proof. Since $\hat{\Lambda}$ is a triangle; this result is immediate by (2.4) and the fact that ℓ_∞, c and c_0 are BK-spaces with their natural norm $\|\cdot\|_\infty$ (Maddox [8, pp.217–218]). To see that, the famous result of Wilansky [16, Theorem 4.3.12, p.63] tells us that bs^λ, cs^λ and cs_0^λ are BK-spaces with the given norm and this completes the proof. □

Theorem 2.2 *The λ-sequence spaces bs^λ, cs^λ and cs_0^λ are isometrically linear-isomorphic to the spaces ℓ_∞, c and c_0, respectively. That is $bs^\lambda \cong \ell_\infty$, $cs^\lambda \cong c$, and $cs_0^\lambda \cong c_0$.*

Proof. To prove this result, we will show that there exists a linear bijection between the spaces bs^λ and ℓ_∞ which preserves the norm. For this, we can use the definition of the space bs^λ to define a linear operator by means of the matrix operator $\hat{\Lambda} : bs^\lambda \to \ell_\infty$ by $x \mapsto \hat{\Lambda}(x)$. Then, it is obvious that $\hat{\Lambda}(x) = 0$ implies $x = 0$, and so $\hat{\Lambda}$ is injective. Also, let $y \in \ell_\infty$ be given and define a sequence $x = (x_j)$ in terms of the sequence y by

$$x_j = \frac{\lambda_j \, \Delta(y_j) - \lambda_{j-1} \, \Delta(y_{j-1})}{\lambda_j - \lambda_{j-1}}; \qquad (j \geq 1),$$

where $y_0 = 0$. Then, it follows by (2.1) that

$$\Lambda_k(x) = \frac{1}{\lambda_k} \sum_{j=1}^{k} [\lambda_j \, \Delta(y_j) - \lambda_{j-1} \, \Delta(y_{j-1})] = \Delta(y_k), \qquad (k \geq 1).$$

Thus, by using (2.3), we find that $\hat{\Lambda}_n(x) = \sum_{k=1}^{n} \Delta(y_k) = y_n$ for all n, which means that $\hat{\Lambda}(x) = y$, but $y \in \ell_\infty$ and so $\hat{\Lambda}(x) \in \ell_\infty$. Thus, we deduce that $x \in bs^\lambda$ such that $\hat{\Lambda}(x) = y$ and hence $\hat{\Lambda}$ is surjective. Further, it is clear by Lemma 2.1 that $\hat{\Lambda}$ is norm preserving, since $\|\hat{\Lambda}(x)\|_\infty = \|x\|_\lambda$ for every $x \in bs^\lambda$. Therefore, the mapping $\hat{\Lambda} : bs^\lambda \to \ell_\infty$ is a linear bijection preserving the norm. That is, our $\hat{\Lambda}$ is an isometry isomorphism between bs^λ and ℓ_∞ which means that $bs^\lambda \cong \ell_\infty$. Similarly, it can be shown that $cs^\lambda \cong c$, and $cs_0^\lambda \cong c_0$. □

Corollary 2.3 *The λ-sequence spaces bs^λ, cs^λ and cs_0^λ are isometrically linear-isomorphic to the spaces bs, cs and cs_0, respectively. That is $bs^\lambda \cong bs$, $cs^\lambda \cong cs$, and $cs_0^\lambda \cong cs_0$.*

Proof. It is immediate by Theorem 2.2 and the facts that $bs \cong \ell_\infty$, $cs \cong c$, and $cs_0 \cong c_0$. □

Remark 2.4 We have already shown in the proof of Theorem 2.2 that the matrix $\hat{\Lambda}$ defines a linear operator from any of the spaces bs^λ, cs or cs_0 into the respective one of the spaces ℓ_∞, c or c_0, is an isometry isomorphism, and this implies the continuity of the matrix operator $\hat{\Lambda}$ which will be used in the sequel.

At the end of this section, we give an example to show that our new spaces bs^λ, cs^λ and cs_0^λ are totally different from the spaces ℓ_∞, c, c_0, bs, cs and cs_0. For simplicity in notations, we will use the symbole μ to denote any of the spaces bs, cs or cs_0 and so μ^λ is the respective one of the spaces bs^λ, cs^λ or cs_0^λ, and μ^* denotes the related space among the spaces ℓ_∞, c or c_0.

Example 2.5 In this example, our aim is to show that our spaces μ^λ are different from all the sequence spaces μ and μ^*. For this, consider the sequence $\lambda = (\lambda_k)$ defined by $\lambda_k = k$ and so $\Delta(\lambda_k) = 1$ for all $k \geq 1$. Then, for any $x \in w$, we have $\Lambda_k(x) = (1/k)\sum_{j=1}^{k} x_j = \sigma_k(x)/k$ and $\hat{\Lambda}_n(x) = \sum_{k=1}^{n} \Lambda_k(x)$ for all $k, n \geq 1$. Thus, our spaces can be defined as $\mu^\lambda = \{x \in w : (\,\sigma_k(x)/k\,) \in \mu\} = \{x \in w : (\,\sum_{k=1}^{n} \sigma_k(x)/k\,) \in \mu^*\}$. Also, define the unbounded sequence $z = (z_k)$ by $z_1 = 1$ and for $k > 1$ by

$$z_k = \begin{cases} k\sqrt{2/(k+1)} + (k-1)\sqrt{2/(k-1)}; & (k \text{ is odd}), \\ -(2k-1)\sqrt{2/k}\,; & (k \text{ is even}). \end{cases}$$

Then, we have $z \notin \ell_\infty$ and so $z \notin \mu^*$ which also implies that $z \notin bs$ and hence $z \notin \mu$ which can independently be obtained from $\sigma_k(z) = k\sqrt{2/(k+1)}$ when k is odd and $\sigma_k(z) = -k\sqrt{2/k}$ when k is even. Further, we have $\Lambda_k(z) = \sqrt{2/(k+1)}$ when k is odd and $\Lambda_k(z) = -\sqrt{2/k}$ when k is even. Thus, we get $\hat{\Lambda}_n(z) = \sqrt{2/(n+1)}$ when n is odd and $\hat{\Lambda}_n(z) = 0$ when n is even. This implies that $\hat{\Lambda}(z) \in c_0$ and so $z \in cs_0^\lambda$ which leads us to $z \in \mu^\lambda$. Hence, we have shown that $z \in \mu^\lambda$ while $z \notin \mu$ as well as $z \notin \mu^*$. Therefore, we deduce that $\mu^\lambda \not\subset \mu$ and $\mu^\lambda \not\subset \mu^*$. On other side, consider the sequence $z' = (z'_k)$ defined by $z'_k = \Delta\,(\,1/\log(1+k)\,)$ for all $k \geq 1$ with noting that $z'_1 = 1/\log 2$. Then, we get $\sigma(z') = (\,1/\log(1+k)\,) \in c_0$ and so $z' \in cs_0$ which implies both $z' \in \mu$ and $z' \in \mu^*$. Besides, we find that $\Lambda(z') = (\,1/(k\log(1+k))\,)$ and so $\hat{\Lambda}_n(z') = \sum_{k=1}^{n} 1/(k\log(1+k))$ which diverges to ∞ as $n \to \infty$ and this means that $z' \notin bs^\lambda$ and so $z' \notin \mu^\lambda$. Hence, we have shown that $z' \notin \mu^\lambda$ while $z' \in \mu$ and $z' \in \mu^*$. Therefore, we deduce that $\mu \not\subset \mu^\lambda$ as well as $\mu^* \not\subset \mu^\lambda$. Consequently, we conclude that the spaces μ^λ are totally different from all the spaces μ and μ^*.

3 Some inclusion relations

In the present section, we establish some new inclusion relations concerning the λ-sequence spaces bs^λ, cs^λ and cs_0^λ. We essentially characterize the case in which the inclusions $bs \subset bs^\lambda$, $cs \subset cs^\lambda$ and $cs_0 \subset cs_0^\lambda$ hold, and discuss their equalities.

Lemma 3.1 *We have the following facts:*

(1) *The inclusions $cs_0^\lambda \subset cs^\lambda \subset bs^\lambda$ strictly hold.*

(2) *The inclusions $\ell_1^\lambda \subset cs^\lambda \subset c_0^\lambda$ and $\ell_1^\lambda \subset bs^\lambda \subset \ell_\infty^\lambda$ strictly hold.*

(3) *The inclusion $cs_0^\lambda \subset c_0^\lambda$ strictly holds.*

(4) *If $1/\lambda \in \ell_1$; then the inclusion $\ell_1 \subset cs^\lambda$ strictly holds, where $1/\lambda = (1/\lambda_j)_{j=1}^{\infty}$.*

(5) *The space ℓ_1 cannot be included in cs_0^λ.*

Proof. (1) the inclusions $cs_0^\lambda \subset cs^\lambda \subset bs^\lambda$ are obviously satisfied (by the well-known inclusions $cs_0 \subset cs \subset bs$). Also, to show that these inclusions are strict, define the sequence $x = (x_j)$ by $x_j = (2^{-j}\,\lambda_j - 2^{-(j-1)}\,\lambda_{j-1})/(\lambda_j - \lambda_{j-1})$ for all $j \geq 1$. Then, by using (2.1), we find that $\Lambda_k(x) = 2^{-k}$ for every $k \geq 1$ and so $\hat{\Lambda}(x) = (1 - 2^{-n}) \in c \setminus c_0$.

This means that $x \in cs^\lambda \setminus cs_0^\lambda$ and so the inclusion $cs_0^\lambda \subset cs^\lambda$ is strict. Also, define the sequence $y = (y_j)$ by $y_j = (-1)^j(\lambda_j + \lambda_{j-1})/(\lambda_j - \lambda_{j-1})$ for all $j \geq 1$. Then, for every $k \geq 1$, we find that $\Lambda_k(y) = (1/\lambda_k)\sum_{j=1}^k(-1)^j(\lambda_j + \lambda_{j-1}) = (-1)^k$ and hence $\hat{\Lambda}_n(y) = -1$ when n is odd or $\hat{\Lambda}_n(y) = 0$ when n is even. Thus, we deduce that $\hat{\Lambda}(y) \in \ell_\infty \setminus c$ which means that $y \in bs^\lambda \setminus cs^\lambda$ and hence the inclusion $cs^\lambda \subset bs^\lambda$ is also strict, and part (1) has been proved. To prove part (2), let $x \in \ell_1^\lambda$. Then, the series $\sum_{k=1}^\infty \Lambda_k(x)$ is absolutely convergent and so it converges which means that $x \in cs^\lambda$ and hence the inclusion $\ell_1^\lambda \subset cs^\lambda$ holds which implies the inclusion $\ell_1^\lambda \subset bs^\lambda$. Also, if $x \in cs^\lambda$; then it follows, from the convergence of the series $\sum_{k=1}^\infty \Lambda_k(x)$, that $\Lambda(x) \in c_0$ and hence $x \in c_0^\lambda$ which means that the inclusion $cs^\lambda \subset c_0^\lambda$ holds. Similarly, we can show that $bs^\lambda \subset \ell_\infty^\lambda$ holds. To show that these inclusions are strict, define the sequence $x = (x_j)$ by $x_j = (-1)^j[\,(\lambda_j/(j+1)) + (\lambda_{j-1}/j)\,]/(\lambda_j - \lambda_{j-1})$ for every $j \geq 1$ Then, it can easily be seen that $\Lambda(x) = (\,(-1)^k/(k+1)\,) \in cs \setminus \ell_1$ and so $x \in cs^\lambda \setminus \ell_1^\lambda$ which means that the inclusion $\ell_1^\lambda \subset cs^\lambda$ is strict, and so is the inclusion $\ell_1^\lambda \subset bs^\lambda$. Further, define the sequence $y = (y_j)$ by $y_j = [\Delta(\,\lambda_j/(j+1)\,)]/(\lambda_j - \lambda_{j-1})$ for every $j \geq 1$. Then, it is easy to show that $\Lambda(y) = (\,1/(k+1)\,) \in c_0 \setminus cs$ which means that $y \in c_0^\lambda \setminus cs^\lambda$ and so the inclusion $cs^\lambda \subset c_0^\lambda$ is strict. Finally, it is clear that $\Lambda(e) = e \in \ell_\infty \setminus bs$ which implies that $e \in \ell_\infty^\lambda \setminus bs^\lambda$ and hence the inclusion $bs^\lambda \subset \ell_\infty^\lambda$ is also strict which ends the proof of part(2). Moerover, part (3) is clear by combining the results of parts (1) and (2). For part (4), suppose $1/\lambda \in \ell_1$. Then, the inclusion $\ell_1 \subset \ell_1^\lambda$ holds (see [12, Theorem 4.12] which tells us that: $\ell_1 \subset \ell_1^\lambda \iff 1/\lambda \in \ell_1$). Thus, the inclusion $\ell_1 \subset cs^\lambda$ is strict by (2). Finally, to prove (5), consider the sequence $e_1 = (1, 0, 0, \cdots)$. Then, it is clear by (2.1) that $\Lambda_k(e_1) = \lambda_1/\lambda_k$ for all $k \geq 1$ and so $\hat{\Lambda}_n(e_1) = \lambda_1\,\sigma_n(1/\lambda) \geq 1$ for all n (as $\lambda_k > 0$ for all k). Thus $\hat{\Lambda}(e_1) \notin c_0$ which means that $e_1 \notin cs_0^\lambda$ while $e_1 \in \ell_1$ and hence $\ell_1 \not\subset cs_0^\lambda$. This completes the proof. □

Remark 3.2 As in part (4) of Lemma 3.1, we will use the convention $1/\lambda = (1/\lambda_j)_{j=1}^\infty$. Also, since λ is a sequence of positive reals; we deduce that $1/\lambda \notin cs_0$, but the sequence of its partial sums $\sigma(1/\lambda)$ is increasing whose positive terms and this leads us to the following equivalences: $1/\lambda \in \ell_1 \iff 1/\lambda \in cs \iff 1/\lambda \in bs$.

Now, in what follows and for simplicity in notations, we will use some conventions to prove our main results concerning the inclusions $bs \subset bs^\lambda$, $cs \subset cs^\lambda$ and $cs_0 \subset cs_0^\lambda$. For this purpose, we are in need to quoting some additional lemmas and terminologies.

For any positive integer n, we define the following two positive real terms:

$$s_k^n = \lambda_k \sum_{j=k}^n \frac{1}{\lambda_j} \quad \text{and} \quad t_k^n = (\lambda_k - \lambda_{k-1}) \sum_{j=k}^n \frac{1}{\lambda_j}, \qquad (1 \leq k \leq n). \tag{3.1}$$

Further, if $1/\lambda \in \ell_1$; then the limits $s_k^n \to s_k$ and $t_k^n \to t_k$ (as $n \to \infty$) exist for each $k \geq 1$. Thus, we can define the following three positive real sequences $s = (s_k)$, $t = (t_k)$ and $u = (u_k)$ as follows:

$$s_k = \lambda_k \sum_{j=k}^\infty \frac{1}{\lambda_j}\,, \quad t_k = \Delta(\lambda_k) \sum_{j=k}^\infty \frac{1}{\lambda_j} \quad \text{and} \quad u_k = \frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\,, \qquad (k \geq 1). \tag{3.2}$$

Moreover, it can easily be deriving the following equalities:

$$s_k = t_k\, u_k \quad (k \geq 1) \quad \text{and} \quad s_k^n = t_k^n\, u_k \quad (1 \leq k \leq n), \tag{3.3}$$

$$t_k = 1 + \Delta(s_k) \quad (k > 1) \quad \text{and} \quad t_k^n = 1 + \Delta(s_k^n) \quad (1 < k \leq n), \tag{3.4}$$

where the difference is taken over k, that is $\Delta(s_k^n) = s_k^n - s_{k-1}^n$ for every $k \leq n$.

Lemma 3.3 *Let $1/\lambda \in \ell_1$ and suppose that $\Delta(u) \in c$. Then, there must exist a positive integer k_0 satisfying all the following:*

(1) $1 < u_k < k$ *for all* $k > k_0$ *and so* $0 \leq \lim_{k\to\infty} \Delta(u_k) < 1$.

(2) *There is a positive real* $\delta < 1/2$ *such that* $-\delta < \Delta(u_k) < 1 - \delta$ *for all* $k > k_0$.

(3) *The difference sequence* $(\,\Delta(\lambda_k)\,)_{k=k_0}^{\infty}$ *is strictly increasing to* ∞.

(4) *If* $\lim_{k\to\infty} \Delta(u_k) = a$; *then* $\lim_{k\to\infty} t_k = 1/(1-a)$ *and* $\lim_{k\to\infty} \Delta(s_k) = a/(1-a)$.

Proof. Suppose that $1/\lambda \in \ell_1$ and $\Delta(u) \in c$ which means that $\lim_{k\to\infty} \Delta(u_k)$ exists. Then $\lim_{k\to\infty} u_k/k$ exists (due to the equality between these two limits). Thus $(u_k/k) \in c \subset \ell_\infty$. Also, we claim that there is a positive integer k_1 such that $u_k/k < 1$ for all $k > k_1$ or $u_{k+1}/(k+1) < 1$ for all $k \geq k_1$ which can equivalently be written as $\lambda_{k+1}/(\lambda_{k+1} - \lambda_k) < k+1$ for all $k \geq k_1$. Otherwise, suppose on contrary that the sequence $\lambda = (\lambda_k)$ has a subsequence $(\lambda_{k_r})_{r=1}^{\infty}$ such that $\lambda_{k_{r+1}}/(\lambda_{k_{r+1}} - \lambda_{k_r}) \geq k_{r+1} \geq r+1$ for all $r \geq 1$. Then, it follows that $\lambda_{k_{r+1}} \leq \lambda_{k_r}\,((r+1)/r)$ and so $\lambda_{k_{r+1}} \leq \lambda_{k_1}\,(r+1)$ for all $r \geq 1$. Thus, we deduce that $1/(r+1) \leq \lambda_{k_1}/\lambda_{k_{r+1}}$ for all $r \geq 1$ and so $(\,1/\lambda_{k_{r+1}}\,) \notin \ell_1$ which contradicts with our hypothesis $1/\lambda \in \ell_1$. Hence, our claim is true (as $u_{k+1} > 1$ for all k). Further, since $\lim_{k\to\infty} \Delta(u_k) = \lim_{k\to\infty} u_k/k$; we find that $0 \leq \lim_{k\to\infty} \Delta(u_k) \leq 1$. Moreover $\lim_{k\to\infty} \Delta(u_k) \neq 1$. For, if $\lim_{k\to\infty} \Delta(u_k) = 1$; we can similarly get $\lambda_k \leq ak$ for some positive real $a > 0$ which is a contradiction with $1/\lambda \in \ell_1$. Therefore, we conclude that $0 \leq \lim_{k\to\infty} \Delta(u_k) < 1$. To prove (2), assume that $a = \lim_{k\to\infty} \Delta(u_k)$, where $0 \leq a < 1$. Then, for every positive real $\epsilon > 0$, there is a positive integer $k' = k'(\epsilon)$ such that $|\Delta(u_{k+1}) - a| < \epsilon$ and so $a - \epsilon < \Delta(u_{k+1}) < a + \epsilon$ for all $k \geq k'$. Now, choose a positive real $\delta < 1/2$ such that $(1-a)/4 < \delta < (1-a)/2$ and so $\delta < (1-a)/2 < 2\delta$. Then, by taking $\epsilon = (1-a)/2 - \delta$ with its $k_2 = k'(\epsilon)$, we get $0 < \epsilon < 1/2$ and find that $a + \epsilon = (1+a)/2 - \delta < 1 - \delta$ and $a - \epsilon \geq -\epsilon = \delta - (1-a)/2 > \delta - 2\delta = -\delta$. Hence, we deduce that $-\delta < \Delta(u_{k+1}) < 1 - \delta$ for all $k \geq k_2$, that is $-\delta < \Delta(u_k) < 1 - \delta$ for all $k > k_2$. To prove (3), we obtain from (2) that $\Delta(u_{k+1}) < 1$ and so $1 + \lambda_k\, \Delta(\,1/\Delta(\lambda_{k+1})\,) < 1$ for all $k \geq k_2$. This implies that $\Delta(\lambda_{k+1}) > \Delta(\lambda_k)$ for all $k \geq k_2$. Thus, the sequence $(\,\Delta(\lambda_k)\,)_{k=k_2}^{\infty}$ is strictly increasing and cannot be bounded (as $1/\lambda \in \ell_1$). Therefore, it must tend to ∞. Also, by taking $k_0 = \max\{k_1, k_2\}$, we get the common integer k_0 in parts (1), (2) and (3). Finally, to prove (4), suppose that $\lim_{k\to\infty} \Delta(u_k) = a$, where $0 \leq a < 1$ by part (1). Then, for every positive real $\epsilon > 0$, there is a positive integer k' such that $|\Delta(u_{k+1}) - a| < \epsilon$ and so $|1 + \lambda_k \Delta(\,1/\Delta(\lambda_{k+1})\,) - a| < \epsilon$ for all $k \geq k'$. Thus $|\,(1-a)/\lambda_k - [\,1/\Delta(\lambda_k) - 1/\Delta(\lambda_k)\,]\,| < \epsilon/\lambda_k$ for all $k \geq k'$ and by taking the summation in both sides from $k = n$ to ∞ and noting that $\Delta(\lambda_k) \to \infty$ as $k \to \infty$ by (3), we get

$$\left|(1-a)\sum_{k=n}^{\infty}\frac{1}{\lambda_k} - \frac{1}{\Delta(\lambda_n)}\right| \leq \sum_{k=n}^{\infty}\left|\frac{1-a}{\lambda_k} - \left(\frac{1}{\Delta(\lambda_k)} - \frac{1}{\Delta(\lambda_{k+1})}\right)\right| < \epsilon\sum_{k=n}^{\infty}\frac{1}{\lambda_k}\,.$$

Dividing both sides by the positive number $\sum_{k=n}^{\infty} 1/\lambda_k$ (as $1/\lambda \in \ell_1$ and $\lambda_k > 0$ for all k); we obtain that $|(1-a) - 1/t_n| < \epsilon$ for all $n \geq k'$, and since ϵ was arbitrary; we deduce that $1/t_n \to 1-a$ or $t_n \to 1/(1-a)$ as $n \to \infty$, and the other limit is immediate by (3.4). This completes the proof. $\square$

Lemma 3.4 *Let $1/\lambda \in \ell_1$ and suppose that $t \in c$. Then, all the following are true:*

(1) $\lim_{k\to\infty} t_k \geq 1$ *and* $\Delta(s) \in c$ *such that* $\lim_{k\to\infty} \Delta(s_k) \geq 0$.

(2) *If* $\lim_{k\to\infty} t_k = b$*; then* $\lim_{k\to\infty} \Delta(u_k) = (b-1)/b$ *and* $\lim_{k\to\infty} \Delta(s_k) = b-1$.

(3) *There exists a positive integer k_0 such that the difference sequence $(\,\Delta(\lambda_k)\,)_{k=k_0}^{\infty}$ is strictly increasing to ∞.*

Proof. Suppose that $1/\lambda \in \ell_1$ and $t \in c$. Then, it follows by (3.4) that $\Delta(s) \in c$ and so $\lim_{k\to\infty} \Delta(s_k) = \lim_{k\to\infty} s_k/k \geq 0$ (since $s_k > 0$ for all k). Again, by (3.4) we get $\lim_{k\to\infty} t_k \geq 1$. To prove (2), we first show that $t \in c$ implies $\Delta(u) \in c$. For this, it follows from $t \in c$ that $\Delta(s) \in c$ and $\Delta(t) \in c$ such that $\lim_{k\to\infty} \Delta(t_k) = \lim_{k\to\infty} t_k/k$. Thus, from (3.3), we find that $\Delta(s_k) = \Delta(t_k u_k) = t_k \Delta(u_k) + u_{k-1}\Delta(t_k)$ which implies that $\lim_{k\to\infty} \Delta(s_k) = \lim_{k\to\infty} t_k(\,\Delta(u_k) + u_{k-1}/k\,)$ exists. Thus $(\,\Delta(u_k) + u_{k-1}/k\,) \in c$. On other side, we have $\lim_{k\to\infty} \Delta(s_k) = \lim_{k\to\infty} \Delta(t_k u_k) = \lim_{k\to\infty}(t_k u_k)/k = \lim_{k\to\infty} t_k(u_k/k)$. Hence $(u_k/k) \in c$, and since $(k/(k-1)) \in c$; we find that $(u_{k-1}/k) \in c$. Therefore, we deduce that $\Delta(u) = (\Delta(u_k)) = (\Delta(u_k) + u_{k-1}/k) - (u_{k-1}/k) \in c$. Now, if $\lim_{k\to\infty} t_k = b$; then by part (4) of Lemma 3.3 we get $\lim_{k\to\infty} \Delta(u_k) = (b-1)/b$ and the other limit is trivial. Finally, part (3) is now immediate by (3) of Lemma 3.3 because $\Delta(u) \in c$. This ends the proof. $\square$

Lemma 3.5 *Suppose that $1/\lambda \in \ell_1$. Then, we have the following equivalences:*

(1) $\Delta(u) \in c$ *if and only if* $t \in c$.

(2) $\Delta(u) \in bv$ *if and only if* $t \in bv$.

(3) $\sup_n \sum_{k=1}^{n-1} |\Delta(t_{k+1}^n)| < \infty$ *if and only if* $\Delta(u) \in bv$.

Proof. Suppose that $1/\lambda \in \ell_1$. Then, the equivalence in part (1) can be obtained by combinig (4) of Lemma 3.3 and (2) of Lemma 3.4. To prove (2), let us first note that $bv \subset c$. Thus, if $\Delta(u) \in bv$ or $t \in bv$; then $\Delta(u) \in c$ as well as $t \in c$. Hence, in both direction of current equivalence, we have $\Delta(u) \in c$ and $t \in c$. Therefore, it follows by (2) of Lemma 3.3 that there are $\delta > 0$ (real) and $k_0 \geq 1$ (integer) such that $\delta < 1 - \Delta(u_k) < 1 + \delta$ for all $k > k_0$. Thus $(\,1 - \Delta(u_{k+1})\,)_{k=k_0}^{\infty}$ is a convergent sequence of positive reals with non-zero limit, that is $(\,1 - \Delta(u_{k+1})\,) \in c \setminus c_0$. Also, it is obvious that t is a convergent sequence of positive reals with non-zero limit, that is $t \in c \setminus c_0$. Further, it follows by (4) of Lemma 3.3 and (2) of Lemma 3.4 that $\lim_{k\to\infty} t_k(1 - \Delta(u_{k+1})) = 1$. Hence, if $\Delta(u) \in bv$ or $t \in bv$; then $(\,t_k(1 - \Delta(u_{k+1}))\,) \in bv$ and so $(\,\Delta[t_k(1 - \Delta(u_{k+1}))]\,) \in \ell_1$. Therefore, we obtain that

$$(\,\Delta[t_k(1 - \Delta(u_{k+1}))]\,) = \Big(\,t_k\Delta(1 - \Delta(u_{k+1})) + (1 - \Delta(u_k))\Delta(t_k)\,\Big) \in \ell_1 \cdot$$

Now, if $t \in bv$; then $\Delta(t) \in \ell_1$ and so $\big((1-\Delta(u_k))\Delta(t_k)\big) \in \ell_1$ which implies that $\big(t_k\Delta(1-\Delta(u_{k+1}))\big) \in \ell_1$ and hence $\big(\Delta(1-\Delta(u_{k+1}))\big) \in \ell_1$ (as $t \in c \backslash c_0$) and this means that $(1-\Delta(u_{k+1})) \in bv$ and so $\Delta(u) \in bv$. Similarly, if $\Delta(u) \in bv$; then $(1-\Delta(u_{k+1})) \in bv$ and so $\big(\Delta(1-\Delta(u_{k+1}))\big) \in \ell_1$ which implies that $\big(t_k\Delta(1-\Delta(u_{k+1}))\big) \in \ell_1$ and hence $\big((1-\Delta(u_k))\Delta(t_k)\big) \in \ell_1$. Thus $\big(\Delta(t_k)\big) \in \ell_1$ (as $(1-\Delta(u_k)) \in c \backslash c_0$), that is $\Delta(t) \in \ell_1$ and so $t \in bv$. Finally, to prove (3), let us first note that $t \in c$ in both direction of current equivanence (as we have already shown in proving (2)) and hence there is an integer $k_0 \geq 1$ such that $(\Delta(\lambda_k))_{k=k_0}^{\infty}$ is strictly increasing to ∞ by (3) of Lemma 3.4. Now, let $n \geq 2$. Then, for every $k < n$, we have $t_{k+1} - t_{k+1}^n = \big(t_{n+1}/\Delta(\lambda_{n+1})\big)\Delta(\lambda_{k+1})$ and so $\Delta(t_{k+1} - t_{k+1}^n) = \big(t_{n+1}/\Delta(\lambda_{n+1})\big)\big(\Delta(\lambda_{k+1}) - \Delta(\lambda_k)\big)$. Thus, it follows that

$$\Big|\, |\Delta(t_{k+1})| - |\Delta(t_{k+1}^n)| \,\Big| \leq |\Delta(t_{k+1}) - \Delta(t_{k+1}^n)| = \frac{t_{n+1}}{\Delta(\lambda_{n+1})}\, |\Delta(\lambda_{k+1}) - \Delta(\lambda_k)|$$

and by taking the summation from $k = 1$ to $n - 1$, we get

$$\left|\sum_{k=1}^{n-1}\big(|\Delta(t_{k+1})| - |\Delta(t_{k+1}^n)|\big)\right| \leq \sum_{k=1}^{n-1}\Big|\, |\Delta(t_{k+1})| - |\Delta(t_{k+1}^n)| \,\Big| = O\Big(\frac{\Delta(\lambda_n)}{\Delta(\lambda_{n+1})}\, t_{n+1}\Big).$$

But $\big(t_{n+1}\Delta(\lambda_n)/\Delta(\lambda_{n+1})\big) \in \ell_\infty$ and so $\Big(\sum_{k=1}^{n-1}|\Delta(t_{k+1})| - \sum_{k=1}^{n-1}|\Delta(t_{k+1}^n)|\Big)_{n=2}^{\infty} \in \ell_\infty$. Therefore, we deduce that $\big(\sum_{k=1}^{n-1}|\Delta(t_{k+1}^n)|\big) \in \ell_\infty \iff \big(\sum_{k=1}^{n-1}|\Delta(t_{k+1})|\big) \in \ell_\infty$, that is $\sup_n \sum_{k=1}^{n-1}|\Delta(t_{k+1}^n)| < \infty \iff \sum_{k=1}^{\infty}|\Delta(t_{k+1})| = \sup_n \sum_{k=1}^{n-1}|\Delta(t_{k+1})| < \infty$ which can equivalently be written as $\sup_n \sum_{k=1}^{n-1}|\Delta(t_{k+1}^n)| < \infty \iff t \in bv$. But $t \in bv \iff \Delta(u) \in bv$ by part (2) and this completes the proof. $\square$

Lemma 3.6 ([15, pp. 3-4]) *For any infinite matrix $A = [a_{nk}]$, we have the following:*

(1) $A \in (cs, c)$ *if and only if the following two conditions hold:*

$$\lim_{n\to\infty} a_{nk} \quad \textit{exists for every} \quad k \geq 1, \tag{3.5}$$

$$\sup_n \sum_{k=1}^{\infty} |a_{nk} - a_{n,k+1}| < \infty. \tag{3.6}$$

(2) $A \in (bs, \ell_\infty)$ *if and only if both* (3.6) *and the following condition hold:*

$$\lim_{k\to\infty} a_{nk} = 0 \quad \textit{for every} \quad n \geq 1. \tag{3.7}$$

Theorem 3.7 *Let $u = (u_k)$ be defined by $u_k = \lambda_k/(\lambda_k - \lambda_{k-1})$ for all $k \geq 1$. Then, we have the following facts:*

(1) *The inclusions $cs \subset cs^\lambda$ and $bs \subset bs^\lambda$ hold if and only if $1/\lambda \in \ell_1$ and $\Delta(u) \in bv$.*

(2) *The equalities $cs^\lambda = cs$ and $bs^\lambda = bs$ hold if and only if $u \in \ell_\infty$ and $\Delta(u) \in bv_0$.*

(3) *The inclusions $cs \subset cs^\lambda$ and $bs \subset bs^\lambda$ strictly hold if and only if $1/\lambda \in \ell_1$, $u \notin \ell_\infty$ and $\Delta(u) \in bv$.*

Proof. To prove (1), suppose that the inclusions $cs \subset cs^\lambda$ and $bs \subset bs^\lambda$ hold. Then, we have $e_1 \in cs$ and $e_1 \in bs$, where $e_1 = (1, 0, 0, \cdots)$. Thus, we must have $e_1 \in cs^\lambda$ as well as $e_1 \in bs^\lambda$. This implies that $\hat{\Lambda}(e_1) \in c$ and $\hat{\Lambda}(e_1) \in \ell_\infty$, respectively. Also, by using (2.2), we find that $\hat{\Lambda}_n(e_1) = \lambda_1 \sigma_n(1/\lambda) = \lambda_1 \sum_{k=1}^n (1/\lambda_k)$ for all $n \geq 1$. Thus, we conclude that $\sigma(1/\lambda) \in c$ and $\sigma(1/\lambda) \in \ell_\infty$ and hence $1/\lambda \in cs$ and $1/\lambda \in bs$, respectively. Therefore, in both cases, we have obtained the same result which is $1/\lambda \in \ell_1$ (see Remark 3.2). That is $1/\lambda \in \ell_1$ is necessary condition for both given inclusions (see Example 2.5). Thus, we assume that $1/\lambda \in \ell_1$ and then it can easily be seen that the inclusions $cs \subset cs^\lambda$ and $bs \subset bs^\lambda$ hold if and only if $\hat{\Lambda} \in (cs, s)$ and $\hat{\Lambda} \in (bs, \ell_\infty)$, respectively. To see that, we have the following equivalences:

$$cs \subset cs^\lambda \iff x \in cs^\lambda \text{ for all } x \in cs \iff \hat{\Lambda}(x) \in c \text{ for all } x \in cs \iff \hat{\Lambda} \in (cs, c),$$

$$bs \subset bs^\lambda \iff x \in bs^\lambda \text{ for all } x \in bs \iff \hat{\Lambda}(x) \in \ell_\infty \text{ for all } x \in bs \iff \hat{\Lambda} \in (bs, \ell_\infty).$$

Thus, to deduce the other necessary and sufficient conditions, we have to use the required conditions for $\hat{\Lambda} \in (cs, s)$ and $\hat{\Lambda} \in (bs, \ell_\infty)$ by means of Lemma 3.6 for the matrix $\hat{\Lambda}$ instead of A. For this, it follows from (3.1) and the definition of our matrix $\hat{\Lambda}$ that $\hat{\lambda}_{nk} = t_k^n$ for $1 \leq k \leq n$ and $\hat{\lambda}_{nk} = 0$ for $k > n$, where $n, k \geq 1$. Thus, by using the intries of $\hat{\Lambda}$, we deduce from condition (3.5) that $\lim_{n\to\infty} \hat{\lambda}_{nk} = \lim_{n\to\infty} t_k^n$ exists for every $k \geq 1$. But these limits actually exist for all $k \geq 1$ (as $1/\lambda \in \ell_1$), where $\lim_{n\to\infty} t_k^n = t_k = \Delta(\lambda_k) \sum_{j=k}^\infty 1/\lambda_j$ for each k. Thus, condition (3.5) is already satisfied for $\hat{\Lambda}$. Also, condition (3.7) trivially holds, since $\hat{\Lambda}$ is a triangle and so $\hat{\lambda}_{nk} = 0$ when $k > n$ for each $n \geq 1$ and this impliess that $\lim_{k\to\infty} \hat{\lambda}_{nk} = 0$ for every $n \geq 1$. Thus, the common condition (3.6) is left, and this condition together with $1/\lambda \in \ell_1$ are the necessary and sufficient conditions for both inclusions. Moreover, for every $n, k \geq 1$ we have

$$\hat{\lambda}_{nk} - \hat{\lambda}_{n,k+1} = \begin{cases} -\Delta(t_{k+1}^n)\,; & (k < n), \\ \Delta(\lambda_n)/\lambda_n\,; & (k = n), \\ 0\,; & (k > n), \end{cases}$$

$$\sum_{k=1}^\infty \left|\hat{\lambda}_{nk} - \hat{\lambda}_{n,k+1}\right| = \frac{\Delta(\lambda_n)}{\lambda_n} + \sum_{k=1}^{n-1} |\Delta(t_{k+1}^n)|,$$

and since $(\Delta(\lambda_n)/\lambda_n) \in \ell_\infty$; we deduce that $\sup_n \sum_{k=1}^\infty |\hat{\lambda}_{nk} - \hat{\lambda}_{n,k+1}| < \infty$ if and only if $\sup_n \sum_{k=1}^{n-1} |\Delta(t_{k+1}^n)| < \infty$. Therefore, condition (3.6) is satisfied for $\hat{\Lambda}$ if and only if $\sup_n \sum_{k=1}^{n-1} |\Delta(t_{k+1}^n)| < \infty$ (or equivalently $\Delta(u) \in bv$ by (3) of Lemma 3.5). Consequently, the inclusions $cs \subset cs^\lambda$ and $bs \subset bs^\lambda$ hold if and only if $1/\lambda \in \ell_1$ and $\Delta(u) \in bv$. To prove (2), we have the equality $x_k - \Lambda_{k-1}(x) = u_k\,[\Lambda_k(x) - \Lambda_{k-1}(x)]$ which is satisfied for any $x \in w$ and every $k \geq 1$ (see [12, Lemma 4.1]). Thus, by taking the summation of both sides from $k = 1$ to $n \geq 1$, we get the following relation:

$$\sigma_n(x) - \hat{\Lambda}_{n-1}(x) = \sum_{k=1}^n u_k\,[\Lambda_k(x) - \Lambda_{k-1}(x)]\,, \qquad (n \geq 1)$$

which can be written as follows:

$$\sigma_n(x) - \hat{\Lambda}_{n-1}(x) = u_{n+1}\,\Lambda_n(x) - \sum_{k=1}^n \Delta(u_{k+1})\,\Lambda_k(x)\,, \qquad (n \geq 1). \tag{3.8}$$

Now, if the equalities $cs^\lambda = cs$ and $bs^\lambda = bs$ hold; we deduce from (3.8) that $u \in \ell_\infty$ and $\Delta(u) \in bv$. But $bv \subset c$ and so $\Delta(u) \in c$ such that $\lim_{k\to\infty} \Delta(u_k) = \lim_{k\to\infty} u_k/k = 0$ (since u is bounded) which implies that $\Delta(u) \in bv_0$, where $bv_0 = bv \cap c_0$. Conversely, if $u \in \ell_\infty$ and $\Delta(u) \in bv_0$; it follows from (3.8) that $x \in cs^\lambda \Leftrightarrow x \in cs$ as well as $x \in bs^\lambda \Leftrightarrow x \in bs$, which means that both equalities $cs^\lambda = cs$ and $bs^\lambda = bs$ are satisfied (we may note that: (i) $u \in \ell_\infty \Rightarrow 1/\lambda \in \ell_1$, (ii) $xy \in cs$ for all $x \in cs \Leftrightarrow y \in bv$, and (iii) $xy \in bs$ for all $x \in bs \Leftrightarrow y \in bv_0$). Finally, part (3) follows from (1) and (2). □

Corollary 3.8 *If the inclusion $cs \subset cs^\lambda$ holds; then for every sequence $x \in cs$ we have $\lim_{n\to\infty} \hat{\Lambda}_n(x) = \lim_{n\to\infty} \sum_{k=1}^n t_k^n x_k = \lim_{n\to\infty} \sum_{k=1}^n t_k x_k$. That is*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Big(\Delta(\lambda_k) \sum_{j=k}^{n} \frac{1}{\lambda_j} \Big) x_k = \lim_{n\to\infty} \sum_{k=1}^{n} \Big(\Delta(\lambda_k) \sum_{j=k}^{\infty} \frac{1}{\lambda_j} \Big) x_k \,.$$

Proof. We have $\hat{\Lambda}_n(x) = \sum_{k=1}^n t_k^n x_k = \sum_{k=1}^n t_k x_k - (\sum_{j=n+1}^\infty 1/\lambda_j)(\sum_{k=1}^n \Delta(\lambda_k) x_k)$ for all n, and since $x_k = \sigma_k(x) - \sigma_{k-1}(x)$; we get $\sum_{k=1}^n \Delta(\lambda_k) x_k = \Delta(\lambda_{n+1})\sigma_n(x) - \sum_{k=1}^n (\Delta(\lambda_{k+1}) - \Delta(\lambda_k))\,\sigma_k(x)$. Thus, we obtain that

$$\hat{\Lambda}_n(x) = \sum_{k=1}^{n} t_k x_k + t_{n+1} \, (\ddot{\sigma}_n(x) - \sigma_n(x)) \,,$$

where $\ddot{\sigma}_n(x) = (1/\Delta(\lambda_{n+1})) \sum_{k=1}^n (\Delta(\lambda_{k+1}) - \Delta(\lambda_k))\,\sigma_k(x)$. That is $\ddot{\sigma}(x) = \ddot{\Lambda}(\sigma(x))$ and $\ddot{\Lambda}$ is the matrix Λ with the sequence $(\Delta(\lambda_{k+1}))$ instead of (λ_k), where $(\Delta(\lambda_{k+1}))_{k=k_0}^\infty$ is strictly increasing to ∞ (for some integer $k_0 \geq 1$ by Lemma 3.3). Hence, we conclude that $\lim_{n\to\infty} \ddot{\sigma}_n(x) = \lim_{n\to\infty} \sigma_n(x)$ by regularity of such matrices. Therefore, our result is now proved by going to the limits in both sides of above equality as $n \to \infty$. □

Corollary 3.9 *The inclusion $cs_0 \subset cs_0^\lambda$ strictly holds if and only if there exists a positive real number $0 < a < 1$ such that $\Delta(u_{k+1}) = a$ for all $k \geq 1$ (equivalently: $cs_0 \subset cs_0^\lambda$ strictly holds if and only if there exists a positive real number $b > 1$ such that $t_k = b$ for all $k \geq 1$). Furthermore, the equality $cs_0^\lambda = cs_0$ cannot be held.*

Proof. Assume $\Delta(u_{k+1}) = a$ $(0 < a < 1)$ for all $k \geq 1$, i.e. $(\Delta(u_2), \Delta(u_3), \cdots)$ is constant. Then $1 + \lambda_k \Delta(1/\Delta(\lambda_{k+1})) = a$ and so $1/\Delta(\lambda_k) - 1/\Delta(\lambda_{k+1}) = (1-a)/\lambda_k$. Thus $\Delta(\lambda)$ is increasing to ∞ and by taking the summation from $k = n$ to ∞ we get $t_n = 1/(1-a)$ for all $n \geq 1$ (t_n is constant). In such case, it is obvious that $1/\lambda \in \ell_1$ and $\Delta(u) \in bv$. Thus, it follows by (1) of Theorem 3.7 that the inclusion $cs \subset cs^\lambda$ holds. Also, for any $x \in cs_0$, we have $x \in cs^\lambda$ (since $cs_0 \subset cs \subset cs^\lambda$). Thus, we deduce from Corollary 3.8 that $\lim_{n\to\infty} \hat{\Lambda}_n(x) = (1/(1-a)) \lim_{n\to\infty} \sigma_n(x) = 0$ which means that $x \in cs_0^\lambda$. Hence, the inclusion $cs_0 \subset cs_0^\lambda$ holds. Conversely, if the inclusion $cs_0 \subset cs_0^\lambda$ holds; then for each $k \geq 1$, we have $\lim_{n\to\infty} \hat{\Lambda}_n(\hat{e}_k) = 0$, where $\hat{e}_k = e_k - e_{k+1} \in cs_0$ for all k. But $\lim_{n\to\infty} \hat{\Lambda}_n(\hat{e}_k) = -\Delta(t_{k+1})$ and so $\Delta(t_{k+1}) = 0$ for all $k \geq 1$. Thus, there exists a positive real $b > 1$ such that $t_k = b$ for all $k \geq 1$ (as $t_1 > 1$). Hence $t_k/\Delta(\lambda_k) - t_{k+1}/\Delta(\lambda_{k+1}) = b/\Delta(\lambda_k) - b/\Delta(\lambda_{k+1})$ and so $1 - 1/b = 1 + \lambda_k \Delta(1/\Delta(\lambda_{k+1}))$ which yields $\Delta(u_{k+1}) = (b-1)/b$ for all $k \geq 1$ and $0 < (b-1)/b < 1$. Further, if the inclusion $cs_0 \subset cs_0^\lambda$ holds; then it must be strict, since the equality can only be held if $a = 0$ (see (2) of Theorem 3.7) which is impossible (as $\Delta(u_2) \neq 0$). □

At the end of this section, we give a general example to support our main results.

Example 3.10 For each non-negative integer $m \geq 0$, we will define other spaces $cs_0^\lambda(m)$, $cs^\lambda(m)$ and $bs^\lambda(m)$ (as particular cases of our spaces) such that the inclusions $cs_0 \subset cs_0^\lambda(m)$, $cs \subset cs^\lambda(m)$ and $bs \subset bs^\lambda(m)$ strictly hold by Corollary 3.9. That is, it will be there an infinitely many number of the spaces according to m. For this, define the sequence $\lambda^{(m)} = (\lambda_k)$ by $\lambda_k = k(k+1)\cdots(k+m+1) = (k+m+1)!/(k-1)!$ for all $k \geq 1$. Then, it can easily be deriving the following $(k, n \geq 1)$:

$$\Delta(\lambda_k) = (m+2)\left[(k+m)!/(k-1)!\right]$$

$$u_k = (k+m+1)/(m+2)\,, \qquad \Delta(u_k) = 1/(m+2) \quad \text{(constant)}$$

$$\frac{1}{\lambda_j} = \frac{1}{j(j+1)\cdots(j+m+1)} = \frac{1}{(m+1)!}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\left[\frac{1}{j+i} - \frac{1}{j+i+1}\right]$$

$$\sum_{j=k}^{n}\frac{1}{\lambda_j} = \frac{1}{(m+1)!}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\left[\frac{1}{k+i} - \frac{1}{n+i+1}\right] = \frac{1}{m+1}\left[\frac{(k-1)!}{(k+m)!} - \frac{n!}{(n+m+1)!}\right]$$

$$t_k^n = \Delta(\lambda_k)\sum_{j=k}^{n}\frac{1}{\lambda_j} = \frac{m+2}{m+1}\left[1 - \binom{k+m}{k-1}\Big/\binom{n+m+1}{n}\right]$$

$$t_k = \Delta(\lambda_k)\sum_{j=k}^{\infty}\frac{1}{\lambda_j} = \frac{m+2}{m+1} \quad \text{(constant)}$$

$$\hat{\Lambda}_n(x) = \frac{m+2}{m+1}\left[\sigma_n(x) - \sum_{k=1}^{n} x_k\binom{k+m}{k-1}\Big/\binom{n+m+1}{n}\right]$$

$$\hat{\Lambda}_n(x) = \frac{m+2}{m+1}\sum_{k=1}^{n}\sigma_k(x)\binom{k+m}{k}\Big/\binom{n+m+1}{n}.$$

Further, from the equality $t_k = (m+2)/(m+1)$; we deduce the following new or known formulae for summation ($m \geq 0$ and $k \geq 1$):

$$\sum_{n=k}^{\infty}\frac{m+1}{n(n+1)\cdots(n+m+1)} = \frac{1}{k(k+1)\cdots(k+m)}$$

$$\sum_{n=k}^{\infty}\frac{1}{n(n+1)\cdots(n+m+1)} = \frac{1}{(m+1)!}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\Big/(k+i)$$

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}\Big/(k+i) = \frac{m!}{k(k+1)\cdots(k+m)}$$

$$\sum_{n=k}^{\infty}\binom{k+m}{k-1}\Big/\binom{n+m+1}{n-1} = \frac{m+2}{m+1}.$$

On other side, we must note that the condition $0 < a < 1$ (or $b > 1$) is necessary in Corollary 3.9 (see Example 2.5 for the case $a = 1$). Also, if $\lambda_k = \alpha^k$ $(\alpha > 1)$; then $\Delta(u_2) \neq 0$ while $\Delta(u_k) = 0$ for $k > 2$ and $t_1 \neq 1$ while $t_k = 1$ for $k > 1$.

4 Schauder bases for the spaces cs_0^λ and cs^λ

In the last section, we construct the Schauder bases for the λ-sequence spaces cs_0^λ and cs^λ, and we conclude their separability.

If a normed space X contains a sequence $(b_k)_{k=1}^\infty$ with the property that for every $x \in X$ there is a unique sequence $(\alpha_k)_{k=1}^\infty$ of scalars such that

$$\lim_{n\to\infty} \|x - (\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n)\| = 0\,;$$

then the sequence $(b_k)_{k=1}^\infty$ is called a Schauder basis for X (or simply a basis for X) and the series $\sum_{k=1}^\infty \alpha_k b_k$ which has the sum x is then called the expansion of x, with respect to the given basis, which can be written as $x = \sum_{k=1}^\infty \alpha_k b_k$, and we then say that x has been uniquely represented in that form. For example, the two sequences $(e_1, e_2, e_3, \cdots)$ and $(e, e_1, e_2, e_3, \cdots)$ are the Schauder bases for the sequence spaces c_0 and c, where $e = (1, 1, 1, \cdots)$ and $e_k = (\delta_{nk})_{n=1}^\infty$ for each $k \geq 1$ [8].

Theorem 4.1 *For each $k \geq 1$, define the sequence $e_k^\lambda = (e_{nk}^\lambda)_{n=1}^\infty$ for every $n \geq 1$ by*

$$e_{nk}^\lambda = \begin{cases} \dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}}\,; & (n = k), \\ -\left(\dfrac{\lambda_{k+1} + \lambda_k}{\lambda_{k+1} - \lambda_k}\right); & (n = k+1), \\ \dfrac{\lambda_{k+1}}{\lambda_{k+2} - \lambda_{k+1}}\,; & (n = k+2), \\ 0\,; & (\text{otherwise}). \end{cases}$$

Then, the sequence $\left(e_k^\lambda\right)_{k=1}^\infty$ is a Schauder basis for the space cs_0^λ and every $x \in cs_0^\lambda$ has a unique representation of the following form:

$$x = \sum_{k=1}^\infty \hat{\Lambda}_k(x)\, e_k^\lambda\,. \tag{4.1}$$

Proof. For each $k \geq 1$, it can easily be seen that

$$e_k^\lambda = \frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\, e_k - \left(\frac{\lambda_{k+1} + \lambda_k}{\lambda_{k+1} - \lambda_k}\right) e_{k+1} + \frac{\lambda_{k+1}}{\lambda_{k+2} - \lambda_{k+1}}\, e_{k+2}\,.$$

Thus, by using (2.1), we find that $\Lambda(e_k^\lambda) = e_k - e_{k+1}$ and so $\hat{\Lambda}(e_k^\lambda) = e_k$. This implies that $\hat{\Lambda}(e_k^\lambda) \in c_0$ and hence $e_k^\lambda \in cs_0^\lambda$ for all $k \geq 1$ which means that $\left(e_k^\lambda\right)_{k=1}^\infty$ is a sequence in cs_0^λ. Further, let $x \in cs_0^\lambda$ be given and for every positive integer m, we put

$$x^{(m)} = \sum_{k=1}^m \hat{\Lambda}_k(x)\, e_k^\lambda\,.$$

Then, by operating $\hat{\Lambda}$ on both sides, we find that

$$\hat{\Lambda}(x^{(m)}) = \sum_{k=1}^m \hat{\Lambda}_k(x)\, \hat{\Lambda}(e_k^\lambda) = \sum_{k=1}^m \hat{\Lambda}_k(x)\, e_k$$

and hence

$$\hat{\Lambda}_n(x - x^{(m)}) = \begin{cases} 0\,; & (1 \leq n \leq m), \\ \hat{\Lambda}_n(x)\,; & (n > m). \end{cases}$$

Now, since $\hat{\Lambda}(x) \in c_0$; for any positive real $\epsilon > 0$, there is a positive integer m_0 such that $|\hat{\Lambda}_m(x)| < \epsilon$ for every $m \geq m_0$. Thus, for any $m \geq m_0$, we have

$$\left\|x - x^{(m)}\right\|_\lambda = \sup_{n>m} |\hat{\Lambda}_n(x)| \leq \sup_{n>m_0} |\hat{\Lambda}_n(x)| \leq \epsilon\,.$$

We therefore deduce that $\lim_{m\to\infty} \|x - x^{(m)}\|_\lambda = 0$ which means that x is represented as in (4.1). Thus, it is remaining to show the uniqueness of the representation (4.1) of x. For this, suppose that $x = \sum_{k=1}^{\infty} \alpha_k\, e_k^\lambda$. Then, we have to show that $\alpha_n = \hat{\Lambda}_n(x)$ for all n, which is immediate by operating $\hat{\Lambda}_n$ on both sides of (4.1) for each $n \geq 1$, where the continuity of $\hat{\Lambda}$ (as we have seen in Remark 2.4) allows us to obtain that

$$\hat{\Lambda}_n(x) = \sum_{k=1}^{\infty} \alpha_k\, \hat{\Lambda}_n(e_k^\lambda) = \sum_{k=1}^{\infty} \alpha_k\, \delta_{nk} = \alpha_n$$

for all $n \geq 1$ and hence the representation (4.1) of x is unique, and this step completes the proof. □

Theorem 4.2 *The sequence* $\left(e^\lambda, e_1^\lambda, e_2^\lambda, \cdots\right)$ *is a Schauder basis for the space* cs^λ *and every* $x \in cs^\lambda$ *has a unique representation in the following form:*

$$x = L\, e^\lambda + \sum_{k=1}^{\infty} \left(\hat{\Lambda}_k(x) - L\right) e_k^\lambda\,, \tag{4.2}$$

where $L = \lim_{n\to\infty} \hat{\Lambda}_n(x)$*, the sequence* $(e_k^\lambda)_{k=1}^{\infty}$ *is as in Theorem 4.1 and* e^λ *is the following sequence:*

$$e^\lambda = e_1 - \left(\frac{\lambda_1}{\lambda_2 - \lambda_1}\right) e_2 = \left(1, -\frac{\lambda_1}{\lambda_2 - \lambda_1}, 0, 0, 0, \cdots\right).$$

Proof. It can easily shown that $\Lambda(e^\lambda) = e_1$ and so $\hat{\Lambda}(e^\lambda) = e \in c$ which means $e^\lambda \in cs^\lambda$. This together with $e_k^\lambda \in cs_0^\lambda \subset cs^\lambda$ imply that $\left(e^\lambda, e_1^\lambda, e_2^\lambda, \cdots\right)$ is a sequence in cs^λ. Also, let $x \in cs^\lambda$ be given. Then $\hat{\Lambda}(x) \in c$ which yields the convergence of the sequence $\hat{\Lambda}(x)$ to a unique limit, say $L = \lim_{n\to\infty} \hat{\Lambda}_n(x)$. Thus, by taking $y = x - L\, e^\lambda$, we get $\hat{\Lambda}(y) = \hat{\Lambda}(x) - L\, e \in c_0$ and so $y \in cs_0^\lambda$. Hence, it follows by Theorem 4.1 that y can be uniquely represented in the following form:

$$y = \sum_{k=1}^{\infty} \hat{\Lambda}_k(y)\, e_k^\lambda = \sum_{k=1}^{\infty} \left(\hat{\Lambda}_k(x) - L\, \hat{\Lambda}_k(e^\lambda)\right) e_k^\lambda = \sum_{k=1}^{\infty} \left(\hat{\Lambda}_k(x) - L\right) e_k^\lambda.$$

Consequently, our x can also be uniquely written as

$$x = L\, e^\lambda + y = L\, e^\lambda + \sum_{k=1}^{\infty} \left(\hat{\Lambda}_k(x) - L\right) e_k^\lambda$$

which proves the unique representation (4.2) of x. □

Corollary 4.3 *We have the following facts:*

(1) *The spaces cs_0^λ and cs^λ are separable BK-spaces.*

(2) *The space bs^λ is a non-separable BK-space and has no a Schauder basis.*

Remark 4.4 We end our work by expressing from now on that the aim of our next paper is to determining the duals of our λ-sequence spaces bs^λ, cs^λ and cs_0^λ, and characterizing some matrix operators between them.

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