On the new λ-difference spaces of convergent and bounded sequences

Abdullah K. Noman and Omar H. Al-Sabri*

Department of Mathematics, Faculty of Education and Science - Rada'a

Albaydha University, Yemen

akanoman@gmail.com;

omar.alsbri@gmail.com

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Abstract

The λ-sequence spaces c_0^λ, c^λ and ℓ_∞^λ have already been studied by Mursaleen and Noman. Next, they have also studied the difference λ-sequence spaces $c_0^\lambda(\Delta)$, $c^\lambda(\Delta)$ and `∞ (Δ) by using the usual manner of difference spaces of sequences. In present paper, we will go away to use another manner in order to introduce the new λ-difference spaces $c_0(\Delta^\lambda)$, $c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$, and then we will study their properties, bases and inclusion relations. Further, we will show that our new spaces are Banach spaces isometrically isomorphic to the related classical sequence spaces c0 ,c and `∞.

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Introduction

We will write w for the linear space of all real or complex sequences. A sequence x ∈ w will be simply written as x = (xk) instead of $x = (x_k)_{k=1}^\infty$. Also, we will use the conventions e = (1,1,1,...) and $e_k = (\delta_{nk})_{n=1}^\infty$ for each k ≥ 1, that is ek is the sequence with zero terms except the k-term only which is 1

Also, any term with non -positive subscript is equal to naught, i.e. x0 = 0 and x−1 = 0. Any linear subspace of w is called a sequence space, and we will write ℓ_∞,c and c0 for the classical sequence spaces of bounded, convergent and null sequences, respectively. Further, we will write $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ for the usual difference spaces, e.g. $\ell_\infty(\Delta) = \{x \in w : (x_k \quad x_{k-1}) \in \ell_\infty\}$. A sequence space X together with a norm $\|\cdot\|$ is called a normed sequence space, and a complete normed sequence space is called a Banach sequence space. By a BK-space, we mean a Banach sequence space with continuous coordinates. An infinite matrix A whose real or complex entries a_{nk} for all $n, k \geq 1$ will be written as $A = [a_{nk}]$ instead of $A = [a_{nk}]_{n,k=1}^\infty$. The act of A on a sequence $x \in w$ is called the A-transform of x, and is defined to be the sequence $A(x) = (A_n(x))_{n=1}^\infty$, where

$$A_n(x) = \sum_{k=1}^\infty a_{nk}\, x_k; \qquad (n \geq 1),$$

provided the series on the right hand side converges for each n, and we then say that $A(x)$ exists or is well-defined. For two sequence spaces X and Y, we say that an infinite matrix A defines a matrix operator form X to Y, which is a linear operator, and we denote it by $A : X \to Y$, if A acts form X to Y, i.e, if for every sequence $x \in X$; the A-transform of x exists and is in Y. Moreover, we will write (X, Y) for the class of all infinite matrices that map X into Y, i.e, $A \in (X, Y)$ if and only if $A(x)$ is well-defined and $A(x) \in Y$ for every $x \in X$.

For an infinite matrix A and a sequence space X, the matrix domain of A in X is denoted by X_A which is a sequence space defined as $X_A = \{x \in w : A(x) \in X\}$. An infinite matrix A is called a triangle if $a_{nk} = 0$ for all $k \geq n$ and $a_{nn} \neq 0$ for all n, where $n, k \geq 1$. If X is a BK-space with its norm $\|\cdot\|$ and A is a triangle, then the matrix domain X_A is also a BK-space with the norm $\|\cdot\|_A$ defined by $\|x\|_A = \|A(x)\|$ for all $x \in X_A$. We will write Δ for the band matrix of difference, that is $\Delta(x) = (x_n - x_{n-1})_{n=1}^\infty = (x_1, x_2 - x_1, x_3 - x_2, \cdots)$ which means that $\Delta(x_k) = x_k - x_{k-1}$ for all k. So that, the difference sequence space $c_0(\Delta), c(\Delta)$ and $\ell_\infty(\Delta)$ can be defined as the matrix domains of Δ in c_0, c and ℓ_∞, respectively. That is $c_0(\Delta) = (c_0)_\Delta$, $c(\Delta) = (c)_\Delta$, and $\ell_\infty(\Delta) = (\ell_\infty)_\Delta$. It is

well-known that c_0, c and ℓ_∞ are BK-spaces with the norm $\|\cdot\|_\infty$ defined by $\|x\|_\infty = \sup_n|x_n|$, where the supremum is taking over all positive integers n. This yields that the difference spaces $c_0(\Delta), c(\Delta)$ and $\ell_\infty(\Delta)$ are BK-spaces with the norm $\|\cdot\|_\Delta$ defined by $\|x\|_\Delta = \sup_n|x_n - x_{n-1}|$.
The idea of constructing a new difference sequence space by means of the matrix domain of a particular triangle has largely been used by several authors, they specially introduce many new difference sequence spaces in different ways. For instance, see [1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 14, 15, 17] and [19].

2 The new λ-difference sequence spaces

In this section, we will introduce the new λ-difference sequence spaces $c_0(\Delta^\lambda), c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$. Throughout this paper, we assume that $\lambda = (\lambda_k)_{k=1}^\infty$ is a strictly increasing sequence of positive reals, that is $0 < \lambda_1 < \lambda_2 < \cdots$. Then, for any $x \in w$; we define the sequence $\Lambda(x) = (\Lambda_n(x))_{n=1}^\infty$ by

$$\Lambda_n(x) = \frac{1}{\lambda_n}\sum_{k=1}^{n}(\lambda_k - \lambda_{k-1})\,x_k; \qquad (n \geq 1). \tag{2.1}$$

In [10] and [12], the λ- sequence spaces have been introduced by Mursaleen and Noman as follows:

$$c_0^\lambda = \{x \in w : \Lambda(x) \in c_0\}, \quad c^\lambda = \{x \in w : \Lambda(x) \in c\} \quad \text{and} \quad \ell_\infty^\lambda = \{x \in w : \Lambda(x) \in \ell_\infty\}.$$

Also, the difference λ-sequence spaces $c_0^\lambda(\Delta)$, $c^\lambda(\Delta)$ and $\ell_\infty^\lambda(\Delta)$ have been studied in [11] as follows:

$$c_0^\lambda(\Delta) = \left(c_0^\lambda\right)_\Delta, \quad c^\lambda(\Delta) = \left(c^\lambda\right)_\Delta \quad \text{and} \quad \ell_\infty^\lambda(\Delta) = \left(\ell_\infty^\lambda\right)_\Delta.$$

Now, we will go away from the technique used in [11] and introduce the λ-difference sequence spaces, which is our contribution in this paper, as follows:

$$c_0\left(\Delta^\lambda\right) = \{x \in w : \Lambda(x) \in c_0(\Delta)\},$$

$$c\left(\Delta^\lambda\right) = \{x \in w : \Lambda(x) \in c(\Delta)\},$$

$$\ell_\infty\left(\Delta^\lambda\right) = \{x \in w : \Lambda(x) \in \ell_\infty(\Delta)\}.$$

Besides, we define the triangle $\tilde{\Lambda} = [\tilde{\lambda}_{nk}]$ for all $n, k \geq 1$ by

$$\tilde{\lambda}_{nk} = \begin{cases} \dfrac{\lambda_n - \lambda_{n-1}}{\lambda_n}; & (n = k), \\ (\lambda_k - \lambda_{k-1})\left(\dfrac{1}{\lambda_n} - \dfrac{1}{\lambda_{n-1}}\right); & (n > k), \\ 0; & (n < k). \end{cases}$$

Then, for any sequence $x \in w$, it can be easily shown that

$$\tilde{\Lambda}_n(x) = \Lambda_n(x) - \Lambda_{n-1}(x) \qquad (n \geq 1)$$

and so $\tilde{\Lambda}(x) = (\Lambda_n(x) - \Lambda_{n-1}(x))_{n=1}^\infty$. Thus, the spaces $c_0(\Delta^\lambda), c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$ can be defined as the matrix domains of the triangle $\tilde{\Lambda}$ in the spaces c_0, c and ℓ_∞, respectively. That is

$$c_0\left(\Delta^\lambda\right) = (c_0)_{\tilde{\Lambda}}, \quad c\left(\Delta^\lambda\right) = (c)_{\tilde{\Lambda}} \quad \text{and} \quad \ell_\infty\left(\Delta^\lambda\right) = (\ell_\infty)_{\tilde{\Lambda}} \tag{2.2}$$

which means that

$$c_0\left(\Delta^\lambda\right) = \left\{x \in w : \lim_{n\to\infty}\tilde{\Lambda}_n(x) = 0\right\},$$

$$c\left(\Delta^\lambda\right) = \left\{x \in w : \lim_{n\to\infty}\tilde{\Lambda}_n(x) \text{ exists}\right\},$$

$$\ell_\infty\left(\Delta^\lambda\right) = \left\{x \in w : \sup_n\left|\tilde{\Lambda}_n(x)\right| < \infty\right\}.$$

It follows that our spaces are sequence spaces of difference type, and we can prove the following results:

Lemma 2.1 *The λ-difference sequence spaces $c_0\left(\Delta^\lambda\right)$, $c\left(\Delta^\lambda\right)$ and $\ell_\infty\left(\Delta^\lambda\right)$ are BK-spaces with the norm $\|\cdot\|_\lambda$ defined by*

$$\|x\|_\lambda = \|\tilde{\Lambda}(x)\|_\infty = \sup_n |\tilde{\Lambda}_n(x)| = \sup_n |\Lambda_n(x) - \Lambda_{n-1}(x)|.$$

Proof. Since c_0, c and ℓ_∞ are BK-spaces with respect to their natural norm (see [7]) and the matrix $\tilde{\Lambda}$ is a triangle; from (2.2) we deduce the fact that $c_0(\Delta^\lambda)$, $c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$ are BK-spaces with the given norm which is obtained by the famous result of Wilansky [18]. □

Theorem 2.2 *The λ-difference sequence spaces $c_0\left(\Delta^\lambda\right)$, $c\left(\Delta^\lambda\right)$ and $\ell_\infty\left(\Delta^\lambda\right)$ are isometrically linear-isomorphic to the spaces c_0, c and ℓ_∞, respectively. That is*

$$c_0\left(\Delta^\lambda\right) \cong c_0, \quad c\left(\Delta^\lambda\right) \cong c \quad \text{and} \quad \ell_\infty\left(\Delta^\lambda\right) \cong \ell_\infty.$$

Proof. To show that $c_0(\Delta^\lambda) \cong c_0$, we will prove the existence of a linear operator between $c_0(\Delta^\lambda)$ and c_0 which is bijective and norm-preserving. For, we define the mapping $\tilde{\Lambda} : c_0(\Delta^\lambda) \to c_0$ by $x \mapsto \tilde{\Lambda}(x)$ for all $x \in c_0(\Delta^\lambda)$. Then, this mapping is clearly a linear operator which is well-defined. Also, it is easy to see that $\tilde{\Lambda}(x) = 0$ implies $x = 0$ which means that $\tilde{\Lambda}$ is injective. Further, to show that $\tilde{\Lambda}$ is surjective, let $y \in c_0$ and define the sequence $x = (x_k)$ by

$$x_k = \frac{1}{\lambda_k - \lambda_{k-1}} \left(\lambda_k \sum_{i=1}^{k} y_i - \lambda_{k-1} \sum_{i=1}^{k-1} y_i \right); \qquad (k \geq 1),$$

where $x_1 = y_1$ (since $\lambda_0 = 0$). Then, for every $n \geq 1$, we have

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) x_k = \frac{1}{\lambda_n} \sum_{k=1}^{n} \left(\lambda_k \sum_{i=1}^{k} y_i - \lambda_{k-1} \sum_{i=1}^{k-1} y_i \right) = \sum_{i=1}^{n} y_i$$

which implies that $\tilde{\Lambda}_n(x) = \Lambda_n(x) - \Lambda_{n-1}(x) = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n-1} y_i = y_n$ for every $n \geq 1$ and this means $\tilde{\Lambda}(x) = y \in c_0$ and so $x \in c_0(\Delta^\lambda)$ such that $\tilde{\Lambda}(x) = y$. This shows that $\tilde{\Lambda}$ is surjective and hence $\tilde{\Lambda}$ is a linear isomorphism. Finally, for any $x \in c_0(\Delta^\lambda)$, we have by Lemma 2.1 that $\|x\|_\lambda = \|\tilde{\Lambda}(x)\|_\infty$ which means that $\tilde{\Lambda}$ is norm-preserving, and so $\tilde{\Lambda}$ is a linear bijection which preserves the norm. Hence, we deduce that $c_0(\Delta^\lambda) \cong c_0$. Similarly, we can show that $c\left(\Delta^\lambda\right) \cong c$ and $\ell_\infty\left(\Delta^\lambda\right) \cong \ell_\infty$. □

Corollary 2.3 *The λ-difference sequence spaces $c_0\left(\Delta^\lambda\right)$, $c\left(\Delta^\lambda\right)$ and $\ell_\infty\left(\Delta^\lambda\right)$ are isometrically linear-isomorphic to the spaces $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$, respectively. That is*

$$c_0\left(\Delta^\lambda\right) \cong c_0(\Delta), \quad c\left(\Delta^\lambda\right) \cong c(\Delta) \quad \text{and} \quad \ell_\infty\left(\Delta^\lambda\right) \cong \ell_\infty(\Delta).$$

Remark 2.4 The matrix operator $\tilde{\Lambda}$ defined on any of the spaces $c_0\left(\Delta^\lambda\right)$, $c\left(\Delta^\lambda\right)$ or $\ell_\infty\left(\Delta^\lambda\right)$ into the corresponding space of c_0, c or ℓ_∞ (respectively) is an isometry linear isomorphism as we have already shown in the proof of Theorem 2.2, and this implies the continuity of the matrix operator $\tilde{\Lambda}$.

At the end of this section, we give an example to show that our new λ-difference spaces of sequences are totally different from the classical sequence spaces and from the well-known λ-sequence spaces. For simplicity in notations, we will use the symbole μ to denote any of the spaces c_0, c or ℓ_∞, and so $\mu(\Delta^\lambda)$ is the respective one of the spaces $c_0(\Delta^\lambda)$, $c(\Delta^\lambda)$ or $\ell_\infty(\Delta^\lambda)$.

Example 2.5 In this example, our aim is to show that the space $\mu(\Delta^\lambda)$ is different from all the sequence spaces μ, $\mu(\Delta)$, μ^λ and $\mu^\lambda(\Delta)$. For this, consider the sequence $\lambda = (\lambda_k)$ defined by $\lambda_k = (2^k - 1)/2^k$ for all $k \geq 1$ which is a strictly increasing sequence of positive reals. Then $\Delta(\lambda_k) = 1/2^{2k-1}$ $(k \geq 1)$ and for any sequence $x \in w$ we have $\tilde{\Lambda}_n(x) = \Lambda_n(x) - \Lambda_{n-1}(x)$ for all $n \geq 1$, where

$$\Lambda_n(x) = \frac{2^n}{2^n - 1} \sum_{k=1}^{n} \frac{x_k}{2^{2k-1}}; \qquad (n \geq 1).$$

Now, consider the unbounded sequence $x = (x_k)$ given by $x_k = 2^{2k-1}(\sqrt{k} - \sqrt{k-1})$ for all $k \geq 1$. Then, it can easily be show that $\Lambda_n(x) = \sqrt{n}(1 + 1/(2^n - 1))$ for all n, and so we obtain that

$$\tilde{\Lambda}_n(x) = \sqrt{n} - \sqrt{n-1} + \frac{\sqrt{n}}{2^n - 1} - \frac{\sqrt{n-1}}{2^{n-1} - 1}\,; \qquad (n > 1)$$

which shows that $\tilde{\Lambda}(x) \in c_0$. Thus $x \in c_0(\Delta^\lambda)$ and hence $x \in \mu(\Delta^\lambda)$ (since $c_0(\Delta^\lambda) \subset c(\Delta^\lambda) \subset \ell_\infty(\Delta^\lambda)$). On other side, it is clear that $x \notin \ell_\infty$ and so $x \notin \mu$. Thus, we have $x \in \mu(\Delta^\lambda)$ while $x \notin \mu$. Consequently, it follows that $\mu(\Delta^\lambda) \neq \mu$. Also, we note that $\Lambda(x) \notin \ell_\infty$ and hence $x \notin \ell_\infty^\lambda$ which means that $x \notin \mu^\lambda$ and so $\mu(\Delta^\lambda) \neq \mu^\lambda$. Further, for every $k \geq 1$, we have $\sqrt{k} + \sqrt{k-1} \geq (\sqrt{k+1} + \sqrt{k})/2$ and hence $\sqrt{k} - \sqrt{k-1} \leq 2(\sqrt{k+1} - \sqrt{k})$. Thus, it follows that $\Delta(x_{k+1}) \geq 2^{2k}(\sqrt{k+1} - \sqrt{k})$ which implies that $\Delta(x_k) \geq x_k/2 \to \infty$ (as $k \to \infty$) and so $\Lambda_n(\Delta(x)) \geq \Lambda_n(x)/2 \to \infty$ (as $n \to \infty$). Hence, we deduce that $x \notin \ell_\infty(\Delta)$ as well as $x \notin \ell_\infty^\lambda(\Delta)$ and so $x \notin \mu(\Delta)$ as well as $x \notin \mu^\lambda(\Delta)$, which means that $\mu(\Delta^\lambda) \neq \mu(\Delta)$ and $\mu(\Delta^\lambda) \neq \mu^\lambda(\Delta)$. Therefore, the space $\mu(\Delta^\lambda)$ is different from all the sequence spaces μ, $\mu(\Delta)$, μ^λ and $\mu^\lambda(\Delta)$.

3 Some inclusion relations

In this section, we derive some interesting inclusion relations between our new λ-difference sequence spaces and the classical sequence spaces (specially, the difference types $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$).

Lemma 3.1 *The inclusions $c_0\left(\Delta^\lambda\right) \subset c\left(\Delta^\lambda\right) \subset \ell_\infty\left(\Delta^\lambda\right)$ strictly hold.*

Proof. These inclusions are immediate from the inclusions $c_0 \subset c \subset \ell_\infty$. To show that these inclusions are strictly, we consider the two sequences x and y defined by

$$x_k = \frac{k\lambda_k - (k-1)\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\;, \qquad y_k = \frac{(-1)^k}{2}\left(\frac{\lambda_k + \lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\right) \qquad (k \geq 1)\,.$$

Then, for any $n \geq 1$, it can be easily seen that

$$\Lambda_n(x) = \frac{1}{\lambda_n}\sum_{k=1}^{n}(k\lambda_k - (k-1)\lambda_{k-1}) = n\,, \qquad \Lambda_n(y) = \frac{1}{2\lambda_n}\sum_{k=1}^{n}(-1)^k(\lambda_k + \lambda_{k-1}) = \frac{(-1)^n}{2}$$

and so $\tilde{\Lambda}_n(x) = \Lambda_n(x) - \Lambda_{n-1}(x) = 1$ and $\tilde{\Lambda}_n(y) = \Lambda_n(y) - \Lambda_{n-1}(y) = (-1)^n$ which imply that $\tilde{\Lambda}_n(x) = e \in c\backslash c_0$ and $\tilde{\Lambda}_n(y) \in \ell_\infty\backslash c$ which maen that $x \in c\left(\Delta^\lambda\right)\backslash c_0\left(\Delta^\lambda\right)$ and $y \in \ell_\infty\left(\Delta^\lambda\right)\backslash c_0\left(\Delta^\lambda\right)$. This completes the proof. □

Lemma 3.2 *The inclusion $c^\lambda \subset c_0\left(\Delta^\lambda\right)$ strictly holds.*

Proof. For any $x \in c^\lambda$, we have $\Lambda(x) \in c$ and so $\tilde{\Lambda}(x) = (\Lambda_n(x) - \Lambda_{n-1}(x)) \in c_0$ which means that $x \in c_0\left(\Delta^\lambda\right)$ and hence $c^\lambda \subset c_0\left(\Delta^\lambda\right)$. Also, to show that this inclusion is strict, define the sequence $x = (x_k)$ by $x_k = (\lambda_k\sqrt{k} - \lambda_{k-1}\sqrt{k-1})/(\lambda_k - \lambda_{k-1})$ for $k \geq 1$. Then $\Lambda_n(x) = \sqrt{n}$ and so $\Lambda(x) = (\sqrt{n}) \notin c$ which means $x \notin c^\lambda$, but $\tilde{\Lambda}(x) = \left(\sqrt{n} - \sqrt{n-1}\right) \in c_0$ which shows that $x \in c_0\left(\Delta^\lambda\right)\backslash c^\lambda$. □

Corollary 3.3 *The spaces c_0, c and c_0^λ are strictly included in $c_0\left(\Delta^\lambda\right)$.*

Corollary 3.4 *The inclusions $c_0^\lambda \subset c_0\left(\Delta^\lambda\right)$, $c^\lambda \subset c\left(\Delta^\lambda\right)$ and $\ell_\infty^\lambda \subset \ell_\infty\left(\Delta^\lambda\right)$ strictly hold.*

Remark 3.5 The spaces ℓ_∞ and $c_0\left(\Delta^\lambda\right)$ overlap, but ℓ_∞ cannot include $c_0\left(\Delta^\lambda\right)$. To see that, we have $c \subset \ell_\infty \cap c_0\left(\Delta^\lambda\right)$, and the sequence x in the proof of Lemma 3.2 is unbounded, since $x_k \geq \sqrt{k}$ for all $k \geq 1$, which means that $x \in c_0\left(\Delta^\lambda\right)\backslash \ell_\infty$.

Theorem 3.6 *The inclusion* $\ell_\infty \subset c_0\left(\Delta^\lambda\right)$ *strictly holds if and only if* $\lim_{n\to\infty} \lambda_{n-1}/\lambda_n = 1$.

Proof. It is clear that $\ell_\infty \subset c_0\left(\Delta^\lambda\right)$ if and only if $\tilde{\Lambda} \in (\ell_\infty, c_0)$ which is equivalent to the condition $\lim_{n\to\infty}\sum_{k=1}^{\infty}|\tilde{\lambda}_{nk}| = 0$ [16]. On other hand, for any $n > 1$, we have

$$\sum_{k=1}^{\infty}\left|\tilde{\lambda}_{nk}\right| = \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)\sum_{k=1}^{n-1}(\lambda_k - \lambda_{k-1}) + \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} = 2\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right) = 2\left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right).$$

Thus, we find that $\lim_{n\to\infty}\sum_{k=1}^{\infty}|\tilde{\lambda}_{nk}| = 0$ if and only if $\lim_{n\to\infty}\lambda_{n-1}/\lambda_n = 1$. This proves that $\ell_\infty \subset c_0\left(\Delta^\lambda\right)$ if and only if $\lim_{n\to\infty}\lambda_{n-1}/\lambda_n = 1$. Also, this inclusion is strict because the equality cannot be satisfied by Remark 3.5. □

Now, in the following results, we will discuss the inclusions $c_0(\Delta) \subset c_0\left(\Delta^\lambda\right)$, $c(\Delta) \subset c\left(\Delta^\lambda\right)$ and $\ell_\infty(\Delta) \subset \ell_\infty\left(\Delta^\lambda\right)$, and for this we need the following Lemmas:

Lemma 3.7 *For any sequence* $x \in w$*, we have*

$$\tilde{\Lambda}_n(x) = \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)\sum_{k=2}^{n}\lambda_{k-1}\ \Delta(x_k) \qquad (n \geq 2)\,. \tag{3.1}$$

Proof. Let $x \in w$. Then, for any $n \geq 2$, we have

$$\tilde{\Lambda}_n(x) = \Lambda_n(x) - \Lambda_{n-1}(x) = \frac{1}{\lambda_n}\sum_{k=1}^{n}(\lambda_k - \lambda_{k-1})x_k - \frac{1}{\lambda_{n-1}}\sum_{k=1}^{n-1}(\lambda_k - \lambda_{k-1})x_k$$

and so we find that

$$\begin{aligned}
\tilde{\Lambda}_n(x) &= \frac{1}{\lambda_n}\sum_{k=1}^{n}(\lambda_k - \lambda_{k-1})x_k - \frac{1}{\lambda_{n-1}}\sum_{k=1}^{n}(\lambda_k - \lambda_{k-1})x_k + \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n-1}}\right)x_n \\
&= \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n-1}}\right)x_n - \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)\sum_{k=1}^{n}(\lambda_k - \lambda_{k-1})x_k \\
&= \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n\lambda_{n-1}}\right)\sum_{k=1}^{n}(\lambda_k x_k - \lambda_{k-1}x_{k-1}) - \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n\lambda_{n-1}}\right)\sum_{k=1}^{n}(\lambda_k - \lambda_{k-1})x_k \\
&= \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n\lambda_{n-1}}\right)\sum_{k=1}^{n}\lambda_{k-1}(x_k - x_{k-1}) \\
&= \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n\lambda_{n-1}}\right)\sum_{k=2}^{n}\lambda_{k-1}\,\Delta(x_k) \\
&= \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)\sum_{k=2}^{n}\lambda_{k-1}\,\Delta(x_k)\,.
\end{aligned}$$

□

Remark 3.8 The two sequences $(k)_{k=1}^{\infty}$ and $(\lambda_k/(\lambda_k - \lambda_{k-1}))_{k=1}^{\infty}$ will be used, and we have the following equalities:

$$\begin{aligned}
&(1)\ \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} = 1 + \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}} && (k \geq 1)\,, \\
&(2)\ \Delta\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right) = \Delta\left(\frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\right) && (k \geq 2)\,, \\
&(3)\ \tilde{\Lambda}_n\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right) = \tilde{\Lambda}_n\left(\frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\right) && (n \geq 2)\,, \\
&(4)\ \tilde{\Lambda}_n(k) = \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)\sum_{k=2}^{n}\lambda_{k-1} && (n \geq 2)\,, \\
&(5)\ \tilde{\Lambda}_n\left(\frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\right) = \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)\sum_{k=2}^{n}\lambda_{k-1}\,\Delta\left(\frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\right) && (n \geq 2)\,.
\end{aligned}$$

Lemma 3.9 *We have the following:*

$$\begin{aligned}
&(1)\quad \tilde{\Lambda}_n(k)+\tilde{\Lambda}_n\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)=1 && (n\geq 1),\\
&(2)\quad \lambda_{k-1}\,\Delta\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\ \leq\ \Delta\left(\frac{\lambda_k\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right) && (k\geq 2),\\
&(3)\quad \left|\tilde{\Lambda}_n\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\right|\leq\ 1 \qquad \tilde{}\ , && (n\geq 1),\\
&(4)\quad 0\ \leq\ \tilde{\Lambda}_n(k)\ \leq\ 2 && (n\geq 1).
\end{aligned}$$

Proof. For(1), we have

$$\tilde{\Lambda}_n(k)+\tilde{\Lambda}_n\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)=\tilde{\Lambda}_n\left(k+\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)=\tilde{\Lambda}_n\left(\frac{k\lambda_k-(k-1)\,\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)=1$$

as we have seen in Lemma 3.1. For (2), it is obvious, for any $k\geq 2$, that

$$\Delta\left(\frac{\lambda_k\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)=\lambda_{k-1}\left(\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}-\frac{\lambda_{k-2}}{\lambda_{k-1}-\lambda_{k-2}}\right)=\lambda_{k-1}\left[1+\Delta\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\right]$$

and we have done by noting that

$$\lambda_{k-1}\left[1+\Delta\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\right]\geq\lambda_{k-1}\Delta\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right).$$

To prove (3), we use (5) of Remark 3.8. Then, for any $n\geq 2$, we have

$$\begin{aligned}
\left|\tilde{\Lambda}_n\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\right| &=\left(\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right)\left|\sum_{k=2}^{n}\lambda_{k-1}\,\Delta\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\right|\\
&\leq\left(\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right)\left|\sum_{k=2}^{n}\Delta\left(\frac{\lambda_k\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\right|\\
&=\left(\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right)\left(\frac{\lambda_n\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}\right).\\
&=1.
\end{aligned}$$

Finally, for (4) we find form (4) of Remark 3.8 that $\tilde{\Lambda}_n(k)\geq 0$ and so the result follows by (3), where

$$\tilde{\Lambda}_n(k)=\left|\tilde{\Lambda}_n(k)\right|=\left|1-\tilde{\Lambda}_n\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\right|\leq 1+\left|\tilde{\Lambda}_n\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\right|. \qquad \square$$

Now, lets define the triangle A as follows:

$$A=\begin{bmatrix}
1 & 0 & 0 & \cdots\\
0 & \lambda_1\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right) & 0 & \cdots\\
0 & \lambda_1\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_3}\right) & \lambda_2\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_3}\right) & \cdots\\
\vdots & \vdots & \vdots &
\end{bmatrix}$$

Then, it follows by Lemma 3.7 that $\tilde{\Lambda}_n(x)=\mathrm{A}_n(\Delta x)$ for all $n\geq 1$ and so $\tilde{\Lambda}_n(x)=\mathrm{A}(\Delta x)$ for every $x\in w$. Also, it is clear that

$$\lim_{n\to\infty}a_{nk}\ =\ \lambda_{k-1}\lim_{n\to\infty}\left(\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right)=0 \qquad (k\geq 2),$$

$$\sum_{k=1}^{\infty}|a_{nk}|\ =\ \sum_{k=1}^{\infty}a_{nk}=\left(\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right)\sum_{k=2}^{n}\lambda_{k-1} \qquad (n\geq 2).$$

Thus, we have $\lim_{n\to\infty} a_{nk} = 0$ for each k ≥ 1 and $\sum_{k=1}^{\infty} |a_{nk}| = \sum_{k=1}^{\infty} a_{nk} = \tilde{\Lambda}_n(k)$ for all n ≥ 1. Hence, by combining these facts with the results of [16] that

$$A \in (\ell_\infty,\ \ell_\infty) \ \Leftrightarrow\ A \in (c_0,\ c_0) \ \Leftrightarrow\ \sup_n \tilde{\Lambda}_n(k) < \infty \ \Leftrightarrow\ \tilde{\Lambda}_n(k) \in \ell_\infty\,, \tag{3.2}$$

$$A \in (c,\ c) \ \Leftrightarrow\ \lim_{n\to\infty} \tilde{\Lambda}_n(k) \ \text{ exists } \ \Leftrightarrow\ \tilde{\Lambda}_{\rm n}(\rm k) \in \rm c\,. \tag{3.3}$$

Theorem 3.10 *We have the following:*

(1) *The inclusions* $c_0(\Delta) \subset c_0\left(\Delta^\lambda\right)$ *and* $\ell_\infty(\Delta) \subset \ell_\infty\left(\Delta^\lambda\right)$ *always hold.*

(2) *The inclusion* $c(\Delta) \subset c\left(\Delta^\lambda\right)$ *holds if and only if* $\tilde{\Lambda}\left(\lambda_k/(\lambda_k - \lambda_{k-1})\right) \in c$.

(3) $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \lim_{n\to\infty} \Delta(x_n)$ *for every* $x \in c(\Delta)$ *if and only if* $\tilde{\Lambda}\left(\lambda_k/(\lambda_k - \lambda_{k-1})\right) \in c_0$.

Proof. For(1), we have $x \in c_0(\Delta)$ if and only if $\Delta x \in c_0$.. Thus, we obtain from the fact $\tilde{\Lambda}(x) = A(\Delta x)$ for all $x \in w$ that:

$$\begin{aligned} c_0(\Delta) \subset c_0\left(\Delta^\lambda\right) &\Leftrightarrow \tilde{\Lambda}(x) \in c_0 && \forall\ x \in c_0(\Delta)\\ &\Leftrightarrow A(\Delta x) \in c_0 && \forall\ \Delta(x) \in c_0\\ &\Leftrightarrow A(y) \in c_0 && \forall\ y \in c_0\\ &\Leftrightarrow A \in (c_0,\ c_0)\,. \end{aligned}$$

Similarly, we can show that $\ell_\infty(\Delta) \subset \ell_\infty\left(\Delta^\lambda\right) \Leftrightarrow A \in (\ell_\infty,\ \ell_\infty)$. Thus, it follows from (3.2) that

$$c_0(\Delta) \subset c_0\left(\Delta^\lambda\right) \ \Leftrightarrow\ \ell_\infty(\Delta) \subset \ell_\infty\left(\Delta^\lambda\right) \ \Leftrightarrow\ \tilde{\Lambda}_n(k) \in \ell_\infty\,.$$

But the condition $\tilde{\Lambda}(k) \in \ell_\infty$ is always satisfied by (4) of Lemma 3.9. Therefore, the inclusions $c_0(\Delta) \subset c_0(\Delta^\lambda)$ and $\ell_\infty(\Delta) \subset \ell_\infty(\Delta^\lambda)$ always hold.
To prove (2), we can use the same technique to show that $c(\Delta) \subset c(\Delta^\lambda) \Leftrightarrow A \in (c,\ c) \Leftrightarrow \tilde{\Lambda}(k) \in c$ which can be obtained with help of (3.3). Thus, by using (1) of Lemma 3.9, we deduce the equivalence $c(\Delta) \subset c\left(\Delta^\lambda\right) \Leftrightarrow \tilde{\Lambda}\left(\lambda_k/(\lambda_k - \lambda_{k-1})\right) \in c$. Finally, to prove (3), let $x \in c(\Delta)$ with $\Delta(x_k) \to L$ as $k \to \infty$. Then $(x_k - Lk) \in c_0(\Delta) \subset c_0(\Delta^\lambda)$. Also, since $\tilde{\Lambda}_n(x) = \tilde{\Lambda}_n(x_k - Lk) + L\tilde{\Lambda}_n(k)$; we find by passing to the limits when $n \to \infty$ that

$$\lim_{n\to\infty} \tilde{\Lambda}_n(x) = L \lim_{n\to\infty} \tilde{\Lambda}_n(k) = L - L \lim_{n\to\infty} \tilde{\Lambda}_n\left(\frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\right).$$

Thus, the regular case $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = \lim_{n\to\infty} \Delta(x_n)$ holds for every $x \in c(\Delta)$ if and only if $\tilde{\Lambda}\left(\lambda_k/(\lambda_k - \lambda_{k-1})\right) \in c_0$, or equivalently $\lim_{n\to\infty} \tilde{\Lambda}_n(k) = 1$ (note that: $\left(\lambda_k/(\lambda_k - \lambda_{k-1})\right) \in c_0(\Delta)$ implies $\tilde{\Lambda}\left(\lambda_k/(\lambda_k - \lambda_{k-1})\right) \in c_0$ but not the converse). □

Remark 3.11 We may note, by Theorem 3.10 and its proof, that the inclusion $c(\Delta) \subset c\left(\Delta^\lambda\right)$ implies both inclusions $c_0(\Delta) \subset c_0\left(\Delta^\lambda\right)$ *and* $\ell_\infty(\Delta) \subset \ell_\infty\left(\Delta^\lambda\right)$, and the inclusion $c(\Delta) \subset c\left(\Delta^\lambda\right)$ holds if and only if $\tilde{\Lambda}(k) \in c$, and this condition can be written by (4) of Remark 3.8 as follows:

$$\lim_{n\to\infty}\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n\lambda_{n-1}}\right)\sum_{k=2}^{n} \lambda_{k-1} \ \text{ exists.}$$

Similarly, by using the same idea, we can show that the inclusion $c_0\left(\Delta^\lambda\right) \subset c_0(\Delta)$ holds if and only if the inclusion $\ell_\infty\left(\Delta^\lambda\right) \subset \ell_\infty(\Delta)$ holds, and the inclusion $c\left(\Delta^\lambda\right) \subset c(\Delta)$ implies both inclusions $c_0\left(\Delta^\lambda\right) \subset c_0(\Delta)$ and $\ell_\infty\left(\Delta^\lambda\right) \subset \ell_\infty(\Delta)$, and these inclusions cannot be strict.

Now, for any sequence $x \in w$, we have the following equality (see [12, Lemma 4.1])

$$\Delta(x_n) - \tilde{\Lambda}_n(x) = \Delta\left(\frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}}\,\tilde{\Lambda}_n(x)\right) \qquad (n \geq 2).$$

Also, by using (2.1) and (3.1), we find for any $n \geq 2$ that

$$\Lambda_n\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\Delta\left(x_k\right)\right)=\frac{1}{\lambda_n}\sum_{k=1}^{n}\lambda_{k-1}\Delta\left(x_k\right)=\frac{1}{\lambda_n}\sum_{k=2}^{n}\lambda_{k-1}\Delta\left(x_k\right)=\frac{\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}\tilde{\Lambda}_n(x)$$

and by operating Δ on both sides and combining the last two equations, we deduce the following equalities for any $x \in w$:

$$\Delta(x_n)-\tilde{\Lambda}_n(x)=\Delta\left(\frac{\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}\tilde{\Lambda}_n(x)\right)=\tilde{\Lambda}_n\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\Delta\left(x_k\right)\right)\qquad(n\geq 2).\qquad(3.4)$$

Theorem 3.12 *We have the following:*

(1) *The equalities* $c_0\left(\Delta^\lambda\right)=c_0\left(\Delta\right)$ *and* $\ell_\infty\left(\Delta^\lambda\right)=\ell_\infty\left(\Delta\right)$ *hold if and only if* $\left(\lambda_k/(\lambda_k-\lambda_{k-1})\right)\in\ell_\infty$.

(2) *The equality* $c\left(\Delta^\lambda\right)=c\left(\Delta\right)$ *holds if and only if* $\left(\lambda_k/(\lambda_k-\lambda_{k-1})\right)\in\ell_\infty\cap c_0(\Delta)$.

Proof. For (1), if $(\lambda_k/(\lambda_k-\lambda_{k-1}))\in\ell_\infty$ and so $(\lambda_{k-1}/(\lambda_k-\lambda_{k-1}))\in\ell_\infty$ by (1) of Remark 3.8; then from (3.4) we find that $x\in c_0(\Delta)\ \Leftrightarrow\ x\in c_0(\Delta^\lambda)$, as well as $x\in\ell_\infty(\Delta)\ \Leftrightarrow\ x\in\ell_\infty(\Delta^\lambda)$, and hence the two equalities in (1) hold. Conversely, if $\ell_\infty(\Delta^\lambda)=\ell_\infty(\Delta)$, or equivalently $c_0(\Delta^\lambda)=c_0(\Delta)$; then it follows from the proof of Lemma 3.1 that $y\in\ell_\infty(\Delta^\lambda)$ and so $y\in\ell_\infty(\Delta)$, where

$$y_k=\frac{(-1)^k}{2}\left(\frac{\lambda_k+\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\qquad(k\geq 1).$$

Thus, we have $\Delta y\in\ell_\infty$,. But, for any $k\geq 2$ we also have

$$|\Delta(y_k)|=\frac{1}{2}\left(\frac{\lambda_k+\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}+\frac{\lambda_{k-1}+\lambda_{k-2}}{\lambda_{k-1}-\lambda_{k-2}}\right)\geq\frac{1}{2}\left(\frac{\lambda_k+\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\geq\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}$$

which implies that $(\lambda_{k-1}/(\lambda_k-\lambda_{k-1}))\in\ell_\infty$ and so $(\lambda_k/(\lambda_k-\lambda_{k-1}))\in\ell_\infty$.
To prove (2), suppose that $(\lambda_k/(\lambda_k-\lambda_{k-1}))\in\ell_\infty\cap c_0(\Delta)$ or equivalently $(\lambda_{k-1}/(\lambda_k-\lambda_{k-1}))\in\ell_\infty\cap c_0(\Delta)$. Then, it follows from (3.4) that $x\in c\left(\Delta^\lambda\right)\Leftrightarrow x\in c(\Delta)$, because of $\lim_{n\to\infty}\tilde{\Lambda}_n(x)=\lim_{n\to\infty}\Delta(x_n)$ for every x in $c(\Delta^\lambda)$ or in $c(\Delta)$. To see that, we have

$$\Delta\left(\frac{\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}\tilde{\Lambda}_n(x)\right)=\frac{\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}\Delta(\tilde{\Lambda}_n(x))+\tilde{\Lambda}_{n-1}(x)\,\Delta\left(\frac{\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}\right)\longrightarrow 0\ \text{ as }\ n\to\infty,$$

$$\Delta\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\Delta\left(x_k\right)\right)=\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\Delta(\Delta(x_k))+\Delta(x_{k-1})\,\Delta\left(\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\right)\longrightarrow 0\ \text{ as }\ k\to\infty.$$

Conversely, suppose that $c\left(\Delta^\lambda\right)=c\left(\Delta\right)$. Then, we must have $\ell_\infty\left(\Delta^\lambda\right)=\ell_\infty\left(\Delta\right)$ and therefore $(\lambda_k/(\lambda_k-\lambda_{k-1}))\in\ell_\infty$. Also, in the proof of Lemma 3.1, we have $x\in c(\Delta^\lambda)$ and so $x\in c(\Delta)$, where

$$x_k=k+\frac{\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}\qquad(k\geq 1)$$

such that $\lim_{k\to\infty}\Delta(x_k)=\lim_{k\to\infty}\tilde{\Lambda}_k(x)=1$. But $\Delta(x_k)=1+\Delta\left(\lambda_{k-1}/(\lambda_k-\lambda_{k-1}\right)$ for all k, which implies that $\lim_{k\to\infty}\Delta\left(\lambda_{k-1}/(\lambda_k-\lambda_{k-1}\right)=0$ and so $\lim_{k\to\infty}\Delta\left(\lambda_k/(\lambda_k-\lambda_{k-1}\right)=0$ by (2) of Remark 3.8. Thus, we deduce that $\left(\lambda_k/(\lambda_k-\lambda_{k-1}\right)\in c_0\left(\Delta\right)$. Finally, we have already shown that $(\lambda_k/(\lambda_k-\lambda_{k-1}))\in\ell_\infty$ as well as $\left(\lambda_k/(\lambda_k-\lambda_{k-1}\right)\in c_0\left(\Delta\right)$, which together imply that $\left(\lambda_k/(\lambda_k-\lambda_{k-1}\right)\in\ell_\infty\cap c_0(\Delta)$ and this completes the proof. □

Corollary 3.13 *We have the following:*

(1) *If* $\left(\lambda_k/(\lambda_k-\lambda_{k-1})\right)\in c$; *then the equality* $c\left(\Delta^\lambda\right)=c\left(\Delta\right)$ *holds.*

(2) *If* $\tilde{\Lambda}(k)\in c_0$; *then all the spaces* $c_0\left(\Delta\right), c\left(\Delta\right)$ *and* $\ell_\infty\left(\Delta\right)$ *are strictly included in* $c_0\left(\Delta^\lambda\right)$.

Proof. (1) is immediate by (2) of Theorem 3.12. To prove (2), let $x \in \ell_\infty(\Delta)$. Then, there exists $M > 0$ such that $|\Delta(x_k)| \leq M$ for all k and so we obtain by (4) of Remark 3.8 that

$$\left|\tilde{\Lambda}_n(x)\right| \leq \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)\sum_{k=2}^{n} \lambda_{k-1}|\Delta(x_k)| \leq M\left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)\sum_{k=2}^{n}\lambda_{k-1} \leq M\,\tilde{\Lambda}_n(k).$$

Thus, we get $0 \leq |\tilde{\Lambda}_n(x)| \leq M\,\tilde{\Lambda}_n(k)$ for all n. Consequently, the result follows by going to the limits when $n \to \infty$. □

At the end of this section, we give some examples of the distinct cases of above results concerning the inclusions $\mu(\Delta) \subset \mu(\Delta^\lambda)$, where μ denotes any of the spaces c_0, c or ℓ_∞. For simplicity in notations, we will use the symbole $u = \lambda/\Delta\lambda$, that is $u_k = \lambda_k/(\lambda_k - \lambda_{k-1})$ for $k \geq 1$.

Example 3.14 The cases of strict inclusions: for $c_0(\Delta) \subsetneqq c_0(\Delta^\lambda)$ and $\ell_\infty(\Delta) \subsetneqq \ell_\infty(\Delta^\lambda)$; it is enough that $u \notin \ell_\infty$, but for $c(\Delta) \subsetneqq c(\Delta^\lambda)$ we must have $u \notin \ell_\infty$ and $\tilde{\Lambda}(u) \in c$. The last case holds when $\Delta(u) \to \infty$ or $\Delta(u) \in c$ with $u \notin \ell_\infty$. Thus, we have the following two cases:

I - When $\Delta(u) \in c$ and $u \notin \ell_\infty$: consider the sequence $\lambda = (\lambda_n)$ defined by $\lambda_n = (n+1)^a$, where $a > 0$ $(n \geq 1)$. Then, we have $u_n \to \infty$, $\Delta(u_n) \to 1/a$, $\tilde{\Lambda}_n(u) \to 1/(1+a)$ and $\tilde{\Lambda}_n(k) \to a/(1+a)$.

II - When $\Delta(u) \to \infty$: it is the strong case of strict inclusions as proved in Corollary 3.13. For example, consider the sequence $\lambda = (\lambda_n)$ given in Example 2.5 or the sequence $\lambda_n = \ln(1+n)$ for $n \geq 1$. Then, we have $u_n \to \infty$, $\Delta(u_n) \to \infty$, $\tilde{\Lambda}_n(u) \to 1$ and $\tilde{\Lambda}_n(k) \to 0$ (the main property of this case is $1/\lambda \notin \ell_p$ for every $p > 0$).

In these two previous cases of this example, it is obvious that:

$$\lim_{n\to\infty} \tilde{\Lambda}_n(u) = \lim_{n\to\infty} \frac{\Delta(u_n)}{1+\Delta(u_n)} \quad \text{and} \quad \lim_{n\to\infty} \tilde{\Lambda}_n(k) = \lim_{n\to\infty} \frac{1}{1+\Delta(u_n)}.$$

Example 3.15 The cases of identities: for $c_0(\Delta^\lambda) = c_0(\Delta)$ and $\ell_\infty(\Delta^\lambda) = \ell_\infty(\Delta)$; it is enough that $u \in \ell_\infty$, but for $c(\Delta^\lambda) = c(\Delta)$ we must have $u \in \ell_\infty \cap c_0(\Delta)$. In the first case, the equality $c(\Delta^\lambda) = c(\Delta)$ may fails as will be shown in the next example. Here, we will consider the the second case (the strong case of regularity). For example, let $\lambda = (\lambda_n)$ be defined by $\lambda_n = (n+1)!$ or $\lambda_n = a^n$, where $a > 1$ $(n \geq 1)$. In such case, we must have $u \in \ell_\infty$, $\Delta(u_n) \to 0$, $\tilde{\Lambda}_n(u) \to 0$ and $\tilde{\Lambda}_n(k) \to 1$.

Example 3.16 The case of non-inclusion between $c(\Delta)$ and $c(\Delta^\lambda)$: that is $c(\Delta) \not\subset c(\Delta^\lambda)$ and $c(\Delta^\lambda) \not\subset c(\Delta)$. In this case, we must have $\tilde{\Lambda}(u) \notin c$ which means that the sequence $\tilde{\Lambda}(u)$ is oscillated (it has no unique limit). The main property of this case is not only that $\Delta(u) \notin c$ (e.g. $\Delta(u) \to \infty$ is not the case) but the limit of $\Delta(u)$ does not exist and it must be oscillated between at least two values (it maybe oscillated through $\pm\infty$). Here also, there are two distinct cases:

I - When $u \in \ell_\infty$ and so it must be oscillated (in this case, the equalities $c_0(\Delta^\lambda) = c_0(\Delta)$ and $\ell_\infty(\Delta^\lambda) = \ell_\infty(\Delta)$ are satisfied): For example, consider the sequence $\lambda = (a, ab, a^2b, a^2b^2, \cdots)$, where $b > a > 1$, that is $\lambda_k = a^{(k+1)/2}\, b^{(k-1)/2}$ when k is odd, or $\lambda_k = a^{k/2}\, b^{k/2}$ when k is even. Then, it can easily be shown that

$$u_k = \begin{cases} a/(a-1)\,; & (k \text{ is odd}) \\ b/(b-1)\,; & (k \text{ is even}) \end{cases} \qquad \Delta(u_k) = \begin{cases} (b-a)/[(a-1)(b-1)]\,; & (k \text{ is odd}) \\ -(b-a)/[(a-1)(b-1)]\,; & (k \text{ is even}) \end{cases}$$

$$\tilde{\Lambda}_n(u) = \begin{cases} \dfrac{b-a}{ab-1} + \dfrac{(a-1)(b+1)}{ab-1}(ab)^{-(n-1)/2}\,; & (n \text{ is odd}) \\ -\dfrac{b-a}{ab-1} + \dfrac{a(b^2-1)}{ab-1}(ab)^{-n/2}\,; & (n \text{ is even}) \end{cases}$$

where $n, k > 1$. Thus, it is clear that all of u, $\Delta(u)$, $\tilde{\Lambda}(u)$ and $\tilde{\Lambda}(k)$ are oscillated.

II - When $u \notin \ell_\infty$ and so $\Delta(u)$ must be oscillated through $+\infty$, $-\infty$ or both (in this case, the inclusions $c_0(\Delta) \subset c_0(\Delta^\lambda)$ and $\ell_\infty(\Delta) \subset \ell_\infty(\Delta^\lambda)$ are strict): For example, consider the sequence

$\lambda = (1, 3, 4, 8, 9, \cdots)$, that is $\lambda_k = (k+1)^2/4$ when k is odd, or $\lambda_k = (k^2+4k)/2$ when k is even. Then, it can easily be seen that

$$u_k = \begin{cases} (k+1)^2/4\,; & (k \text{ is odd}) \\ (k+4)/4\,; & (k \text{ is even}) \end{cases} \qquad \Delta(u_k) = \begin{cases} (k^2+k-2)/4\,; & (k \text{ is odd}) \\ (-k^2+k+4)/4\,; & (k \text{ is even}) \end{cases}$$

$$\tilde{\Lambda}_n(u) = \begin{cases} (3n-1)/(3n+3)\,; & (n \text{ is odd}) \\ (-n+4)/(3n)\,; & (n \text{ is even}) \end{cases}$$

where $n, k > 1$. Hence, we have $u_k \to \infty$, $\Delta(u)$ is oscillated between $\pm\infty$, $\tilde{\Lambda}(u)$ is oscillated between 1 and $-1/3$, and so $\tilde{\Lambda}(k)$ will be oscillated between 0 and $4/3$.

4 The Schauder bases for $c_0\left(\Delta^\lambda\right)$ and $c\left(\Delta^\lambda\right)$

In the last section, we construct the Schauder bases for the λ-difference spaces $c_0\left(\Delta^\lambda\right)$ and $c\left(\Delta^\lambda\right)$ and conclude their separability.

If a normed space X contains a sequence $(b_k)_{k=1}^\infty$ with the property that for every $x \in X$ there is a unique sequence $(\alpha_k)_{k=1}^\infty$ of scalars such that $\lim_{n\to\infty} \|x - (\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n)\| = 0$; then the sequence $(b_k)_{k=1}^\infty$ is called a Schauder basis for X (or simply a basis for X) and the series $\sum_{k=1}^\infty \alpha_k b_k$ which has the sum x is then called the expansion of x, with respect to the given basis, which can be written as $x = \sum_{k=1}^\infty \alpha_k b_k$, and we then say that x has been uniquely represented in that form. For example, the sequences $(e_1, e_2, e_3, \cdots)$ and $(e, e_1, e_2, \cdots)$ are the Schauder bases for the sequence spaces c_0 and c, respectively, where $e = (1, 1, 1, \ldots)$ and $e_k = (\delta_{nk})_{n=1}^\infty$ for each k, but the space ℓ_∞ is non-separable and so it has no a Schauder basis [7].

Theorem 4.1 *For each $k \geq 1$, define the sequence $e_k^\lambda = (e_{nk}^\lambda)_{n=1}^\infty$ by*

$$e_{nk}^\lambda = \begin{cases} 0\,; & (n \leq k), \\ \dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}}\,; & (n = k), \qquad (n \geq 1) \\ 1\,; & (n > k), \end{cases}$$

Then, the sequence $\left(e_k^\lambda\right)_{k=1}^\infty$ is a Schauder basis for the space $c_0(\Delta^\lambda)$ and every $x \in c_0(\Delta^\lambda)$ has a unique representation in the following form:

$$x = \sum_{k=1}^\infty \tilde{\Lambda}_k(x)\, e_k^\lambda\,. \tag{4.1}$$

Proof. It is clear that $\Lambda_n(e_k^\lambda) = 0$ for $1 \leq n < k$ and $\Lambda_n(e_k^\lambda) = 1$ for $n \geq k$ and so $\tilde{\Lambda}_n(e_k^\lambda) = \delta_{nk}$ for all $n, k \geq 1$. Thus $\tilde{\Lambda}(e_k^\lambda) = e_k \in c_0$ and hence $e_k^\lambda \in c_0(\Delta^\lambda)$ for all $k \geq 1$. This means that $\left(e_k^\lambda\right)_{k=1}^\infty$ is a sequence in $c_0(\Delta^\lambda)$. Further, let $x \in c_0(\Delta^\lambda)$ be given and for every positive integer m, we put

$$x^{(m)} = \sum_{k=1}^m \tilde{\Lambda}_k(x)\, e_k^\lambda\,.$$

Then, we find that

$$\tilde{\Lambda}(x^{(m)}) = \sum_{k=1}^m \tilde{\Lambda}_k(x)\, \tilde{\Lambda}(e_k^\lambda) = \sum_{k=1}^m \tilde{\Lambda}_k(x)\, e_k$$

and hence

$$\tilde{\Lambda}_n(x - x^{(m)}) = \begin{cases} 0\,; & (1 \leq n \leq m), \\ \tilde{\Lambda}_n(x)\,; & (n > m). \end{cases}$$

Now, for any positive real $\epsilon > 0$, there is a positive integer m_0 such that $|\tilde{\Lambda}_m(x)| < \epsilon$ for all $m \geq m_0$. Thus, for every $m \geq m_0$, we have

$$\|x - x^{(m)}\|_\lambda = \sup_{n>m} |\tilde{\Lambda}_n(x)| \leq \sup_{n>m_0} |\tilde{\Lambda}_n(x)| \leq \epsilon\,.$$

We therefore deduce that $\lim_{m\to\infty} \|x - x^{(m)}\|_\lambda = 0$ which means that x is represented as in (4.1). Thus, it is remaining to show the uniqueness of the representation (4.1) of x. For this, suppose that $x = \sum_{k=1}^{\infty} \alpha_k\, e_k^\lambda$. Then, we have to show that $\alpha_n = \tilde{\Lambda}_n(x)$ for all n, which is immediate by operating $\tilde{\Lambda}_n$ on both sides of (4.1) for each $n \geq 1$, where the continuity of $\tilde{\Lambda}$ (as we have seen in Remark 2.4) allows us to obtain that

$$\tilde{\Lambda}_n(x) = \sum_{k=1}^{\infty} \alpha_k\, \tilde{\Lambda}_n(e_k^\lambda) = \sum_{k=1}^{\infty} \alpha_k\, \delta_{nk} = \alpha_n$$

for all $n \geq 1$ and hence the representation (4.1) of x is unique, and this completes the proof. □

Theorem 4.2 *The sequence* $\left(e^\lambda, e_1^\lambda, e_2^\lambda, \cdots\right)$ *is a Schauder basis for the space* $c(\Delta^\lambda)$ *and every* $x \in c(\Delta^\lambda)$ *has a unique representation in the following form:*

$$x = L\, e^\lambda + \sum_{k=1}^{\infty} \left(\tilde{\Lambda}_k(x) - L\right) e_k^\lambda\,, \tag{4.2}$$

where $L = \lim_{n\to\infty} \tilde{\Lambda}_n(x)$, *the sequence* $\left(e_k^\lambda\right)_{k=1}^{\infty}$ *is as in Theorem 4.1 and* e^λ *is the following sequence:*

$$e^\lambda = \left(\frac{n\lambda_n - (n-1)\,\lambda_{n-1}}{\lambda_n - \lambda_{n-1}}\right)_{n=1}^{\infty}.$$

Proof. It was already shown in the proof of Lemma 3.1 that $\tilde{\Lambda}(e^\lambda) = e \in c$ and so $e^\lambda \in c(\Delta^\lambda)$. This together with $e_k^\lambda \in c_0(\Delta^\lambda) \subset c(\Delta^\lambda)$ imply that that $\left(e^\lambda, e_1^\lambda, e_2^\lambda, \cdots\right)$ is a sequence in $c(\Delta^\lambda)$. Also, let $x \in c(\Delta^\lambda)$ be given. Then $\tilde{\Lambda}(x) \in c$ which yields the convergence of the sequence $\tilde{\Lambda}(x)$ to a unique limit, say $L = \lim_{n\to\infty} \tilde{\Lambda}_n(x)$. Thus, by taking $y = x - L\, e^\lambda$, we get $\tilde{\Lambda}(y) = \tilde{\Lambda}(x) - L\, e \in c_0$ and so $y \in c_0(\Delta^\lambda)$. Hence, it follows by Theorem 4.1 that y can be uniquely represented in the following form:

$$y = \sum_{k=1}^{\infty} \tilde{\Lambda}_k(y)\, e_k^\lambda = \sum_{k=1}^{\infty} \left(\tilde{\Lambda}_k(x) - L\, \tilde{\Lambda}_k(e^\lambda)\right) e_k^\lambda = \sum_{k=1}^{\infty} \left(\tilde{\Lambda}_k(x) - L\right) e_k^\lambda.$$

Consequently, our x can also be uniquely written as

$$x = L\, e^\lambda + y = L\, e^\lambda + \sum_{k=1}^{\infty} \left(\tilde{\Lambda}_k(x) - L\right) e_k^\lambda$$

which proves the unique representation (4.2) of x. □

Example 4.3 To give an example of the unique representation of a single sequence in particular spaces of $c_0(\Delta^\lambda)$ and $c(\Delta^\lambda)$, consider the sequence $\lambda = (\lambda_k)$ given by $\lambda_k = k(k+1)$ for $k \geq 1$. Then, we have $\tilde{\Lambda}_n(x) = \Lambda_n(x) - \Lambda_{n-1}(x)$ for all $n \geq 1$ and every $x \in w$, where

$$\Lambda_n(x) = \frac{2}{n(n+1)} \sum_{k=1}^{n} k\, x_k\,; \qquad (n \geq 1).$$

This yields the following particular cases of the general spaces of λ-difference sequences:

$$c_0(\Delta^\lambda) = \left\{ x = (x_k) \,:\, \left(\frac{2}{n(n+1)} \sum_{k=1}^{n} k\, x_k\right)_{n=1}^{\infty} \in c_0(\Delta) \right\},$$

$$c(\Delta^\lambda) = \left\{ x = (x_k) \,:\, \left(\frac{2}{n(n+1)} \sum_{k=1}^{n} k\, x_k\right)_{n=1}^{\infty} \in c(\Delta) \right\}.$$

Further, with help of Theorems 4.1 and 4.2, the Schauder bases for these two spaces are respectively the two sequences $\left(e_1^\lambda, e_2^\lambda, e_3^\lambda, \cdots\right)$ and $\left(e^\lambda, e_1^\lambda, e_2^\lambda, \cdots\right)$, where

$$e_1^\lambda = (1,\, 1,\, 1,\, \cdots),\ e_2^\lambda = (0,\, 3/2,\, 1,\, 1, \cdots),\ e_3^\lambda = (0,\, 0,\, 2,\, 1,\, 1, \cdots),\ e_4^\lambda = (0,\, 0,\, 0,\, 5/2,\, \cdots),\, \ldots$$

$$\text{and } e^\lambda = \big((3k-1)/2\big)_{k=1}^\infty = (1,\, 5/2,\, 4,\, 11/2,\, \cdots).$$

Now, consider the sequence $y = (y_k) \in c_0(\Delta^\lambda)$ defined by $y_k = (k+1)\sqrt{k} - (k-1)\sqrt{k-1}$ for $k \geq 1$. Then, we have $\Lambda_n(y) = 2\sqrt{n}$ and so $\tilde{\Lambda}_n(y) = 2(\sqrt{n} - \sqrt{n-1})$ for all $n \geq 1$. Thus, our sequence y has the unique representation $y = 2\sum_{n=1}^\infty (\sqrt{n} - \sqrt{n-1})\, e_n^\lambda$ in terms of the Schauder basis (e_n^λ) of the space $c_0(\Delta^\lambda)$. In addition, if we define $x = (x_k)$ by $x_k = 1 - 3k + y_k$ for $k \geq 1$. Then, we find that $\tilde{\Lambda}_n(x) = -2 + \tilde{\Lambda}_n(y)$ for all $n \geq 1$. Thus $x \in c(\Delta^\lambda)$ such that $\lim_{n\to\infty} \tilde{\Lambda}_n(x) = -2$. Hence, by applying Theorem 4.2, the sequence x has also a unique representation $x = -2\, e^\lambda + 2\sum_{n=1}^\infty (\sqrt{n} - \sqrt{n-1})\, e_n^\lambda$ in terms of the Schauder basis $\left(e^\lambda, e_1^\lambda, e_2^\lambda, \cdots\right)$ of the space $c(\Delta^\lambda)$.

Corollary 4.4 *We have the following facts:*

(1) *The spaces $c_0(\Delta^\lambda)$ and $c(\Delta^\lambda)$ are separable BK-spaces.*

(2) *The space $\ell_\infty(\Delta^\lambda)$ is a non-separable BK-space and has no a Schauder basis.*

Remark 4.5 We end our work by expressing from now on that our aim of the next paper is to determining the duals of our difference λ-sequence spaces and characterizing some matrix operators between them.

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