On some new difference λ-sequence spaces of p-absolute type

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Abstract.

In the present paper, as a natural continuation of the λ-sequence spaces ℓ_p^λ ($1 \leq p < \infty$) which have already been studied by Mursaleen and Noman in 2011, we will introduce the difference λ-sequence spaces $\ell_p^\lambda(\Delta)$. Also, we will study their properties, inclusion relations and bases. Moreover, we will show that our new spaces $\ell_p^\lambda(\Delta)$ are Banach spaces which are isometrically isomorphic to the classical sequence spaces ℓ_p. Furthermore, among all the spaces $\ell_p^\lambda(\Delta)$, we will show that the space $\ell_2^\lambda(\Delta)$ is only a Hilbert space.

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Introduction

Throughout this paper, we will denote by w for the vector space of all real or complex valued sequences, and any sequence $x \in w$ will be simply written as $x = (x_k)$ instead of $x = (x_k)_{k=1}^{\infty}$. Any vector subspace of w is called a sequence space, and a Banach sequence space is a normed sequence space which is complete. By a *BK*-space, we mean a Banach sequence space with continuous coordinates (see [20]). We shall write ℓ_∞, c and c_0 for the classical sequence spaces of all bounded, convergent and null sequences, respectively, which are BK-spaces with their usual norm $\|\cdot\|_\infty$ defined by $\|x\|_\infty = \sup_k |x_k|$, where the supremum is taken over all $k \geq 1$. Also, by ℓ_p (for each $1 \leq p < \infty$), we denote the sequence space of all sequences associated with p-absolutely convergent series, which is a BK-space with its natural norm $\|\cdot\|_p$ given by $\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}$. Also, we will write bv_p for the sequence space of p-bounded variation, i.e. $bv_p = \{x \in w : (x_n - x_{n-1}) \in \ell_p\}$. For simplicity in notation, here and in sequel, we will use the conventions $e = (1, 1, 1, \ldots)$, $e_k = (\delta_{nk})_{n=1}^{\infty}$ for each $k \geq 1$ and any term with non-positive subscript is equal to naught, e.g. $x_0 = 0$ and $y_{-1} = 0$. If a normed space X contains a sequence $(b_k)_{k=1}^{\infty}$ with the property that for every $x \in X$ there is a unique sequence $(\alpha_k)_{k=1}^{\infty}$ of scalars such that $\lim_{n\to\infty} \|x - (\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n)\| = 0$; then the sequence $(b_k)_{k=1}^{\infty}$ is called a Schauder basis for X (or simply a basis for X) and the series $\sum_{k=1}^{\infty} \alpha_k b_k$ which has the sum x is then called the expansion of x, with respect to the given basis, which can be written as $x = \sum_{k=1}^{\infty} \alpha_k b_k$, and we then say that x has been uniquely represented in that form. For example, the sequence $(e_1, e_2, e_3, \cdots)$ is the Schauder basis for the sequence spaces ℓ_p $(1 \le p < \infty)$. Also, a normed space X is said to be separable if it has a countable dense subset, and it is well-known that a Banach space with Schauder basis must be separable [8]. An infinite matrix A with real or complex entries a_{nk} for $n, k \ge 1$ will be simply written as $A = [a_{nk}]$ instead of $A = [a_{nk}]_{n,k=1}^{\infty}$ and for any sequence $x \in w$ we denote the A-transform of x by $A(x)$ which is defined to be the sequence $A(x) = (A_n(x))_{n=1}^{\infty}$, where

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk}\, x_k\,; \qquad (n \ge 1)$$

provided the convergence of series on the right hand side for each $n \ge 1$, and we then say that $A(x)$ exists, i.e. $A(x) \in w$.

For any two sequence spaces X and Y, we say that an infinite matrix A defines a matrix transformation from X to Y, and we denote it by $A : X \to Y$, which is a linear operator of X into Y, if for every sequence $x \in X$ the sequence $A(x)$ of A-transform of x exists and is in Y. By $(X : Y)$, we denote the class of all infinite matrices A that map X into Y, i.e., $A \in (X : Y)$ if and only if $A(x)$ exists and $Ax \in Y$ for every $x \in X$. For any sequence space X and an infinite matrix A, we will write X_A for the matrix domain of A in X which is a sequence space defined by $X_A = \{x \in w : A(x) \in X\}$. For instance, we note that $bv_p = (\ell_p)_\Delta$, where Δ is the band matrix of difference, i.e. $\Delta(x) = (x_n - x_{n-1})_{n=1}^{\infty} = (x_1, x_2 - x_1, x_3 - x_2, \cdots)$ which means $\Delta(x_n) = x_n - x_{n-1}$ for all $n \ge 1$. An infinite matrix A is called a triangle if $a_{nk} = 0$ for all $k > n$ and $a_{nn} \ne 0$ for all n, where $n, k \ge 1$. It is well-known that if X is a BK-space with a norm $\|\cdot\|$ and A is a triangle; then the matrix domain X_A is also a BK-space with the norm $\|\cdot\|_A$ defined by $\|x\|_A = \|A(x)\|$ for every $x \in X_A$.

The approach constructing a new difference sequence space by means of the matrix domain of a particular triangle has recently been employed by several authors, see for example [1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 14, 15, 16, 17, 18, 19] and [21]. In the present paper, following [13, 14] and [15], we will introduce the difference λ-sequence spaces $\ell_p^\lambda(\Delta)$ for $1 \le p < \infty$ and deduce some related results. Also, we will derive some new inclusion relations between our spaces and construct their bases.

2 The difference λ-sequence spaces $\ell_p^\lambda(\Delta)$

In this section, we introduce the difference λ-sequence spaces $\ell_p^\lambda(\Delta)$ $(1 \le p < \infty)$ and show that these spaces are BK-spaces which are isometrically isomorphic to the spaces ℓ_p.

Here and in what follows, we shall assume throughout that $\lambda = (\lambda_k)_{k=1}^{\infty}$ is a strictly increasing sequence of positive reals tending to ∞, that is $0 < \lambda_1 < \lambda_2 < \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$. For any $x \in w$, we define the sequence $\Lambda(x) = (\Lambda_n(x))_{n=1}^{\infty}$ by

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})\, x_k\,; \qquad (n \ge 1). \tag{2.1}$$

The λ-sequence spaces ℓ_p^λ $(1 \le p < \infty)$ have been introduced by Mursaleen and Noman [12] as follows:

$$\ell_p^\lambda = \Big\{x \in w : \ \Lambda(x) \in \ell_p\Big\} = \Big\{x \in w : \ \sum_{n=1}^{\infty} |\Lambda_n(x)|^p < \infty\Big\}$$

and as a natural continuation of these spaces, our contribution is the new spaces $\ell_p^\lambda(\Delta)$ for $1 \le p < \infty$, which are defined by $\ell_p^\lambda(\Delta) = \big\{x \in w : \ \Delta(x) \in \ell_p^\lambda\big\}$, that is

$$\ell_p^\lambda(\Delta) = \left\{x = (x_k) \in w \ : \ \sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})\, \Delta(x_k) \right|^p < \infty \right\}.$$

Besides, we define the triangle $\bar{\Lambda} = [\bar{\lambda}_{nk}]$ for every $n, k \ge 1$ by

$$\bar{\lambda}_{nk} = \begin{cases} \dfrac{(\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k)}{\lambda_n}\,; & (k < n), \\ \dfrac{\lambda_n - \lambda_{n-1}}{\lambda_n}\,; & (k = n), \\ 0\,; & (k > n). \end{cases}$$

Then, by using (2.1), we have the equality $\bar{\Lambda}(x) = \Lambda\left(\Delta(x)\right)$ which cab be written as

$$\bar{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})\, \Delta(x_k) \qquad (n \ge 1) \tag{2.2}$$

for every $x \in w$ [13, p. 605]. Thus, the difference λ-sequence spaces $\ell_p^\lambda(\Delta)$ can be defined as the matrix domains of $\bar{\Lambda}$ in ℓ_p, that is

$$\ell_p^\lambda(\Delta) = (\ell_p)_{\bar{\Lambda}} = \left\{x \in w \ : \ \sum_{n=1}^{\infty} \left|\bar{\Lambda}_n(x)\right|^p < \infty \right\}. \tag{2.3}$$

It follows that our $\ell_p^\lambda(\Delta)$ are sequence spaces for $1 \le p < \infty$, and we may begin now with the following result which is essential in the text:

Lemma 2.1 *The difference λ-sequence spaces $\ell_p^\lambda(\Delta)$ $(1 \le p < \infty)$ are BK-spaces with the norm $\|\cdot\|_{\lambda p}$ defined for all $x \in \ell_p^\lambda(\Delta)$ by*

$$\|x\|_{\lambda p} = \|\bar{\Lambda}(x)\|_p = \left(\sum_{n=1}^{\infty} \left|\bar{\Lambda}_n(x)\right|^p \right)^{1/p}.$$

Proof. Since $\bar{\Lambda}$ is a triangle; this result is immediate by (2.3) and the fact that every space ℓ_p (for each $1 \le p < \infty$) is a BK-space with its natural norm $\|\cdot\|_p$ (see Maddox [8, pp.217–218]). To see that, the famous result of Wilansky [20, Theorem 4.3.12, p.63] tells us that $\ell_p^\lambda(\Delta)$ is a BK-space with the given norm which completes the proof. □

Theorem 2.2 *For each $1 \leq p < \infty$, the difference λ-sequence space $\ell_p^\lambda(\Delta)$ is isometrically linear-isomorphic to ℓ_p. That is $\ell_p^\lambda(\Delta) \cong \ell_p$ for $1 \leq p < \infty$.*

Proof. For each $1 \leq p < \infty$, we have to prove the existence of a linear bijection between the spaces $\ell_p^\lambda(\Delta)$ and ℓ_p which preserves the norm. For this, we can use the definition of the space $\ell_p^\lambda(\Delta)$ to define a linear operator by means of the matrix transformation $\bar{\Lambda} : \ell_p^\lambda(\Delta) \to \ell_p$ defined by $x \mapsto \bar{\Lambda}(x)$. Then, it is obvious that $x = 0$ whenever $\bar{\Lambda}(x) = 0$, and so $\bar{\Lambda}$ is injective. Also, let $y = (y_k) \in \ell_p$ be given and define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=1}^{k} \left(\frac{\lambda_j\, y_j - \lambda_{j-1}\, y_{j-1}}{\lambda_j - \lambda_{j-1}} \right) \qquad (k \geq 1).$$

Then, for any $k \geq 1$, we have $\Delta(x_k) = (\lambda_k\, y_k - \lambda_{k-1}\, y_{k-1})/(\lambda_k - \lambda_{k-1})$. Thus, it follows by (2.2) that

$$\bar{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k\, y_k - \lambda_{k-1}\, y_{k-1}) = y_n$$

for every $n \geq 1$ which means that $\bar{\Lambda}(x) = y$, but $y \in \ell_p$ and so $\bar{\Lambda}(x) \in \ell_p$. Thus, we deduce that $x \in \ell_p^\lambda(\Delta)$ such that $\bar{\Lambda}(x) = y$ and hence $\bar{\Lambda}$ is surjective. Further, it is clear by Lemma 2.1 that $\bar{\Lambda}$ is norm preserving, since $\|\bar{\Lambda}(x)\|_p = \|x\|_{\lambda p}$ for every $x \in \ell_p^\lambda(\Delta)$. Therefore, the mapping $\bar{\Lambda}$ is a linear bijection preserving the norm. That is, our $\bar{\Lambda}$ is an isometry isomorphism between $\ell_p^\lambda(\Delta)$ and ℓ_p which means that $\ell_p^\lambda(\Delta) \cong \ell_p$. □

Corollary 2.3 *For each $1 \leq p < \infty$, we have the following:*

(1) *The space $\ell_p^\lambda(\Delta)$ is isometrically linear-isomorphic to bv_p, i.e. $\ell_p^\lambda(\Delta) \cong bv_p$.*

(2) *The space $\ell_p^\lambda(\Delta)$ is isometrically linear-isomorphic to ℓ_p^λ, i.e. $\ell_p^\lambda(\Delta) \cong \ell_p^\lambda$.*

Theorem 2.4 *Among all spaces $\ell_p^\lambda(\Delta)$ $(1 \leq p < \infty)$, the space $\ell_2^\lambda(\Delta)$ is the only Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_\lambda$ defined by $\langle x, y \rangle_\lambda = \langle \bar{\Lambda}(x), \bar{\Lambda}(y) \rangle_2$ for all $x, y \in \ell_2^\lambda(\Delta)$, where $\langle \cdot, \cdot \rangle_2$ is the inner product on the space ℓ_2.*

Proof. It is well-known in the elementary functional analysis that except the case $p = 2$, all the spaces ℓ_p $(1 \leq p < \infty)$ are not inner product spaces and hence not Hilbert spaces, where only the norm on ℓ_2 can be induced from an inner product on ℓ_2. Consequently, the present result is immediate by Theorem 2.2. □

Remark 2.5 We have already shown in the proof of Theorem 2.2 that the matrix $\bar{\Lambda}$ defines a linear operator from $\ell_p^\lambda(\Delta)$ into ℓ_p which is an isometry isomorphism $(1 \leq p < \infty)$, and this implies the continuity of the matrix operator $\bar{\Lambda}$ which will be used in the sequel.

At the end of this section, we give an example to show that our new spaces $\ell_p^\lambda(\Delta)$ $(1 \leq p < \infty)$ are different from the sequence spaces ℓ_p, bv_p and ℓ_p^λ.

Example 2.6 Let $1 \le p < \infty$ and define the sequence λ=(λ_k) by $\lambda_k = k$ and so $\Delta(\lambda_k) = 1$ for all $k \ge 1$. Then, for every $x \in w$, we have $\bar{\Lambda}_n(x) = (1/n)\sum_{k=1}^{n}\Delta(x_k) = x_n/n$ $(n \ge 1)$ and $\ell_p^\lambda(\Delta) = \{x \in w : \bar{\Lambda}(x) \in \ell_p\} = \{x \in w : \sum_{n=1}^{\infty}(|x_n|/n)^p < \infty\}$. Also, consider the the sequence $z = (z_k)$ defined by $z_k = (-1)^k/k^{1/p}$ and so $|\Delta(z_k)| \ge 2/k^{1/p}$ for all $k \ge 1$ which implies that $\Delta(z) \notin \ell_p$. Thus $z \notin bv_p$ and so $z \notin \ell_p$. Further, as we have seen above, it is clear that $\bar{\Lambda}_n(z) = (-1)^n/n^{(p+1)/p}$ for all $n \ge 1$ and so $\bar{\Lambda}(z) \in \ell_p$ which means that $z \in \ell_p^\lambda(\Delta)$. Hence, we have shown that $z \in \ell_p^\lambda(\Delta)$ while $z \notin \ell_p$ as well as $z \notin bv_p$ which means that $\ell_p^\lambda(\Delta) \ne \ell_p$ and $\ell_p^\lambda(\Delta) \ne bv_p$, where $p \ge 1$. On other side, we have $\Lambda_n(z) = (1/n)\sum_{k=1}^{n}(-1)^k/k^{1/p}$ $(n \ge 1)$. But, it is obvious that the series $\sum_{k=1}^{\infty}(-1)^k/k^{1/p}$ is conditionally convergent to non-zero sum. Thus, there must exist a positive real $a > 0$ such that $|\sum_{k=1}^{n}(-1)^k/k^{1/p}| \ge a$ and so $|\Lambda_n(z)| \ge a/n$ for all $n \ge 1$ which implies that $\Lambda(z) \notin \ell_1$ and hence $z \notin \ell_1^\lambda$. That is $z \in \ell_1^\lambda(\Delta)$ while $z \notin \ell_1^\lambda$ and this means that $\ell_1^\lambda(\Delta) \ne \ell_1^\lambda$. Similarly, we can define a sequence $z' \in \ell_p^\lambda(\Delta)$ such that $z \notin \ell_p^\lambda$ for $p > 1$ which yeilds that $\ell_p^\lambda(\Delta) \ne \ell_p^\lambda$ for $p > 1$. Consequently, we conclude that our spaces $\ell_p^\lambda(\Delta)$ $(1 \le p < \infty)$ are totally different from the sequence spaces ℓ_p, bv_p and ℓ_p^λ.

3 Some inclusion relations

In the present section, we derive some new inclusion relations concerning the spaces $\ell_p^\lambda(\Delta)$ for $1 \le p < \infty$. We essentially characterize the case in which the inclusion $bv_p \subset \ell_p^\lambda(\Delta)$ and the equality $\ell_p^\lambda(\Delta) = bv_p$ hold for each $1 \le p < \infty$.

Lemma 3.1 *If $1 \le p < q < \infty$; then the inclusion $\ell_p^\lambda(\Delta) \subset \ell_q^\lambda(\Delta)$ strictly holds.*

Proof. Suppose that $1 \le p < q < \infty$. Then, it is clear, by the well-known inclusion $\ell_p \subset \ell_q$, that the inclusion $\ell_p^\lambda(\Delta) \subset \ell_q^\lambda(\Delta)$ holds. To see that, let $x \in \ell_p^\lambda(\Delta)$. Then $\bar{\Lambda}(x) \in \ell_p$ and so $\bar{\Lambda}(x) \in \ell_q$ which implies that $x \in \ell_q^\lambda(\Delta)$. Also, define the sequence $y = (y_k)$ by $y_k = \sum_{j=1}^{k}\Delta\left(\lambda_j(1+j)^{-1/p}\right)/\Delta(\lambda_j)$ and so $\Delta(y_k) = \Delta\left(\lambda_k(1+k)^{-1/p}\right)/\Delta(\lambda_k)$ for all $k \ge 1$. Then, we find that $\bar{\Lambda}_n(y) = (1/\lambda_n)\sum_{k=1}^{n}\Delta\left(\lambda_k(1+k)^{-1/p}\right) = (1+n)^{-1/p}$ for all $n \ge 1$. Thus $\bar{\Lambda}(y) \in \ell_q \setminus \ell_p$ (since $q/p > 1$) which means that $y \in \ell_q^\lambda(\Delta) \setminus \ell_p^\lambda(\Delta)$. That is, the sequence y is in $\ell_q^\lambda(\Delta)$ but not in $\ell_p^\lambda(\Delta)$. Consequently, the inclusion $\ell_p^\lambda(\Delta) \subset \ell_q^\lambda(\Delta)$ is strict and this ends the proof. □

Lemma 3.2 *The inclusion $\ell_p^\lambda(\Delta) \subset c_0^\lambda(\Delta)$ strictly holds, for every $1 \le p < \infty$.*

Proof. If $x \in \ell_p^\lambda(\Delta)$; then $\bar{\Lambda}(x) \in \ell_p \subset c_0$ and hence $\bar{\Lambda}(x) \in c_0$ which means that $x \in c_0^\lambda(\Delta)$, where $c_0^\lambda(\Delta) = \{x \in w : \bar{\Lambda}(x) \in c_0\}$ (see [13, p. 604]). This implies the inclusion $\ell_p^\lambda(\Delta) \subset c_0^\lambda(\Delta)$. Further, to show that this inclusion is strict, consider the sequence $x = (x_k)$ defined by

$$x_k = \sum_{j=1}^{k}\frac{1}{(j+1)^{1/p}}; \qquad (k \ge 1).$$

Then, we have $\Delta(x) = \big(1/(k+1)^{1/p}\big) \in c_0$ and so $\Delta(x) \in c_0^\lambda$, because $c_0 \subset c_0^\lambda$. Thus $x \in c_0^\lambda(\Delta)$. On the other hand, for every $n \geq 1$ we have

$$|\bar{\Lambda}_n(x)| = \frac{1}{\lambda_n}\sum_{k=1}^{n}\frac{\lambda_k - \lambda_{k-1}}{(k+1)^{1/p}} \geq \frac{1}{\lambda_n(n+1)^{1/p}}\sum_{k=1}^{n}(\lambda_k - \lambda_{k-1}) = \frac{1}{(n+1)^{1/p}}.$$

This shows that $\bar{\Lambda}(x) \notin \ell_p$ and hence $x \notin \ell_p^\lambda(\Delta)$. Thus, our sequence x is in $c_0^\lambda(\Delta)$ but not in $\ell_p^\lambda(\Delta)$. Therefore, the inclusion $\ell_p^\lambda(\Delta) \subset c_0^\lambda(\Delta)$ is strict for $1 \leq p < \infty$. □

Now, we are in need to quoting the following two lemmas (see [12, Corollary 4.14] and [12, Lemma 4.2]) which are helpful in proving our main results.

Lemma 3.3 *The inclusion $\ell_p \subset \ell_p^\lambda$ holds if and only if $1/\lambda \in \ell_p$, where $1 \leq p < \infty$.*

Lemma 3.4 *We have the following equivalence:*

$$\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right)_{k=1}^{\infty} \in \ell_\infty \text{ if and only if } \liminf_{k\to\infty}\frac{\lambda_{k+1}}{\lambda_k} > 1.$$

It is obvious that Lemma 3.4 still holds if the sequence $(\lambda_k/(\lambda_k - \lambda_{k-1}))$ is replaced by the sequence $(\lambda_{k-1}/(\lambda_k - \lambda_{k-1}))$, because the difference between their terms is 1. Also, we may note that $1/\lambda = (1/\lambda_k)_{k=1}^{\infty}$ in Lemma 3.3.

Theorem 3.5 *For each $1 \leq p < \infty$, we have the following facts:*

(1) *The inclusion $bv_p \subset \ell_p^\lambda(\Delta)$ holds if and only if $1/\lambda \in \ell_p$.*

(2) *The inclusion $bv_p \subset \ell_p^\lambda(\Delta)$ strictly holds if and only if the inclusion $\ell_p \subset \ell_p^\lambda$ strictly holds.*

(3) *The inclusion $bv_p \subset \ell_p^\lambda(\Delta)$ strictly holds if and only if $1/\lambda \in \ell_p$ as well as $\liminf_{k\to\infty}\lambda_{k+1}/\lambda_k = 1$.*

Proof. Let $1 \leq p < \infty$. To prove (1), suppose that the inclusion $bv_p \subset \ell_p^\lambda(\Delta)$ holds. Then, by taking the sequence $e = (1,1,1,\cdots)$, we get $\Delta(e) = e_1 = (1,0,0,\cdots)$ and so $\Delta(e) \in \ell_p$ which means that $e \in bv_p$ and hence $e \in \ell_p^\lambda(\Delta)$ by assumption. Thus, we have $\bar{\Lambda}(e) \in \ell_p$ and so $\sum_{n=1}^{\infty}|\bar{\Lambda}_n(e)|^p < \infty$, where $\bar{\Lambda}_n(e) = \lambda_1/\lambda_n$ for every $n \geq 1$, which can be obtained by using (2.2) and $\Delta(e) = (1,0,0,\cdots)$. Thus, we get the following

$$\lambda_1^p\sum_{n=1}^{\infty}\frac{1}{\lambda_n^p} = \sum_{n=1}^{\infty}|\bar{\Lambda}_n(e)|^p < \infty$$

which implies that $1/\lambda \in \ell_p$. Conversely, suppose that $1/\lambda \in \ell_p$. Then, the inclusion $\ell_p \subset \ell_p^\lambda$ holds by Lemma 3.3. Thus, for any $x \in bv_p$, we have $\Delta(x) \in \ell_p$ and so $\Delta(x) \in \ell_p^\lambda$ by hypothesis. This means $x \in \ell_p^\lambda(\Delta)$. Hence, we deduce that the inclusion $bv_p \subset \ell_p^\lambda(\Delta)$ holds and this completes the proof of part (1). For part (2), it follows by combining Lemma 3.3 with part (1) that the inclusion $bv_p \subset \ell_p^\lambda(\Delta)$ holds if and only if the inclusion $\ell_p \subset \ell_p^\lambda$ holds. Further, if the inclusion $bv_p \subset \ell_p^\lambda(\Delta)$ is strict; then there exists a sequence $x \in \ell_p^\lambda(\Delta) \setminus bv_p$, i.e. $x \in \ell_p^\lambda(\Delta)$ but $x \notin bv_p$. Thus, we deduce that $\Delta(x) \in \ell_p^\lambda$ while $\Delta(x) \notin \ell_p$. This means that the difference sequence $\Delta(x)$ is in ℓ_p^λ but not in ℓ_p, i.e. $\Delta(x) \in \ell_p^\lambda \setminus \ell_p$ and hence the inclusion $\ell_p \subset \ell_p^\lambda$ is also strict. Conversely, if the inclusion $\ell_p \subset \ell_p^\lambda$ is strict; then there exists a sequence $y \in \ell_p^\lambda \setminus \ell_p$, i.e. $y \in \ell_p^\lambda$ while $x \notin bv_p$. Let us now define a sequence x, in terms of the sequence y, by $x_k = \sum_{j=1}^k y_j$ for every $k \geq 1$. Then $\Delta(x_k) = y_k$ for all k which means that $\Delta(x) = y$. Thus, we deduce that the sequence $\Delta(x)$ is in ℓ_p^λ but not in ℓ_p which means that $x \in \ell_p^\lambda(\Delta)$ while $x \notin bv_p$, i.e. $x \in \ell_p^\lambda(\Delta) \setminus bv_p$ and hence the inclusion $bv_p \subset \ell_p^\lambda(\Delta)$ is also strict, and this ends the proof of part (2). Finally, part (3) is now an immediate consequence of part (2) and the results in [12, Theorem 4.18] which states that the given two conditions are necessary and sufficient for the inclusion $\ell_p \subset \ell_p^\lambda$ to be strictly held, and the proof is now complete. □

Remark 3.6 It well-known that the inclusion $\ell_p \subset bv_p$ strictly holds $(1 \leq p < \infty)$. Thus, if $1/\lambda \in \ell_p$; then the inclusion $\ell_p \subset \ell_p^\lambda(\Delta)$ strictly holds (where $1 \leq p < \infty$), but for this inclusion, the given condition is only sufficient and not necessary. For example, consider the particular case $\lambda_n = n$ (see Example 2.6). Then, the inclusion $\ell_1 \subset \ell_1^\lambda(\Delta)$ holds while $1/\lambda \notin \ell_1$. To see that, it has been shown that $\bar{\Lambda}_n(x) = x_n/n$ and so $|\bar{\Lambda}_n(x)| \leq |x_n|$ for all $n \geq 1$. Thus $\sum_{n=1}^\infty |\bar{\Lambda}_n(x)| \leq \sum_{n=1}^\infty |x_n| < \infty$ whenever $x \in \ell_1$ which shows that $\ell_1 \subset \ell_1^\lambda(\Delta)$.

Theorem 3.7 *The equality* $\ell_p^\lambda(\Delta) = bv_p$ *holds if and only if* $\liminf_{k\to\infty} \lambda_{k+1}/\lambda_k > 1$, *where* $1 \leq p < \infty$.

Proof. Let $1 \leq p < \infty$. Then, to prove the necessity of the given condition, suppose that the equality $\ell_p^\lambda(\Delta) = bv_p$ holds. Then, the inclusion $bv_p \subset \ell_p^\lambda(\Delta)$ holds and so $1/\lambda \in \ell_p$ by part (1) of Theorem 3.5. Also, the inclusion $bv_p \subset \ell_p^\lambda(\Delta)$ cannot be strict and this leads us with part (3) of Theorem 3.5 to conclude that $\liminf_{k\to\infty} \lambda_{k+1}/\lambda_k \neq 1$ and hence $\liminf_{k\to\infty} \lambda_{k+1}/\lambda_k > 1$, since $\lambda_{k+1}/\lambda_k > 1$ for all $k \geq 1$. Conversely, for the sufficiency, assume that $\liminf_{k\to\infty} \lambda_{k+1}/\lambda_k > 1$. Then, there exists a real number $\alpha > 1$ such that $\lambda_{k+1}/\lambda_k \geq \alpha$ for all $k \geq 1$. Thus, for any $k \geq 1$, we deduce that $\lambda_k \geq \alpha\, \lambda_{k-1} \geq \alpha^2\, \lambda_{k-2} \geq \cdots \geq \alpha^{k-1}\, \lambda_1$, that is $\lambda_k \geq \alpha^{k-1}\, \lambda_1$ and so $1/\lambda_k \leq 1/(\alpha^{k-1}\lambda_1)$ for all k which means that $1/\lambda \in \ell_1 \subset \ell_p$, because $\alpha > 1$. Therefore, we have $1/\lambda \in \ell_p$ and it follows by part (1) of Theorem 3.5 that the inclusion $bv_p \subset \ell_p^\lambda(\Delta)$ holds. So, it is remaining to prove the converse inclusion $\ell_p^\lambda(\Delta) \subset bv_p$. For this, we have the following relation which is satisfied for any $x \in w$ (see [12, Lemma 4.1])

$$x_n - \Lambda_n(x) = \frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} \left[\Lambda_n(x) - \Lambda_{n-1}(x)\right], \qquad (n \geq 1).$$

Thus, by replacing x by $\Delta(x)$ in this relation and noting that $\Lambda(\Delta(x)) = \bar{\Lambda}(x)$, we establish the following relation

$$\Delta(x_n) - \bar{\Lambda}_n(x) = \frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} \left[\bar{\Lambda}_n(x) - \bar{\Lambda}_{n-1}(x)\right], \qquad (n \geq 1)$$

for any sequence $x \in w$. Finally, let $x \in \ell_p^\lambda(\Delta)$ be given. Then, we have $\bar{\Lambda}(x) \in \ell_p$ and so $\big(\bar{\Lambda}_n(x) - \bar{\Lambda}_{n-1}(x)\big) \in \ell_p$. Besides, since $\liminf_{k\to\infty} \lambda_{k+1}/\lambda_k > 1$; it follows by Lemma 3.4 that $(\lambda_n/(\lambda_n - \lambda_{n-1})) \in \ell_\infty$ and so $(\lambda_{n-1}/(\lambda_n - \lambda_{n-1})) \in \ell_\infty$. Therefore, we deduce from the above relation that $\Delta(x) \in \ell_p$ and hence $x \in bv_p$ which proves the inclusion $\ell_p^\lambda(\Delta) \subset bv_p$. This inclusion together with the previous one lead us to the equality $\ell_p^\lambda(\Delta) = bv_p$, and the proof is now complete. □

Finally, we support our results in Theorems 3.5 and 3.7 by the following example:

Example 3.8 Let $1 \le p < \infty$. Then, we have the following two distinct cases:

I - The case of strict inclusion $bv_p \subset \ell_p^\lambda(\Delta)$: consider the sequence $\lambda = (\lambda_k)$ defined by $\lambda_k = k^a$ for all k, where $a > 1$. Then, it can easily be shown that $\lambda_k/\Delta(\lambda_k) \ge k/a$ for all k. In such case, we will generally obtain that $\lambda/\Delta(\lambda) \notin \ell_\infty$ and so $bv_p \subsetneqq \ell_p^\lambda(\Delta)$ by Theorem 3.5.

II - The case of equality $\ell_p^\lambda(\Delta) = bv_p$: consider the sequence $\lambda = (\lambda_k)$ defined by $\lambda_k = k!$ for all k (or $\lambda_k = a^k$, where $a > 1$). Then, it can easily be shown that $\lambda_k/\Delta(\lambda_k) = k/(k-1)$ for all k (or $\lambda_k/\Delta(\lambda_k) = a$). In such case, we will generally find that $\lambda/\Delta(\lambda) \in \ell_\infty$ and so $\ell_p^\lambda(\Delta) = bv_p$ by Theorem 3.7.

4 Schauder basis for the spaces $\ell_p^\lambda(\Delta)$

In this final section, we construct a common sequence in all the spaces $\ell_p^\lambda(\Delta)$ $(1 \le p < \infty)$ which forms the Schauder basis for each space of them.

If a normed space X contains a sequence $(b_k)_{k=1}^\infty$ with the property that for every $x \in X$ there is a unique sequence $(\alpha_k)_{k=1}^\infty$ of scalars such that

$$\lim_{n\to\infty} \|x - (\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n)\| = 0\,;$$

then the sequence $(b_k)_{k=1}^\infty$ is called a Schauder basis for X (or simply a basis for X) and the series $\sum_{k=1}^\infty \alpha_k b_k$ which has the sum x is then called the expansion of x, with respect to the given basis, which can be written as $x = \sum_{k=1}^\infty \alpha_k b_k$, and we then say that x has been uniquely represented in that form. For example, the sequence $(e_1, e_2, e_3, \cdots)$ is the Schauder basis for each space of the sequence spaces ℓ_p $(1 \le p < \infty)$, where $e_k = (\delta_{nk})_{n=1}^\infty$ for each $k \ge 1$. Also, a normed space X is said to be separable if it has a countable dense subset, and it is well-known that a Banach space with Schauder basis must be separable [8].

Theorem 4.1 *For each $k \ge 1$, define the sequence $e_k^\lambda = (e_{nk}^\lambda)_{n=1}^\infty$ by*

$$e_{nk}^\lambda = \begin{cases} 0\,; & (n < k), \\ \dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}}\,; & (n = k), \qquad (n \ge 1) \\ \dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}} - \dfrac{\lambda_k}{\lambda_{k+1} - \lambda_k}\,; & (n > k). \end{cases}$$

Then, the sequence $\left(e_k^\lambda\right)_{k=1}^\infty$ *is a Schauder basis for the space* $\ell_p^\lambda(\Delta)$ $(1 \le p < \infty)$ *and every* $x \in \ell_p^\lambda(\Delta)$ *has a unique representation of the following form:*

$$x = \sum_{k=1}^{\infty} \bar{\Lambda}_k(x)\, e_k^\lambda\,. \tag{4.1}$$

Proof. Let $1 \le p < \infty$. Then, for each $k \ge 1$, it can easily be seen that $\Delta(e_k) = \left(\Delta(e_{nk}^\lambda)\right)_{n=1}^\infty$, where

$$\Delta(e_{nk}^\lambda) = \begin{cases} \dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}}\,; & (n = k), \\ -\dfrac{\lambda_k}{\lambda_{k+1} - \lambda_k}\,; & (n = k+1), \qquad (n \ge 1) \\ 0\,; & (\text{otherwise}). \end{cases}$$

Thus, by using (2.2) we get $\bar{\Lambda}_n(e_k^\lambda) = \delta_{nk}$ for all $n, k \ge 1$. That is $\bar{\Lambda}(e_k^\lambda) = e_k \in c_0$ and hence $e_k^\lambda \in \ell_p^\lambda(\Delta)$ for all $k \ge 1$. This means that $\left(e_k^\lambda\right)_{k=1}^\infty$ is a sequence in $\ell_p^\lambda(\Delta)$. Further, let $x \in \ell_p^\lambda(\Delta)$ be given and for every positive integer m, we put

$$x^{(m)} = \sum_{k=1}^{m} \bar{\Lambda}_k(x)\, e_k^\lambda\,.$$

Then, by operating $\bar{\Lambda}$ on both sides, we find that

$$\bar{\Lambda}(x^{(m)}) = \sum_{k=1}^{m} \bar{\Lambda}_k(x)\, \bar{\Lambda}(e_k^\lambda) = \sum_{k=1}^{m} \bar{\Lambda}_k(x)\, e_k$$

and hence

$$\bar{\Lambda}_n(x - x^{(m)}) = \begin{cases} 0\,; & (1 \le n \le m), \\ \bar{\Lambda}_n(x)\,; & (n > m). \end{cases}$$

Now, since $\bar{\Lambda}(x) \in \ell_p$; for any positive real $\epsilon > 0$, there is a positive integer m_0 such that $\sum_{n=m_0+1}^\infty |\bar{\Lambda}_n(x)| < \epsilon^p$. Thus, for any $m \ge m_0$, we have

$$\left\| x - x^{(m)} \right\|_{\lambda p} = \left(\sum_{n=m+1}^{\infty} |\bar{\Lambda}_n(x)| \right)^{1/p} \le \left(\sum_{n=m_0+1}^{\infty} |\bar{\Lambda}_n(x)| \right)^{1/p} < \epsilon\,.$$

We therefore deduce that $\lim_{m\to\infty} \|x - x^{(m)}\|_{\lambda p} = 0$ which means that x is represented as in (4.1). Thus, it is remaining to show the uniqueness of the representation (4.1) of x. For this, suppose that $x = \sum_{k=1}^\infty \alpha_k\, e_k^\lambda$. Then, we have to show that $\alpha_n = \bar{\Lambda}_n(x)$ for all n, which is immediate by operating $\bar{\Lambda}_n$ on both sides of (4.1) for each $n \ge 1$, where the continuity of $\bar{\Lambda}$ (as we have seen in Remark 2.5) allows us to obtain that

$$\bar{\Lambda}_n(x) = \sum_{k=1}^{\infty} \alpha_k\, \bar{\Lambda}_n(e_k^\lambda) = \sum_{k=1}^{\infty} \alpha_k\, \delta_{nk} = \alpha_n$$

for all $n \ge 1$ and hence the representation (4.1) of x is unique, and this step completes the proof. □

Corollary 4.2 *The difference λ-sequence spaces $\ell_p^\lambda(\Delta)$ are separable BK-spaces for $1 \leq p < \infty$.*

Proof. Since $\ell_p^\lambda(\Delta)$ $(1 \leq p < \infty)$ are Banach spaces begin BK-spaces as we have shown in Lemma 2.1 and these spaces have Schauder basis by Theorem 4.1, they must be separable. □

Conclusion: In the present paper, we have essentially introduced some new Banach and Hilbert spaces of sequences and constract their bases, which will give a scope for further research and study in future. So, we end our work by expressing from now on that the aim of our next paper is to determining the duals of the difference λ-sequence spaces $\ell_p^\lambda(\Delta)$ and characterizing some matrix transformations between them, where $1 \leq p < \infty$.

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