On the new λ-sequence spaces of p-bounded variation

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Abstract.

The λ-sequence spaces ℓ_p^λ $(1 \le p < \infty)$ have been introduced by Mursaleen and Noman.

In the present paper, we will use the same technique to introduce the new λ-sequence spaces bv_p^λ of p-bounded variation and study their properties. Also, we will show that the new spaces bv_p^λ are Banach spaces which are isometrically isomorphic to the classical sequence spaces ℓ_p for $1 \le p < \infty$. Further, we will establish their bases and derive some inclusion relations between them. Moreover, we will prove that among all our spaces, the space bv_2^λ is the only Hilbert space.

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1 Introduction

By w, we denote the vector space of all real or complex valued sequences and any vector subspace of w is called a sequence space. Also, any sequence $x \in w$ will be written as $x = (x_k)$ instead of $x = (x_k)_{k=1}^\infty$ and we will use the conventions $e = (1,1,1,...)$ and $e_k = (\delta_{nk})_{n=1}^\infty$ for each $k \in \mathbb{N}$, that is e_k is the sequence whose only one non-zero term which is the k-term and is equal to 1. A sequence space X with a linear topology is called a Kspace provided each of the maps $P_n : X \to \mathbb{F}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$, where $\mathbb{F}$ denotes the scalar field and $\mathbb{N} = \{1,2,3,...\}$. A K-space X is called an FK-space provided X is a complete linear metric space. An FK-space whose topology is normable is called a BK-space. We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively, which are BK-spaces with the sup-norm given by $\|x\| = \sup_k |x_k|$, where, here and in the sequel, the supremum $_\infty$

is taken over all $k \in \mathbb{N}$. Also, by ℓ_p $(1 \le p < \infty)$, we denote the sequence space of $\|x\|_p = \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p}$ all sequences associated with p-absolutely convergent series, which is a BK-space with the p-norm defined by(see [11, pp. 217-218]). If a normed space X contains a sequence $(b_k)_{k=1}^\infty$ with the property that for every $x \in X$ there is a unique sequence $(\alpha_k)_{k=1}^\infty$ of scalars such that $\lim_{n\to\infty} \|x - (\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n)\| = 0$; then the sequence $(b_k)_{k=1}^\infty$ is called a Schauder basis for X (or simply a basis for X) and the series $\sum_{k=1}^\infty \alpha_k b_k$ which has the sum x is then called the expansion of x, with respect to the given basis, which can be written as $x = \sum_{k=1}^\infty \alpha_k b_k$, and we then say that x has been uniquely represented in that form. For example, the sequence $(e_1, e_2, e_3, \cdots)$ is the Schauder basis for the sequence spaces ℓ_p $(1 \le p < \infty)$. Also, a normed space X is said to be separable if it has a countable dense subset, and it is well-known that a Banach space with Schauder basis must be separable [11]. Also, let X and Y be sequence spaces and $A = [a_{nk}]$ an infinite matrix of real or complex entries a_{nk} for $n,k \in \mathbb{N}$. Then, we say that A defines a matrix transformation from X into Y if for every sequence $x = (x_k) \in X$ the sequence $A(x) = (A_n(x))$, called as the A-transform of x, exists and is in Y, where

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk}\, x_k\,, \qquad (n \in \mathbb{N})$$

provided the convergence of each series. By $(X : Y)$, we denote the class of all infinite matrices that map X into Y. Thus $A \in (X : Y)$ if and only if the A-transform of every sequence $x \in X$ exists such that $A(x) \in Y$ for all $x \in X$. If $A \in (X : Y)$; then A defines a linear operator $A : X \to Y$ by $x \mapsto A(x)$. For a sequence space X, the matrix domain of an infinite matrix A in X is a sequence space defined by $X_A = \{x \in w : A(x) \in X\}$. An infinite matrix A is called a triangle if $a_{nn} \neq 0$ for all n and $a_{nk} = 0$ for every $k > n$, where $n, k \in \mathbb{N}$. The matrix domain of a triangle A in a BK-space X is also a BK-space with the norm defined by $\|x\|_A = \|A(x)\|$ for all $x \in X_A$, where $\|\cdot\|$ is the norm on X. For example, the sequence space bv_p of p-bounded variation $(1 \le p < \infty)$ is defined as the matrix domain of the matrix Δ in the sequence space ℓ_p, where the triangle Δ is the band matrix of difference, that is $\Delta(x) = (x_1, x_2 - x_1, x_3 - x_2, \cdots)$ which means that $\Delta(x_k) = x_k - x_{k-1}$ for all $k \in \mathbb{N}$ with using the convention that any term with non-positive subscript is equal to zero, i.g. $x_0 = x_{-1} = 0$. Thus $bv_p = (\ell_p)_\Delta = \{x \in w : \Delta(x) \in \ell_p\}$, and so bv_p is a BK-space with the norm given by $\|x\|_{\Delta p} = \big(\sum_{k=1}^{\infty} |x_k - x_{k-1}|^p\big)^{1/p}$, where $1 \le p < \infty$. Recently, many authors in several research papers have used the idea of matrix domain to introduce new sequence spaces as the matrix domains of some particular triangles via different manners, and some of them have introduced sequence spaces of bounded variation and other types, see for instance [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18] and [20]. In this paper, we will use the same technique to introduce the new λ-sequence spaces bv_p^λ of p-bounded variation and study their properties. Also, we will show that the new spaces bv_p^λ are Banach spaces which are isometrically isomorphic to the classical sequence spaces ℓ_p for $1 \le p < \infty$. Further, we will establish their bases and derive some inclusion relations between them. Moreover, we will prove that among all our spaces, the space bv_2^λ is the only Hilbert space.

2 The new λ-sequence spaces bv_p^λ of p-bounded variation

In this section, we introduce the new λ-sequence spaces bv_p^λ of p-bounded variation $(1 \le p < \infty)$ and show that these spaces are BK-spaces isometrically isomorphic to the classical sequence spaces ℓ_p.

Here and in what follows, we shall assume throughout that $\lambda = (\lambda_k)_{k=1}^{\infty}$ is a strictly increasing sequence of positive reals and p is a positive real greater than or equal to 1, that is $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ and $1 \le p < \infty$. Also, we define the infinite matrix $\Lambda = [\lambda_{nk}]$ for every $n, k \in \mathbb{N}$ by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n}\,; & (1 \le k \le n), \\ 0\,; & (k > n \ge 1). \end{cases}$$

Then, for any sequence $x = (x_k) \in w$, we obtain the sequence $\Lambda(x) = (\Lambda_n(x))_{n=1}^{\infty}$, where

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})\, x_k\, ; \qquad (n \in \mathbb{N}). \tag{2.1}$$

The idea of λ-sequence spaces c_0^λ, c^λ, ℓ_∞^λ and ℓ_p^λ have been introduced by by Mursaleen and Noman [12, 14] as the matrix domains of the triangle Λ in the classical sequence spaces c_0, c, ℓ_∞ and ℓ_p, respectively. For example, the spaces c^λ and ℓ_p^λ have been defined as follows:

$$\begin{aligned} c^\lambda &= (c)_\Lambda = \{x \in w : \ \Lambda(x) \in c\} = \{x \in w : \ \lim_{n\to\infty} \Lambda_n(x) \ \text{exists}\}\,, \\ \ell_p^\lambda &= (\ell_p)_\Lambda = \{x \in w : \ \Lambda(x) \in \ell_p\} = \{x \in w : \ \textstyle\sum_{n=1}^{\infty} |\Lambda_n(x)|^p < \infty\}\,. \end{aligned}$$

Now, as a natural continuation of their work, we follow them to introduce the new space bv_p^λ for each p $(1 \le p < \infty)$ as the matrix domain of the triangle Λ in the space bv_p. That is, our contribution is the following new λ-sequence space of p-bounded variation:

$$bv_p^\lambda = (bv_p)_\Lambda = \big\{x \in w : \ \Lambda(x) \in bv_p\big\} = \big\{x \in w : \ (\,\Lambda_n(x) - \Lambda_{n-1}\,)_{n=1}^{\infty} \in \ell_p\big\}.$$

Besides, we define the triangle $\tilde{\Lambda} = [\tilde{\lambda}_{nk}]$ for every $n, k \in \mathbb{N}$ by

$$\tilde{\lambda}_{nk} = \begin{cases} \dfrac{\lambda_n - \lambda_{n-1}}{\lambda_n}\,; & (n = k), \\ (\lambda_k - \lambda_{k-1}) \left(\dfrac{1}{\lambda_n} - \dfrac{1}{\lambda_{n-1}} \right)\,; & (n > k), \\ 0\,; & (n < k). \end{cases}$$

Then, for any sequence $x = (x_k) \in w$, we have $\tilde{\Lambda}_1(x) = x_1$ and

$$\tilde{\Lambda}_n(x) = \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \right) x_n + \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}} \right) \sum_{k=1}^{n-1} (\lambda_k - \lambda_{k-1})\, x_k\,, \qquad (n \ge 2). \tag{2.2}$$

Thus, it can easily be shown, for every sequence $x \in w$, that

$$\tilde{\Lambda}_n(x) = \ \Delta(\,\Lambda_n(x)\,) = \ \Lambda_n(x) - \Lambda_{n-1}(x)\,, \qquad (n \in \mathbb{N}) \tag{2.3}$$

and so $\tilde{\Lambda}(x) = (\,\Lambda_n(x) - \Lambda_{n-1}(x)\,)_{n=1}^{\infty}$. It follows that our spaces bv_p^λ can equivalently be redefined as the matrix domains of the triangle $\tilde{\Lambda}$ in the respective spaces ℓ_p for $1 \le p < \infty$. That is

$$bv_p^\lambda = (\ell_p)_{\tilde{\Lambda}} = \left\{ x \in w : \ \tilde{\Lambda}(x) \in \ell_p \right\}, \qquad (1 \le p < \infty) \tag{2.4}$$

which can be written as follows:

$$bv_p^\lambda = \left\{ x \in w : \ \sum_{n=1}^{\infty} \left| \tilde{\Lambda}_n(x) \right|^p < \infty \right\} = \left\{ x \in w : \ \sum_{n=1}^{\infty} \left| \Lambda_n(x) - \Lambda_{n-1}(x) \right|^p < \infty \right\}.$$

We therefore deduce that bv_p^λ $(1 \le p < \infty)$ are sequence spaces and we may begin now with the following result which is essential in the text:

Lemma 2.1 *For each p $(1 \leq p < \infty)$, the λ-sequence space bv_p^λ of p-bounded variation is a BK-space with the norm $\|\cdot\|_{\lambda p}$ defined for every $x \in bv_p^\lambda$ by*

$$\|x\|_{\lambda p} = \|\tilde{\Lambda}(x)\|_p = \left(\sum_{n=1}^{\infty} \left| \tilde{\Lambda}_n(x) \right|^p \right)^{1/p}.$$

Proof. For each p $(1 \leq p < \infty)$, we know that ℓ_p is a BK-space with its natural p-norm $\|\cdot\|_p$ (Maddox [11, pp. 217–218]), and since $\tilde{\Lambda}$ is a triangle; it follows from (2.4) that bv_p^λ is the matrix domain of the triangle $\tilde{\Lambda}$ in the BK-spaces ℓ_p. Thus, our reslut is immediate by the famous result of Wilansky [19, Theorem 4.3.12, p. 63] which tells us that bv_p^λ is a BK-space with the given norm. □

Theorem 2.2 *For each p $(1 \leq p < \infty)$, the λ-sequence space bv_p^λ of p-bounded variation is isometrically linear-isomorphic to the space ℓ_p. That is $bv_p^\lambda \cong \ell_p$ for $1 \leq p < \infty$.*

Proof. Let $1 \leq p < \infty$. Then, we will prove the existence of a linear bijection between the spaces bv_p^λ and ℓ_p which preserves the norm. For this, by using the definition of the space bv_p^λ, we have the matrix transformation $\tilde{\Lambda} \in (bv_p^\lambda : \ell_p)$ which defines a linear operator $\tilde{\Lambda} : bv_p^\lambda \to \ell_p$ by $x \mapsto \tilde{\Lambda}(x)$. Also, it is obvious that $x = 0$ whenever $\tilde{\Lambda}(x) = 0$, and so $\tilde{\Lambda}$ is injective. Further, let $y \in \ell_p$ be given and define a sequence $x = (x_k)$ in terms of the sequence y by

$$x_k = \frac{1}{\lambda_k - \lambda_{k-1}} \left(\lambda_k \sum_{j=1}^{k} y_j - \lambda_{k-1} \sum_{j=1}^{k-1} y_j \right); \qquad (k \in \mathbb{N}),$$

where $x_1 = y_1$ (since $\lambda_0 = 0$). Then, for every $n \in \mathbb{N}$, it follows by (2.1) that

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) x_k = \frac{1}{\lambda_n} \sum_{k=1}^{n} \left(\lambda_k \sum_{j=1}^{k} y_j - \lambda_{k-1} \sum_{j=1}^{k-1} y_j \right) = \sum_{j=1}^{n} y_j$$

which together with (2.3) imply that $\tilde{\Lambda}_n(x) = \Lambda_n(x) - \Lambda_{n-1}(x) = y_n$ for all $n \in \mathbb{N}$ and this means $\tilde{\Lambda}(x) = y \in \ell_p$. Thus $x \in bv_p^\lambda$ such that $\tilde{\Lambda}(x) = y$. This shows that $\tilde{\Lambda}$ is surjective and hence $\tilde{\Lambda}$ is a linear isomorphism. Finally, we have by Lemma 2.1 that $\|\tilde{\Lambda}(x)\|_p = \|x\|_{\lambda p}$ for all $x \in bv_p^\lambda$ which means that $\tilde{\Lambda}$ is norm-preserving, and so $\tilde{\Lambda}$ is a linear bijection which preserves the norm. Hence, we deduce that $bv_p^\lambda \cong \ell_p$. This completes the proof. □

Corollary 2.3 *For each $1 \leq p < \infty$, we have the following facts:*

(1) *The space bv_p^λ is isometrically linear-isomorphic to the space bv_p, that is $bv_p^\lambda \cong bv_p$.*

(2) *The space bv_p^λ is isometrically linear-isomorphic to the space ℓ_p^λ, that is $bv_p^\lambda \cong \ell_p^\lambda$.*

Proof. It is immediate by Theorem 2.2 and the facts that $bv_p \cong \ell_p$ and $\ell_p^\lambda \cong \ell_p$ for each p, where $1 \leq p < \infty$. □

Corollary 2.4 *Among all the spaces* bv_p^λ $(1 \le p < \infty)$, *the space* bv_2^λ *is the only Hilbert space equipped with the inner product* $\langle \cdot, \cdot \rangle_\lambda$ *defined by* $\langle x, y\rangle_\lambda = \langle \tilde{\Lambda}(x), \tilde{\Lambda}(y)\rangle_2$ *for all* $x, y \in bv_2^\lambda$, *where* $\langle \cdot, \cdot \rangle_2$ *is the inner product on the space* ℓ_2.

Proof. It is well-known in the elementary functional analysis that except the case $p = 2$, all the spaces ℓ_p $(1 \le p < \infty)$ are not inner product spaces and hence not Hilbert spaces, where only the norm on ℓ_2 can be induced from an inner product on ℓ_2. Consequently, the present result is immediate by Theorem 2.2. □

Remark 2.5 In the proof of Theorem 2.2, we have already shown that the matrix $\tilde{\Lambda}$ defines a linear operator from bv_p^λ into ℓ_p $(1 \le p < \infty)$ which is an isometry isomorphism , and this implies the continuity of the matrix mapping $\tilde{\Lambda}$ which will be used in the sequel.

Remark 2.6 It is noted that we can similarly define the space $bv_\infty^\lambda = \{x \in w : \Lambda(x) \in bv_\infty\} = \{x \in w : \tilde{\Lambda}(x) \in \ell_\infty\}$ which is a BK-space with the norm defined by $\|x\|_{\lambda\infty} = \sup_n |\tilde{\Lambda}_n(x)|$ for all $x \in bv_\infty^\lambda$ and is isometrically linear-isomorphic to each of the spaces bv_∞, ℓ_∞ and ℓ_∞^λ, where $bv_\infty = \{x \in w : \Delta(x) \in \ell_\infty\}$.

At the end of this section, we give a general example to show that our new spaces bv_p^λ $(1 \le p < \infty)$ are different from the classical sequence spaces ℓ_p and bv_p.

Example 2.7 For each pair of (fixed) reals $a > 0$ and $p \ge 1$, we will define the space $bv_p^\lambda(a)$ (as a particular case of bv_p^λ) such that $bv_p^\lambda(a_1) \ne bv_p^\lambda(a_2)$ whenever $a_1 \ne a_2$, as well as $bv_{p_1}^\lambda(a) \ne bv_{p_2}^\lambda(a)$ whenever $p_1 \ne p_2$. That is, it will be there an infinitely many number of the spaces $bv_p^\lambda(a)$ according to both a and p, and all these spaces are different from the spaces ℓ_p and bv_p. For this, let $a > 0$ and $1 \le p < \infty$, and define the sequence $\lambda^{(a)} = (\lambda_k)$ by $\lambda_k = k^a$ and so $\Delta(\lambda_k) = k^a - (k-1)^a$ for all $k \in \mathbb{N}$. Then, for every $x \in w$, we have $\Lambda_n(x) = (1/n^a) \sum_{k=1}^n (k^a - (k-1)^a) x_k$ $(n \in \mathbb{N})$ and $bv_p^\lambda(a) = \{x \in w : \Lambda_n(x) \in bv_p\}$. Now, consider the following two distinct cases:

I - When $p > 1$: in this case, we define a sequence $z = (z_k)$ by $z_1 = -1$ and

$$z_k = (-1)^k \left(\frac{\lambda_k^{(a-1)/a} + \lambda_{k-1}^{(a-1)/a}}{\lambda_k - \lambda_{k-1}} \right) = (-1)^k \left(\frac{k^{a-1} + (k-1)^{a-1}}{k^a - (k-1)^a} \right), \quad (k > 1)\cdot$$

Then $z_k = (-1)^k |z_k|$ for all k and we have

$$\lim_{k\to\infty} |z_k| = \lim_{k\to\infty} \left(\frac{k^{a-1}}{k^a - (k-1)^a} + \frac{(k-1)^{a-1}}{k^a - (k-1)^a} \right) = \frac{1}{a} + \frac{1}{a} = \frac{2}{a}\cdot$$

Thus $(|z_k|) \in c \setminus c_0$ and so $z \notin c$ (z is oscillated between $\pm 2/a$). Hence $z \notin \ell_p$ and $z \notin bv_p$. On other side, we have $\Lambda_n(z) = (1/n^a) \sum_{k=1}^n (-1)^k (k^{a-1} + (k-1)^{a-1}) = (-1)^n/n$ for all $n \in \mathbb{N}$. Thus $\Lambda(z) \in bv_p$ and so $z \in bv_p^\lambda(a)$. Hence, we find that $z \in bv_p^\lambda(a)$ while $z \notin \ell_p$ and $z \notin bv_p$ which means that $bv_p^\lambda(a) \ne \ell_p$ and $bv_p^\lambda(a) \ne bv_p$, where $p > 1$.

II - When $p = 1$: consider the sequence $z' = (z'_k)$ defined by $z'_1 = -1$ and

$$z'_k = (-1)^k \left(\frac{\lambda_k^{(a-2)/a} + \lambda_{k-1}^{(a-2)/a}}{\lambda_k - \lambda_{k-1}} \right) = (-1)^k \left(\frac{k^{a-2} + (k-1)^{a-2}}{k^a - (k-1)^a} \right), \quad (k > 1)\cdot$$

Then $z'_k = (-1)^k|z'_k|$ for all k and $\lim_{k\to\infty} k|z'_k| = \lim_{k\to\infty}|z_k| = 2/a$ which means that $\lim_{k\to\infty}|z'_k| = \lim_{k\to\infty} 2/(ak)$. But $((-1)^k/k) \notin bv_1$ and so $z' \notin bv_1$ as well as $z' \notin \ell_1$. On other side, we have $\Lambda_n(z') = (1/n^a)\sum_{k=1}^n (-1)^k(k^{a-2} + (k-1)^{a-2}) = (-1)^n/n^2$ for all $n \in \mathbb{N}$. Thus $\Lambda(z') \in bv_1$ and so $z' \in bv_1^\lambda(a)$. Hence, we find that $z' \in bv_1^\lambda(a)$ while $z' \notin \ell_1$ and $z' \notin bv_1$ which means that $bv_1^\lambda(a) \neq \ell_1$ as well as $bv_1^\lambda(a) \neq bv_1$. Consequently, we conclude that $bv_p^\lambda(a) \neq \ell_p$ and $bv_p^\lambda(a) \neq bv_p$ for $1 \leq p < \infty$.

3 Some inclusion relations

In the present section, we establish some new inclusion relations concerning the λ-sequence spaces bv_p^λ $(1 \leq p < \infty)$. We essentially prove the inclusion $bv_p \subset bv_p^\lambda$ and characterize the case in which the equality $bv_p^\lambda = bv_p$ will be held.

Lemma 3.1 *If $1 \leq p < q < \infty$; then the inclusion $bv_p^\lambda \subset bv_q^\lambda$ strictly holds.*

Proof. Suppose that $1 \leq p < q < \infty$. Then, the inclusion $bv_p^\lambda \subset bv_q^\lambda$ clearly holds by the well-known inclusion $bv_p \subset bv_q$. Also, to show that this inclusion is strict, define a sequence $x = (x_k)$ by

$$x_k = \frac{1}{\lambda_k - \lambda_{k-1}}\,\Delta\left(\lambda_k \sum_{j=1}^{k} \frac{1}{(1+j)^{1/p}}\right) \qquad (k \in \mathbb{N}).$$

Then, for every $n \in \mathbb{N}$, it follows from (2.1) that

$$\Lambda_n(x) = \frac{1}{\lambda_n}\sum_{k=1}^{n} \Delta\left(\lambda_k \sum_{j=1}^{k} \frac{1}{(1+j)^{1/p}}\right) = \sum_{k=1}^{n} \frac{1}{(1+k)^{1/p}}$$

and by using (2.3) we get $\tilde{\Lambda}_n(x) = 1/(1+n)^{1/p}$ for all $n \in \mathbb{N}$ and so $\tilde{\Lambda}(x) \in \ell_q \setminus \ell_p$ which means that $x \in bv_q^\lambda \setminus bv_p^\lambda$ and hence the inclusion $bv_p^\lambda \subset bv_q^\lambda$ is strict, and this ends the proof. □

Theorem 3.2 *Let $1 \leq p < \infty$. Then, we have the following:*

(1) *The inclusion $\ell_p^\lambda \subset bv_p^\lambda$ strictly holds.*

(2) *The spaces bv_p^λ and c_0^λ overlap, but none of them includes the other..*

(3) *The inclusion $bv_1^\lambda \subset c^\lambda$ strictly holds.*

Proof. Let $1 \leq p < \infty$ and consider the first part (1). If $x \in \ell_p^\lambda$; then $\Lambda(x) \in \ell_p$ and so $\Lambda(x) \in bv_p$ (since $\ell_p \subset bv_p$). This means that $x \in bv_p^\lambda$ and hence the inclusion $\ell_p^\lambda \subset bv_p^\lambda$ holds. Also, to show that this inclusion is strict, consider the sequence $e = (1, 1, 1, \cdots)$. Then, it can easily be seen that $\Lambda(e) = e$ and so $\tilde{\Lambda}(e) = e_1$, where $e_1 = (1, 0, 0, \cdots)$. Thus $e \in bv_p^\lambda \setminus \ell_p^\lambda$ which means that our inclusion $\ell_p^\lambda \subset bv_p^\lambda$ is strict. For (2), it is obvious that bv_p^λ and c_0^λ overlap, since $\ell_p^\lambda \subset bv_p^\lambda \cap c_0^\lambda$. Further, we have shown in the proof of part (1) that $\Lambda(e) = e \in bv_p \setminus c_0$ and hence $e \in bv_p^\lambda \setminus c_0^\lambda$. Thus, we have $bv_p^\lambda \not\subset c_0^\lambda$. On other side, let us define a sequence $x = (x_k)$ by

$$x_k = \frac{(-1)^k}{\lambda_k - \lambda_{k-1}} \left(\frac{\lambda_k}{\log(k+2)} + \frac{\lambda_{k-1}}{\log(k+1)} \right), \qquad (k \in \mathbb{N}).$$

Then, for every $n \in \mathbb{N}$, it is easy to show that

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} (-1)^k \left(\frac{\lambda_k}{\log(k+2)} + \frac{\lambda_{k-1}}{\log(k+1)} \right) = \frac{(-1)^n}{\log(n+2)}$$

and so $\Lambda(x) = \big((-1)^n / \log(n+2) \big) \in c_0 \setminus bv_p$. To see that, we may note that $|\tilde{\Lambda}_n(x)| = 1/\log(n+2) + 1/\log(n+1) \geq 2/\log(n+2)$ for every $n \in \mathbb{N}$ and $\big(2/\log(n+2) \big) \notin \ell_p$ for all positive reals p. This means that $x \in c_0^\lambda \setminus bv_p^\lambda$. Thus, we have $c_0^\lambda \not\subset bv_p^\lambda$. Finally, to prove (3), let $x \in bv_1^\lambda$ be given, i.e. $\tilde{\Lambda}(x) \in \ell_1$. Then, the absolute convergence of the series $\sum_{n=1}^{\infty} \tilde{\Lambda}_n(x)$ implies its convergence, and hence the limit $\lim_{m\to\infty} \big(\sum_{n=1}^{m} \tilde{\Lambda}_n(x) \big)$ exists which means that $\lim_{m\to\infty} \Lambda_m(x)$ exists, since $\Lambda_m(x) = \sum_{n=1}^{m} \tilde{\Lambda}_n(x)$. Thus $\Lambda(x) \in c$ and so $x \in c^\lambda$ which leads us to deduce the inclusion $bv_1^\lambda \subset c^\lambda$. Also, this inclusion is strict, becauce of the sequence x defined in the proof of part (2) above, where $\Lambda(x) \in c \setminus bv_1$ and so $x \in c^\lambda \setminus bv_1^\lambda$ which completes the proof. $\square$

Corollary 3.3 *If $p > 1$; then we have the following:*

(1) *The spaces bv_p^λ and ℓ_∞^λ overlap, but none of them contains the other.*

(2) *The spaces bv_p^λ and c^λ overlap, but none of them contains the other.*

Proof. Suppose that $p > 1$. Then, with help of Theorem 2.2 and its proof, we deduce the present result. For (1), it is obvious that bv_p^λ and ℓ_∞^λ overlap, since $\ell_p^\lambda \subset bv_p^\lambda \cap \ell_\infty^\lambda$. Also, we have $x \in \ell_\infty^\lambda \setminus bv_p^\lambda$, where x is the sequence defined in the proof of Theorem 2.2 above. Thus, we have $\ell_\infty^\lambda \not\subset bv_p^\lambda$. On other side, choose any real m such that $1 < m < p$ and define a sequence $y = (y_k)$ by

$$y_k = \frac{1}{\lambda_k - \lambda_{k-1}} \left(\lambda_k (k+2)^{1-\frac{1}{m}} - \lambda_{k-1} (k+1)^{1-\frac{1}{m}} \right), \qquad (k \in \mathbb{N}).$$

Then, we find that $\Lambda_n(y) = (n+2)^{1-\frac{1}{m}}$ $(n \in \mathbb{N})$ and so $\Lambda(y) \notin \ell_\infty$ which means that $y \notin \ell_\infty^\lambda$. Besides, by using the following two inequalities

$$(n+1)^{\frac{1}{m}} (n+2)^{1-\frac{1}{m}} - (n+1)^{1-\frac{1}{m}} (n+2)^{\frac{1}{m}} \leq (n+2)^{\frac{1}{m}} (n+2)^{1-\frac{1}{m}} - (n+1)^{1-\frac{1}{m}} (n+1)^{\frac{1}{m}} = 1\,,$$

$$(n+2)^{\frac{1}{m}} + (n+1)^{\frac{1}{m}} \geq 2\,(n+1)^{\frac{1}{m}},$$

we obtain for every $n \in \mathbb{N}$ that

$$\begin{aligned} \left| \tilde{\Lambda}_n(y) \right| &= (n+2)^{1-\frac{1}{m}} - (n+1)^{1-\frac{1}{m}} \\ &= \frac{1 + (n+1)^{\frac{1}{m}} (n+2)^{1-\frac{1}{m}} - (n+1)^{1-\frac{1}{m}} (n+2)^{\frac{1}{m}}}{(n+2)^{\frac{1}{m}} + (n+1)^{\frac{1}{m}}} \\ &\leq \frac{1}{(n+1)^{\frac{1}{m}}} \end{aligned}$$

which shows that $\sum_{n=1}^{\infty} |\tilde{\Lambda}_n(y)|^p < \infty$ (since $p/m > 1$). Thus $\tilde{\Lambda}(y) \in \ell_p$ and so $y \in bv_p^{\lambda}$ which means that $y \in bv_p^{\lambda} \setminus \ell_{\infty}^{\lambda}$. We therefore deduce that $bv_p^{\lambda} \not\subset \ell_{\infty}^{\lambda}$. Finally, to prove (2), it is clear that $bv_p^{\lambda} \not\subset \ell_{\infty}^{\lambda}$ implies $bv_p^{\lambda} \not\subset c^{\lambda}$, and $c_0^{\lambda} \not\subset bv_p^{\lambda}$ implies $c^{\lambda} \not\subset bv_p^{\lambda}$. $\square$

Now, to prove our main results concerning the inclusion $bv_p \subset bv_p^{\lambda}$ and the equality $bv_p^{\lambda} = bv_p$, we are in need to quoting the following lemmas:

Lemma 3.4 *For any sequence $x \in w$, we have the following equality:*

$$\tilde{\Lambda}_n(x) = \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right) \sum_{k=2}^{n} \lambda_{k-1} \, \Delta(x_k) \qquad (n \geq 2)\,. \tag{3.1}$$

Proof. Let $x \in w$. Then, for any $n \geq 2$, we have

$$\tilde{\Lambda}_n(x) = \Lambda_n(x) - \Lambda_{n-1}(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) x_k - \frac{1}{\lambda_{n-1}} \sum_{k=1}^{n-1} (\lambda_k - \lambda_{k-1}) \, x_k$$

and so we find that

$$\begin{aligned}
\tilde{\Lambda}_n(x) &= \frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) x_k - \frac{1}{\lambda_{n-1}} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) x_k + \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n-1}}\right) x_n \\
&= \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n-1}}\right) x_n - \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right) \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) \, x_k \\
&= \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}}\right) \sum_{k=1}^{n} (\lambda_k x_k - \lambda_{k-1} x_{k-1}) - \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}}\right) \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) \, x_k \\
&= \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}}\right) \sum_{k=1}^{n} \lambda_{k-1} (x_k - x_{k-1}) \\
&= \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}}\right) \sum_{k=2}^{n} \lambda_{k-1} \, \Delta(x_k) \\
&= \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right) \sum_{k=2}^{n} \lambda_{k-1} \, \Delta(x_k)\,.
\end{aligned}$$

$\square$

Lemma 3.5 *We have the following:*

(1) $$\lambda_{k-1} \, \Delta\left(\frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\right) \leq \Delta\left(\frac{\lambda_k \lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\right) \qquad (k \geq 2)\,.$$

(2) $$\sum_{k=2}^{n} \lambda_{k-1} \leq 2\left(\frac{\lambda_n \lambda_{n-1}}{\lambda_n - \lambda_{n-1}}\right) \qquad (n \geq 2)\,.$$

Proof. For (1), it is obvious, for any $k \geq 2$, that

$$\Delta\left(\frac{\lambda_k \lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\right) = \lambda_{k-1}\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} - \frac{\lambda_{k-2}}{\lambda_{k-1} - \lambda_{k-2}}\right) = \lambda_{k-1}\left[1 + \Delta\left(\frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\right)\right]$$

and we have done by noting that

$$\lambda_{k-1}\left[1+\Delta\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\right] \geq \lambda_{k-1}\,\Delta\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right).$$

To prove (2), let $n \geq 2$. Then, we have

$$\begin{aligned}
\sum_{k=2}^{n}\lambda_{k-1} &= \sum_{k=2}^{n}(\lambda_k-\lambda_{k-1})\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\\
&= \sum_{k=2}^{n}(\lambda_k-\lambda_{k-1})\sum_{j=2}^{k}\Delta\left(\frac{\lambda_{j-1}}{\lambda_j-\lambda_{j-1}}\right)\\
&= \sum_{j=2}^{n}\Delta\left(\frac{\lambda_{j-1}}{\lambda_j-\lambda_{j-1}}\right)\sum_{k=j}^{n}(\lambda_k-\lambda_{k-1})\\
&= \sum_{j=2}^{n}\Delta\left(\frac{\lambda_{j-1}}{\lambda_j-\lambda_{j-1}}\right)(\lambda_n-\lambda_{j-1}).
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
\sum_{k=2}^{n}\lambda_{k-1} &= \lambda_n\sum_{k=2}^{n}\Delta\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)-\sum_{k=2}^{n}\lambda_{k-1}\,\Delta\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\\
&= \frac{\lambda_n\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}-\sum_{k=2}^{n}\lambda_{k-1}\,\Delta\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right).
\end{aligned}$$

Now, by usig part (1) and noting that $\lambda_k > 0$ for all k, we find that

$$\begin{aligned}
\sum_{k=2}^{n}\lambda_{k-1} &= \left|\sum_{k=2}^{n}\lambda_{k-1}\right|\\
&\leq \frac{\lambda_n\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}+\left|\sum_{k=2}^{n}\lambda_{k-1}\,\Delta\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\right|\\
&\leq \frac{\lambda_n\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}+\sum_{k=2}^{n}\Delta\left(\frac{\lambda_k\,\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\\
&= \frac{\lambda_n\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}+\frac{\lambda_n\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}\\
&= 2\left(\frac{\lambda_n\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}\right). \qquad \square
\end{aligned}$$

Lemma 3.6 *For each $1 \leq p < \infty$ and every sequence $x \in w$, we have the following inequality:*

$$\left|\tilde{\Lambda}_n(x)\right|^p \leq 2^{p-1}\left(\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right)\sum_{k=2}^{n}\lambda_{k-1}\,|\Delta(x_k)|^p, \qquad (n \geq 2).$$

Proof. For the case $p = 1$, the given inequality is immediately obtained by applying the triangle inequality to (3.1) of Lemma 3.4. This gives us the following:

$$\left|\tilde{\Lambda}_n(x)\right| \leq \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right) \sum_{k=2}^{n} \lambda_{k-1} \left|\Delta(x_k)\right| , \qquad (n \geq 2)$$

which is the required inequality in case of $p = 1$, and we have nothing to do in this case. So, suppose now that $1 < p < \infty$ and let us go to the power p in both sides, we find that

$$\left|\tilde{\Lambda}_n(x)\right|^p \leq \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}}\right)^p \left[\sum_{k=2}^{n} \lambda_{k-1} \left|\Delta(x_k)\right|\right]^p \qquad (n \geq 2).$$

On other side, by applying the Hölder inequality and then using (2) of Lemma 3.5, it follows that

$$\begin{aligned} \left[\sum_{k=2}^{n} \lambda_{k-1} \left|\Delta(x_k)\right|\right]^p &\leq \left(\sum_{k=2}^{n} \lambda_{k-1}\right)^{p-1} \left(\sum_{k=2}^{n} \lambda_{k-1} \left|\Delta(x_k)\right|^p\right) \\ &\leq 2^{p-1}\left(\frac{\lambda_n \lambda_{n-1}}{\lambda_n - \lambda_{n-1}}\right)^{p-1} \sum_{k=2}^{n} \lambda_{k-1} \left|\Delta(x_k)\right|^p \end{aligned}$$

and we have done by combining the last two inequalities. □

Theorem 3.7 *The inclusion $bv_p \subset bv_p^\lambda$ holds for every $1 \leq p < \infty$.*

Proof. Let $1 \leq p < \infty$ and take any $x \in bv_p$. Then, we have $\Delta(x) \in \ell_p$ and so $\sum_{n=1}^{\infty} |\Delta(x_n)|^p < \infty$. Further, by using Lemma 3.6, we have

$$\left|\tilde{\Lambda}_n(x)\right|^p \leq 2^{p-1}\left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right) \sum_{k=2}^{n} \lambda_{k-1} \left|\Delta(x_k)\right|^p , \qquad (n \geq 2).$$

Thus, by taking the summation to both sides from $n = 2$ to m $(m \geq 2)$, we find that

$$\begin{aligned} \sum_{n=2}^{m} \left|\tilde{\Lambda}_n(x)\right|^p &\leq 2^{p-1} \sum_{n=2}^{m} \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right) \sum_{k=2}^{n} \lambda_{k-1} \left|\Delta(x_k)\right|^p \\ &= 2^{p-1} \sum_{k=2}^{m} \lambda_{k-1} \left|\Delta(x_k)\right|^p \sum_{n=k}^{m} \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right) \\ &= 2^{p-1} \sum_{k=2}^{m} \lambda_{k-1} \left|\Delta(x_k)\right|^p \left(\frac{1}{\lambda_{k-1}} - \frac{1}{\lambda_m}\right) \\ &= 2^{p-1} \left(\sum_{k=2}^{m} \left|\Delta(x_k)\right|^p - \frac{1}{\lambda_m} \sum_{k=2}^{m} \lambda_{k-1} \left|\Delta(x_k)\right|^p\right) \\ &\leq 2^{p-1} \sum_{k=2}^{m} \left|\Delta(x_k)\right|^p \end{aligned}$$

and by going to the limits when $m \to \infty$, we get $\sum_{n=2}^{\infty} |\tilde{\Lambda}_n(x)|^p \leq 2^{p-1} \sum_{k=2}^{\infty} |\Delta(x_k)|^p < \infty$ which means that $\tilde{\Lambda}(x) \in \ell_p$ and hence $x \in bv_p^\lambda$. Consequently, the inclusion $bv_p \subset bv_p^\lambda$ holds. □

Corollary 3.8 *The inclusion $\ell_p \subset bv_p^\lambda$ strictly holds for every $1 \le p < \infty$.*

Proof. This result follows immediately from Theorem 3.7, since $\ell_p \subset bv_p$ for every $1 \le p < \infty$. □

Finally, we end this section with the following relation satisfied for any $x \in w$ (see [14, Lemma 4.1])

$$x_n - \Lambda_n(x) = \frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} \, \tilde{\Lambda}_n(x) \, , \qquad (n \in \mathbb{N}).$$

Thus, by operating Δ on both sides, we establish the following relation:

$$\Delta(x_n) - \tilde{\Lambda}_n(x) = \Delta\left(\frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} \, \tilde{\Lambda}_n(x) \right), \qquad (n \in \mathbb{N}) \tag{3.2}$$

for any sequence $x \in w$, which can be written as follows:

$$\Delta(x_n) - \tilde{\Lambda}_n(x) = \frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} \left[\tilde{\Lambda}_n(x) - \tilde{\Lambda}_{n-1}(x) \right] + \Delta\left(\frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} \right) \tilde{\Lambda}_{n-1}(x) \, , \quad (n \ge 2) \tag{3.3}$$

Therefore, by noting that $(\lambda_n/(\lambda_n - \lambda_{n-1}))_{n=1}^\infty$ is a sequence of positive reals and $\lambda_n/(\lambda_n - \lambda_{n-1}) = 1 + \lambda_{n-1}/(\lambda_n - \lambda_{n-1})$ for all $n \in \mathbb{N}$; we conclude by using (3.2) and (3.3) the following result:

Corollary 3.9 *For each $1 \le p < \infty$, we have the following equivalences:*

(1) *The equality $bv_p^\lambda = bv_p$ holds if and only if $(\lambda_n/(\lambda_n - \lambda_{n-1}))_{n=1}^\infty \in \ell_\infty$.*

(2) *The inclusion $bv_p \subset bv_p^\lambda$ strictly holds if and only if $(\lambda_n/(\lambda_n - \lambda_{n-1}))_{n=1}^\infty \notin \ell_\infty$.*

Proof. Suppose that $1 \le p < \infty$. For (1), if the equality $bv_p^\lambda = bv_p$ holds; then $bv_p^\lambda \subset bv_p$ and so $x \in bv_p$ for all $x \in bv_p^\lambda$. Thus $\Delta(x) \in \ell_p$ whenever $\tilde{\Lambda}(x) \in \ell_p$ and hence $\Delta(x) - \tilde{\Lambda}(x) \in \ell_p$ whenever $\tilde{\Lambda}(x) \in \ell_p$. Thus, from (3.3), we must have $(\lambda_{n-1}/(\lambda_n - \lambda_{n-1})) \in \ell_\infty$ and so $(\lambda_n/(\lambda_n - \lambda_{n-1})) \in \ell_\infty$ and this prove the necessity. Conversely, to prove the sufficiency, assume that $(\lambda_n/(\lambda_n - \lambda_{n-1})) \in \ell_\infty$ and so $(\lambda_{n-1}/(\lambda_n - \lambda_{n-1})) \in \ell_\infty$. Then, for any $x \in bv_p^\lambda$, we have $\tilde{\Lambda}(x) \in \ell_p$ and hence $(\Delta(\lambda_{n-1}\tilde{\Lambda}_n(x)/(\lambda_n - \lambda_{n-1}))) \in \ell_p$. Thus, it follows from (3.2) that $(\Delta(x_n) - \tilde{\Lambda}_n(x)) \in \ell_p$ which implies that $\Delta(x) \in \ell_p$ (since $\tilde{\Lambda}(x) \in \ell_p$ by assumption). Thus, we get $x \in bv_p$ which yields the inclusion $bv_p^\lambda \subset bv_p$. Consequently, with help of Theorem 3.7, we deduce the equality $bv_p^\lambda = bv_p$. Finally, the result of part (2) follows immediately from part (1) and Theorem 3.7. □

Remark 3.10 In the light of Remark 2.6, it can easily be seen that all results of this section still hold for the space bv_∞^λ. For example, all the spaces c_0, c, ℓ_∞, ℓ_p, bv_p, c_0^λ, c^λ, ℓ_∞^λ, ℓ_p^λ and bv_p^λ are properly contained in the space bv_∞^λ $(1 \le p < \infty)$, and the inclusion $bv_\infty \subset bv_\infty^\lambda$ holds with equality if and only if $(\lambda_n/(\lambda_n - \lambda_{n-1}))_{n=1}^\infty \in \ell_\infty$.

Finally, we support our results in Corollary 3.9 by the following example:

Example 3.11 Let $1 \leq p < \infty$. Then, we have the following two distinct cases:

I - The case of strict inclusion $bv_p \subset bv_p^\lambda$: consider the sequence $\lambda = (\lambda_k)$ defined by $\lambda_k = \log(1+k)$ for all k (or $\lambda_k = k^a$, where $a > 0$, see Example 2.7). Then, it can easily be shown that $\lambda_k/\Delta(\lambda_k) \geq k \log(1+k)$ for all k (or $\lambda_k/\Delta(\lambda_k) \geq k/a$). In such case, we will generally obtain that $\lambda/\Delta(\lambda) \notin \ell_\infty$ and so $bv_p \subsetneqq bv_p^\lambda$ by Corollary 3.9.

II - The case of equality $bv_p\lambda = bv_p$: consider the sequence $\lambda = (\lambda_k)$ defined by $\lambda_k = k!$ for all k (or $\lambda_k = a^k$, where $a > 1$). Then, it can easily be shown that $\lambda_k/\Delta(\lambda_k) = k/(k-1)$ for all k (or $\lambda_k/\Delta(\lambda_k) = a$). In such case, we will generally find that $\lambda/\Delta(\lambda) \in \ell_\infty$ and so $bv_p^\lambda = bv_p$ by Corollary 3.9.

4 Schauder basis for the spaces bv_p^λ $(1 \leq p < \infty)$

In the last section, we construct a common sequence to be the Schauder basis for the λ-sequence spaces bv_p^λ of p-bounded variation, and we conclude their separability, where $1 \leq p < \infty$.

It is well-known that the sequence $(e_1, e_2, e_3, \cdots)$ is the Schauder basis for the sequence spaces ℓ_p $(1 \leq p < \infty)$, where $e_k = (\delta_{nk})_{n=1}^\infty$ for each $k \in \mathbb{N}$, and every $x = (x_k) \in \ell_p$ has the unique representation $x = \sum_{k=1}^\infty x_k\, e_k$ which leads us to the fact that the spaces ℓ_p are separable BK-spaces [11]. This fact can be used with Theorem 2.2 to deduce the following result:

Theorem 4.1 *For each $k \geq 1$, define the sequence $e_k^\lambda = (e_{nk}^\lambda)_{n=1}^\infty$ by*

$$e_{nk}^\lambda = \begin{cases} 0\,; & (n \leq k), \\ \dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}}\,; & (n = k), \\ 1\,; & (n > k), \end{cases} \qquad (n \in \mathbb{N})$$

Then, the sequence $\left(e_k^\lambda\right)_{k=1}^\infty$ is a Schauder basis for the space bv_p^λ $(1 \leq p < \infty)$ and every $x \in bv_p^\lambda$ has a unique representation of the following form:

$$x = \sum_{k=1}^\infty \tilde{\Lambda}_k(x)\, e_k^\lambda\,. \tag{4.1}$$

Proof. It is clear that $\Lambda_n(e_k^\lambda) = 0$ for $1 \leq n < k$ and $\Lambda_n(e_k^\lambda) = 1$ for $n \geq k$ and so $\tilde{\Lambda}_n(e_k^\lambda) = \delta_{nk}$ for all $n, k \in \mathbb{N}$. Thus $\tilde{\Lambda}(e_k^\lambda) = e_k \in \ell_p$ and hence $e_k^\lambda \in bv_p^\lambda$ for all $k \in \mathbb{N}$. This means that $\left(e_k^\lambda\right)_{k=1}^\infty$ is a sequence in bv_p^λ. Further, let $x \in bv_p^\lambda$ be given and for every positive integer m, we put $x^{(m)} = \sum_{k=1}^m \tilde{\Lambda}_k(x)\, e_k^\lambda$. Then, we find that

$$\tilde{\Lambda}(x^{(m)}) = \sum_{k=1}^m \tilde{\Lambda}_k(x)\, \tilde{\Lambda}(e_k^\lambda) = \sum_{k=1}^m \tilde{\Lambda}_k(x)\, e_k$$

and hence

$$\tilde{\Lambda}_n(x - x^{(m)}) = \begin{cases} 0\,; & (1 \leq n \leq m), \\ \tilde{\Lambda}_n(x)\,; & (n > m). \end{cases}$$

Now, since $\tilde{\Lambda}(x) \in \ell_p$; for any positive real $\epsilon > 0$, there is a positive integer m_0 such that $\sum_{n=m_0+1}^{\infty} |\tilde{\Lambda}_n(x)| < \epsilon^p$. Thus, for any $m \geq m_0$, we have

$$\left\| x - x^{(m)} \right\|_{\lambda p} = \left(\sum_{n=m+1}^{\infty} |\tilde{\Lambda}_n(x)| \right)^{1/p} \leq \left(\sum_{n=m_0+1}^{\infty} |\tilde{\Lambda}_n(x)| \right)^{1/p} < \epsilon \, .$$

We therefore deduce that $\lim_{m\to\infty} \|x - x^{(m)}\|_{\lambda p} = 0$ which means that x is represented as in (4.1). Thus, it is remaining to show the uniqueness of the representation (4.1) of x. For this, suppose that $x = \sum_{k=1}^{\infty} \alpha_k \, e_k^{\lambda}$. Then, we have to show that $\alpha_n = \tilde{\Lambda}_n(x)$ for all n, which is immediate by operating $\tilde{\Lambda}_n$ on both sides of (4.1) for each $n \in \mathbb{N}$, where the continuity of $\tilde{\Lambda}$ (as we have seen in Remark 2.5) allows us to obtain that

$$\tilde{\Lambda}_n(x) = \sum_{k=1}^{\infty} \alpha_k \, \tilde{\Lambda}_n(e_k^{\lambda}) = \sum_{k=1}^{\infty} \alpha_k \, \delta_{nk} \, = \, \alpha_n$$

for all $n \in \mathbb{N}$ and hence the representation (4.1) of x is unique, and this completes the proof. $\square$

Corollary 4.2 *We have the following facts:*

(1) *The spaces bv_p^{λ} are separable BK-spaces for all $1 \leq p < \infty$.*

(2) *The space bv_{∞}^{λ} is a non-separable BK-space and has no a Schauder basis.*

Conclusion: In the present work, we have essentially introduced some new Banach and Hilbert spaces of sequences and constract their bases, which will give a scope for further research and study in future. So, we end our work by expressing from now on that the aim of our next paper is to determining the duals of our spaces bv_p^{λ}, and characterizing some matrix operators between them.

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