Exact Solution of One–Dimension Damping Wave Equation Using Laplace Transforms

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Abstract:

In this work, the exact solution of vibrating problem described by one dimensional damped wave equation using Laplace Transform is presented. The motion of an elastic fixed ends string which are subjected to a drag force has been investigated. In addition, the role of the initial displacement has been discussed and the exact solution obtained is illustrated and graphically presented.
Keywords: *Laplace transforms - damping waves.*

Introduction

Partial differential equations are equations which contain one or more partial derivatives of functions of more one independent variables. Moreover, partial differential equation describe all types of physical phenomena in engineering and science such as vibrating strings, acoustics, transport phenomena electromagnetic theory and blowing winds. So partial differential equations are very essential in solving engineering and theoretical physics problems. And the it is important to understand at least the main principles of the approximate solution of partial differential equations. Wave equation is example for physical problem governing by partial differential equation.

Laplace transform method has been known to be a powerful device for solving many functional equations as algebraic equations, ordinary and partial differential equations, integral equations and so on. In this paper we used this method for solving the partial differential equations. The method has the ability of solving systems of both linear and nonlinear partial differential equations. Extension of the method for solving of partial differential equations is active and excellent opportunity for future research.

Wave equation is an extremely important evolution model and it is widely used by physicist in describing the propagation of wave, electromagnetic waves, sound wave oscillatory wave. etc. One-dimensional wave occurs in many problems in science engineering, economics, sociology, physiology, and management.

In this work we consider a one-dimensional damping wave equation with a damping coefficient. The Laplace Transform is employed for solving this problem the crux of the subject matter of this work is using the inversion formula we obtain the final solution.

Statement of the problem

Consider a flexible string of length L is lightly stretched along the x-axis, string fixed at each end points, but free to move horizontally. so that $x = 0$, $x = L$. We assume that the set-up has damping. Then, the vertical displacement of the string for $0 < x < L$ and at any time is given by the displacement function $u(x,t)$. It satisfies the one–dimensional damped wave equation which in the presence of resistance proportional to velocity becomes, see [1], [2], [3] and [4]

$$a^2 u_{xx} = u_{tt} + \alpha u_t \ , \alpha \neq 0 \quad ; 0 < x < L \quad , t > 0 \qquad (1)$$

The equation is subject to boundary conditions (BC)

$$\mathrm{u}(0,\mathrm{t}) = 0, \ \ \mathrm{u}(\mathrm{l},0) = 0 \qquad (2)$$

and Initial conditions (IC)

$$\mathrm{u}(\mathrm{x},0) = \mathrm{f}(\mathrm{x}), \quad \mathrm{u_t}(\mathrm{x},0) = 0 \qquad (3)$$

The term αu_t represents a damping force proportional to the velocity u_t.

Laplace Transform Method

In our recent study we will solve the problem defined by $(1) - (3)$, by using the Laplace transform technique. By take Laplace transform of both sides of equation (1), we refer to [4], [5] and [6]

$$\mathcal{L}\{a^2 u_{xx}\} = \mathcal{L}\{u_{tt} + \alpha u_t\}$$

implies that

$$a^2 \mathcal{L}\{u_{xx}\} = s^2 \mathcal{L}\{u\} - su(x,0) - u_t(x,0) + \alpha(s\mathcal{L}(u) - u(x,0))$$

$$a^2 \frac{d^2}{dx^2}(\mathcal{L}\{u\}) = s^2 \mathcal{L}\{u\} - sf(x) - 0 + \alpha s \mathcal{L}\{u\} - \alpha\, f(x)$$

Since

$$\mathcal{L}\{u\} = U(x,s) \quad , \quad \text{puting } \ U(x,s) = Y(x)$$

The equation become,

$$\frac{d^2Y(x)}{dx^2} - \beta^2 Y(x) = -\frac{(s+\alpha)}{a^2} f(x)$$

$$where \quad \beta^2 = \frac{s^2+\alpha s}{a^2}$$

To solve this ODE we will apply Laplace Transform method again, we have

$$\mathcal{L}\left\{\frac{d^2Y(x)}{dx^2} - \beta^2 Y(x)\right\} = -\mathcal{L}\left\{\frac{s+\alpha}{a^2} f(x)\right\}$$

This leads to the equation

$$p^2\mathcal{L}\{Y\} - p\,Y(0) - \dot{Y}(0) - \beta^2\mathcal{L}\{Y\} = -\frac{s+\alpha}{a^2}\,\mathcal{L}\{f(x)\}$$

Use the fact that $Y(0) = U(0,S) = 0$ Since $u(0,t) = 0$ and simplifying, we get

$$\mathcal{L}\{Y\} = \frac{\dot{Y}(0)}{P^2-\beta^2} - \frac{s+\alpha}{a^2(p^2-\beta^2)}\,F(p)$$

By taken the inverse of Laplace Transform we obtained

$$Y(x) = \dot{Y}(0)\mathcal{L}^{-1}\left\{\frac{1}{p^2-\beta^2}\right\} - \frac{s+\alpha}{a^2}\mathcal{L}^{-1}\left\{\frac{F(p)}{p^2-\beta^2}\right\}$$

$$= \left(\frac{\sinh\beta x}{\beta}\right)\dot{Y}(0) - \frac{s+\alpha}{a^2}\frac{1}{\beta}\int_0^x f(u)\,\sinh\beta(x-u)\,du \qquad (4)$$

$$Where \quad \beta = \frac{\sqrt{s^2+\alpha s}}{a}.$$

Using the initial conditions, $Y(L) = U(L,S) = 0 \;\Rightarrow Y(L) = 0,$ and simplifying, we Find

$$\dot{Y}(0) = \frac{(s+\alpha)}{a^2}\frac{1}{\sinh\beta L}\int_0^L f(u)\sinh\beta(L-u)du$$

Substitute of $\dot{Y}(0)$ into equation (4), and simplifying, we find

$$Y(x) = \frac{s+\alpha}{\beta a^2}\left[\int_0^L \frac{f(u)\sinh\beta x\sinh\beta(L-u)du}{\sinh\beta L} - \int_0^x f(u)\sinh\beta(x-u)du\right] \qquad (5)$$

Employing some algebraic manipulations and use the fact that

$\int_0^L = \int_0^x + \int_x^L$ and $\sinh(k_1 \pm k_2) = \sinh k_1\cosh k_2 \pm \cosh k_1 \sinh k_2,$

Equation (5) becomes

$$Y(x) = U(x,s) = \int_0^x f(u)\frac{\sinh\beta(L-x)\sinh\beta u}{s\sinh\beta L}du + \int_x^L f(u)\frac{\sinh\beta x\sinh\beta(L-u)}{s\sinh\beta L}du$$

We now proceed to evaluate the Laplace inverse of $Y(x)$ using the complex inversion formula [4]

$$\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} e^{st}\left[\int_0^x f(u)\frac{\sinh\frac{\sqrt{s(s+\alpha)}}{a}(L-x)\sinh\frac{\sqrt{s(s+\alpha)}}{a}u}{s\sinh\frac{\sqrt{s(s+\alpha)}}{a}L}du + \int_x^L f(u)\frac{\sinh\frac{\sqrt{s(s+\alpha)}}{a}(x)\cdot\sinh\frac{\sqrt{s(s+\alpha)}}{a}(L-u)}{s\sinh\frac{\sqrt{s(s+\alpha)}}{a}L}du\right]ds = \sum Res(s_k) \qquad (6)$$

It is clear that the $Res(s_k)$ are residue of the function $e^{st}\,U(x,s)$. Since $U(x,s)$ has a removeable singulasrity of poles and this will lead immediately to the conclusion that $e^{st}\;U(x,s)$, which also has a removable singularity of poles.

Thus, the function $e^{st}U(x,s)$ is seen to have simple poles at, see [4]

$$s = 0\,, \quad s = 0\,, \quad s = -\alpha \;\text{ and }\; s = \frac{-\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \frac{a^2n^2\pi^2}{L^2}}$$

In case $\frac{\alpha^2}{4} < \frac{a^2n^2\pi^2}{L^2}$, we can write $s = \frac{-\alpha}{2} \pm i\,\mu_n$, where $\mu_n = \sqrt{\frac{a^2n^2\pi^2}{L^2} - \frac{\alpha^2}{4}}$

It is clear that $s_0 = 0$ is pole of order 2,

For the pole $s = 0$, the Residue is given by .

$$Res(e^{st}U(x,s),0)$$
$$= \lim_{s\to 0}\frac{1}{(2-1)!}\frac{\partial}{\partial s}\left[s^2e^{st}\int_0^x f(u)\frac{sinh\frac{\sqrt{s(s+\alpha)}}{a}(L-x)}{s\,sinh\frac{\sqrt{s(s+\alpha)}}{a}L}sinh\frac{\sqrt{s(s+\alpha)}}{a}u\,du\right.$$
$$\left.+s^2e^{st}\int_x^L f(u)\frac{sinh\frac{\sqrt{s(s+\alpha)}}{a}x\,sinh\frac{\sqrt{s(s+\alpha)}}{a}(L-u)du}{s\;sinh\frac{\sqrt{s(s+\alpha)}}{a}L}\right]$$

By apply the first derivative with respect to s and applying L'Hospital's rule of the limit at $s\to 0$, we get,

$$\mathrm{Res}(e^{-st}U(x,s),0)=0 \qquad (7)$$

In similar way, For $s_1=-\alpha$ the Residue is given by

$$Res(e^{st}U(x,s),-\alpha)=0 \qquad (8)$$

For $s=\frac{-\alpha}{2}\pm i\,\mu_n$ by putting $m_1=\frac{-\alpha}{2}+i\,\mu_n$ *and* $m_2=\frac{-\alpha}{2}-i\,\mu_n$

The residue for $m_1=\frac{-\alpha}{2}+i\,\mu_n$ is given by

$$Res(e^{st}U(x,s),m_1)$$
$$=\lim_{s\to m_1}\left[(s-m_1)e^{st}\int_0^x f(u)\frac{sinh\frac{\sqrt{s(s+\alpha)}}{a}(L-x)\,sinh\frac{\sqrt{s(s+\alpha)}}{a}u}{s\,sinh\frac{s(s+\alpha)}{a^2}L}du\right.$$
$$\left.+(s-m_1)e^{st}\int_x^L f(u)\,sinh\frac{\sqrt{s(s+\alpha)}}{a}x\,sinh\frac{\sqrt{s(s+\alpha)}}{a}(L-u)\,du\right]$$

We recast $Res(e^{st}U(x,s),m_1)$ as

$$Res(e^{st}U(x,s),m_1)$$
$$=\lim_{s\to m_1}\frac{s-m_1}{sinh\frac{\sqrt{s(s+\alpha)}}{a}L}\left[\lim_{s\to m_1}e^{st}\int_0^x f(u)\frac{sinh\frac{\sqrt{s(s+\alpha)}}{a}(L-x)\;sinh\frac{\sqrt{s(s+\alpha)}}{a}u}{s}du\right.$$
$$\left.+\lim_{s\to m_1}e^{st}\int_x^L f(u)\,sinh\frac{\sqrt{s(s+\alpha)}}{a}x\;sinh\frac{\sqrt{s(s+\alpha)}}{a}(L-u)du\right] \qquad (9)$$

Taking the indicated limits and employing some algebraic manipulations, equation (9) becomes

$$Res(e^{st}U(x,s),m_1)=\left(\frac{n\pi}{2\mu_n}\frac{a^2}{L}\;\frac{e^{m_1t}}{m_1}\right)\left(\frac{2}{L}\int_0^L f(u)\,sin\frac{n\pi}{L}u\;du\right)sin\frac{n\pi x}{L} \qquad (10)$$

In similar fashion, we get

$$Res(e^{st}U(x,s),m_2)=\left(-\frac{n\pi}{2\mu_n}\frac{a^2}{L}\;\frac{e^{m_2t}}{m_2}\right)\left(\frac{2}{L}\int_0^L f(u)\,sin\frac{n\pi}{L}u\;du\right)sin\frac{n\pi x}{L} \qquad (11)$$

Using equations (7), (8), (9), (10) and (11) in equation (6) gives us the summation of the residues

$$\sum_{n=1}^{\infty}Res(s_n)=\sum_{n=1}^{\infty}\frac{n\pi}{2\mu_n}\frac{a^2}{L}\left(\frac{e^{tm_1}}{m_1}-\frac{e^{tm_2}}{m_2}\right)\left(\frac{2}{L}\int_0^L f(u)\,sin\frac{n\pi}{L}u\;du\right)sin\frac{n\pi x}{L}$$

Thus the general solution is

$$u(x,t)=\sum_{n=1}^{\infty}\frac{n\pi}{2\mu_n}\frac{a^2}{L}\left(\frac{e^{tm_1}}{m_1}-\frac{e^{tm_2}}{m_2}\right)\left(\frac{2}{L}\int_0^L f(u)\,sin\frac{n\pi}{L}u\;du\right)sin\frac{n\pi x}{L} \qquad (12)$$

By simplifying, expression (12) becomes

$$u(x,t)\;=\sum_{n=1}^{\infty}e^{\frac{-\alpha}{2}t}\frac{L}{n\,\pi}a_n\left(\cos\mu_n t+\frac{\alpha}{2\mu_n}\sin\mu_n t\right)sin\frac{n\pi x}{L} \qquad (13)$$

Where $\mu_n = \sqrt{\left|\frac{a^2 n^2 \pi^2}{L^2} - \frac{\alpha^2}{4}\right|}$, and $a_n = \frac{2}{L}\int_0^L f(u)\, sin\frac{n\pi}{L}u\ du$

In case $\alpha^2 > \frac{4a^2n^2\pi^2}{L^2}$,

$$u(x,t) = \sum_{n=1}^{\infty} e^{\frac{-\alpha}{2}t}\frac{L}{n\,\pi}a_n\left(\cosh\mu_n t + \frac{\alpha}{2\mu_n}\sinh\mu_n t\right) sin\frac{n\pi x}{L}$$

In both cases the solutions are exactly the same as that obtained by the separation of variables method.

We note that the solution $u(x,t)$ corresponding to the eigenvalues λ_n are oscillatory if, $\frac{\alpha^2}{4} < \frac{a^2n^2\pi^2}{L^2}$, $\forall$ n . i.e.

$$\frac{\alpha^2}{4} < \min_{n\geq 1}\frac{a^2n^2\pi^2}{L^2} = \frac{a^2\pi^2}{L^2}$$

Taking the square root of both sides and rearranging gives the criterion that make oscillatory,

$$\frac{\alpha L}{2a\pi} < 1 \quad or \quad L < \frac{2a\pi}{\alpha}$$

Clearly, in such case the length of string is directly proportional to the material constant and inversely to the damping parameter. Now for apply this result directly and in order to analyze the behavior of the solutions of Eq. (13), we need to specify the parameters and initial conditions. To this end, initial displacement of the most simple form (i.e. a plucked string at the center) is taken,

$$u(x,0) = \begin{cases} \frac{2A}{L}x, & 0 < x \leq \frac{L}{2} \\ \frac{2A}{L}(L-x), & \frac{L}{2} \leq x < L \end{cases} \qquad (14)$$

For our analyze, we have considered a string of length $L = \pi$, and the distinct values of constants. For the initial displacement (14), the obtained constants under these conditions are

$$a_n = \frac{8\,A}{n^2\pi^2}(-1)^{n-1}, \quad n = 1,2,3, \dots \dots$$

The exact solution $u(x,t)$ as given by equations (13), with this particular set of initial conditions and different values of the damping parameter is shown in Figures 1-5, some three-dimensional figures have been depicted in some special cases over two period using MatLab [1]. The analytical results and profiles obtained in this contribution provide us a physical interpretation for the considered equation.

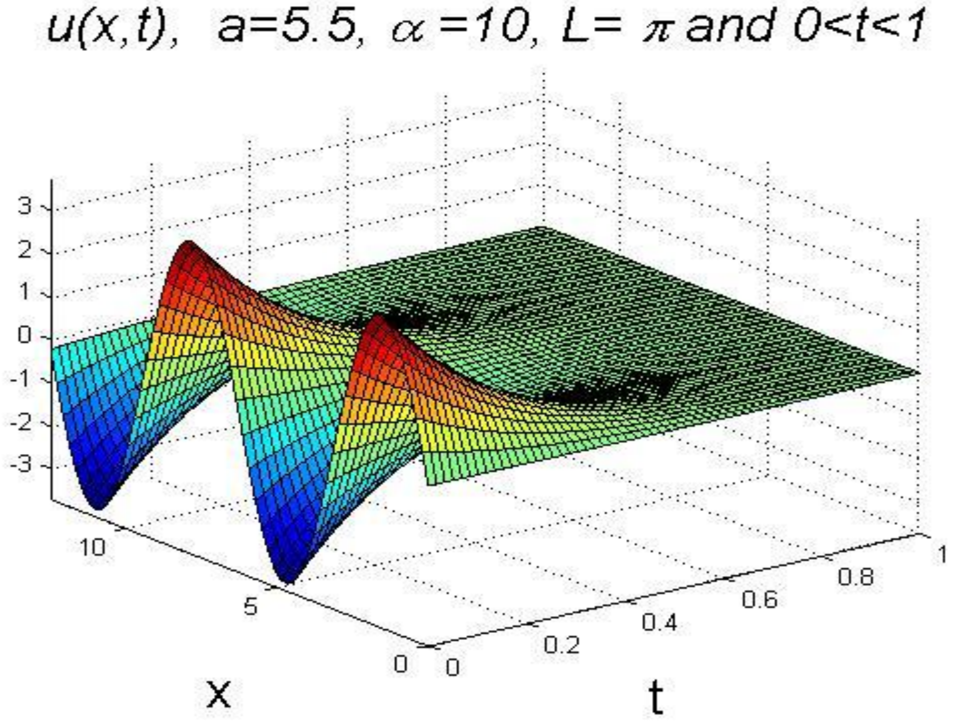

Figure 1: The two-dimensional plot of the vibration of a string u(x, t) over 2 period at various times 0<t<1 with strong damped effect, α=10.

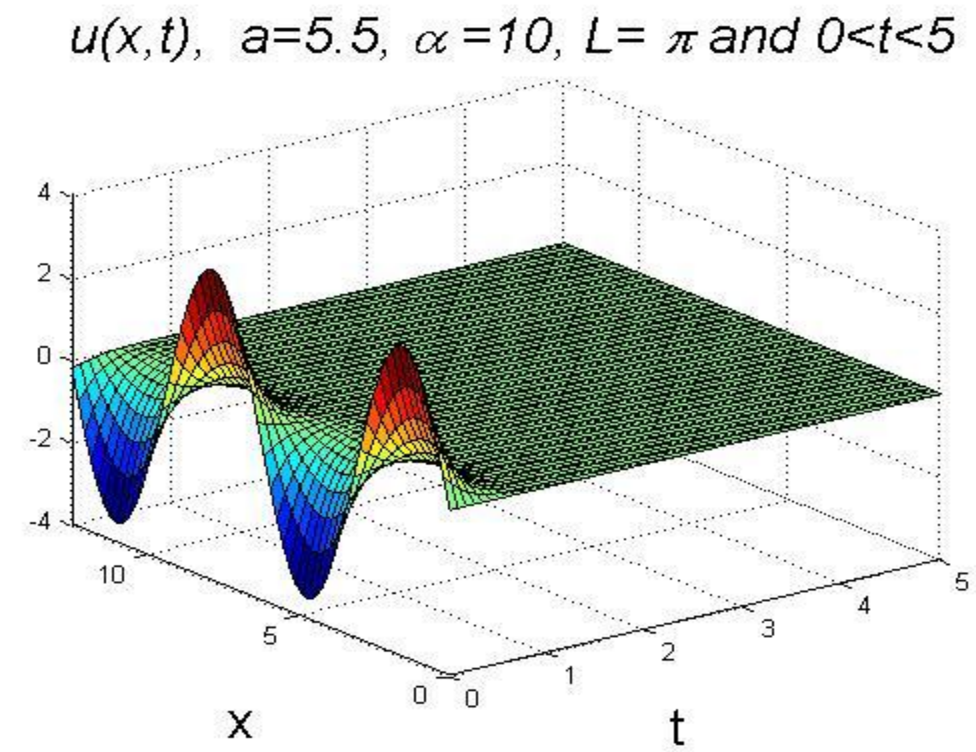

Figure 2:The two-dimensional plot of the vibration of a string u(x, t) over 2 period at various times 0<t<5 with strong damped effect, α=10.

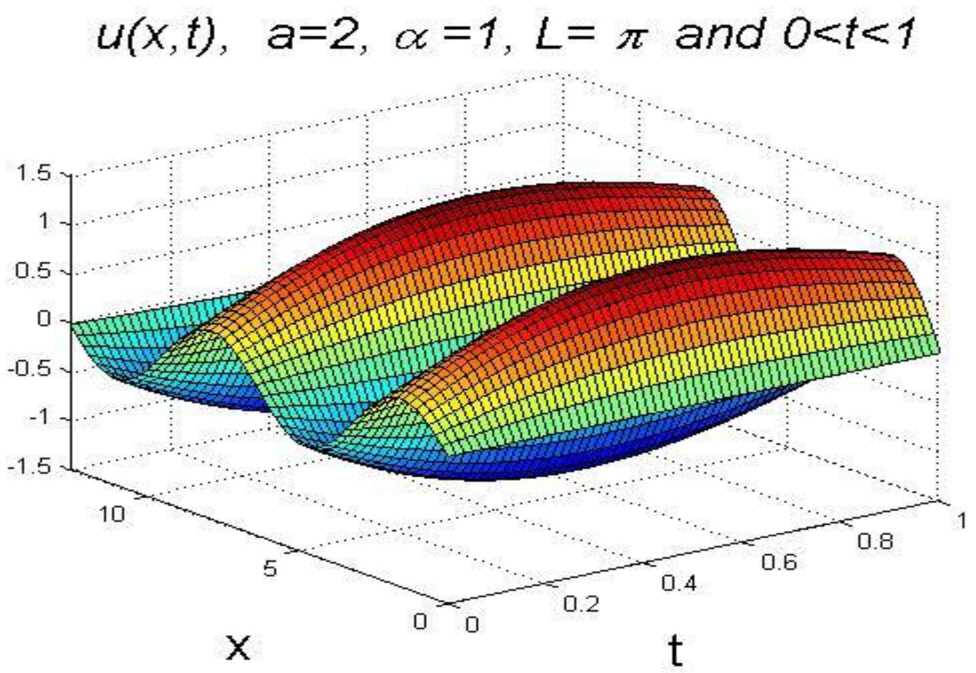

Figure 3: The two-dimensional plot of the vibration of a string u(x, t) over 2 period at various times 0<t<1 with strong damped effect, α=1.

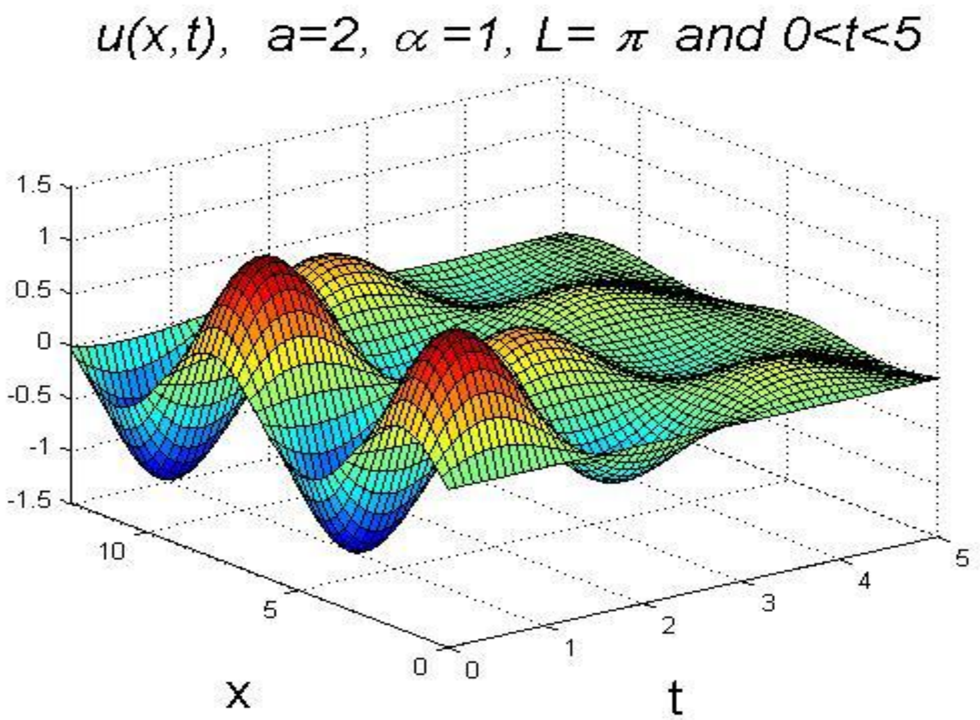

Figure 4: The two-dimensional plot of the vibration of a string u(x, t) over 2 period at various times 0<t<5 with strong damped effect, α=1.

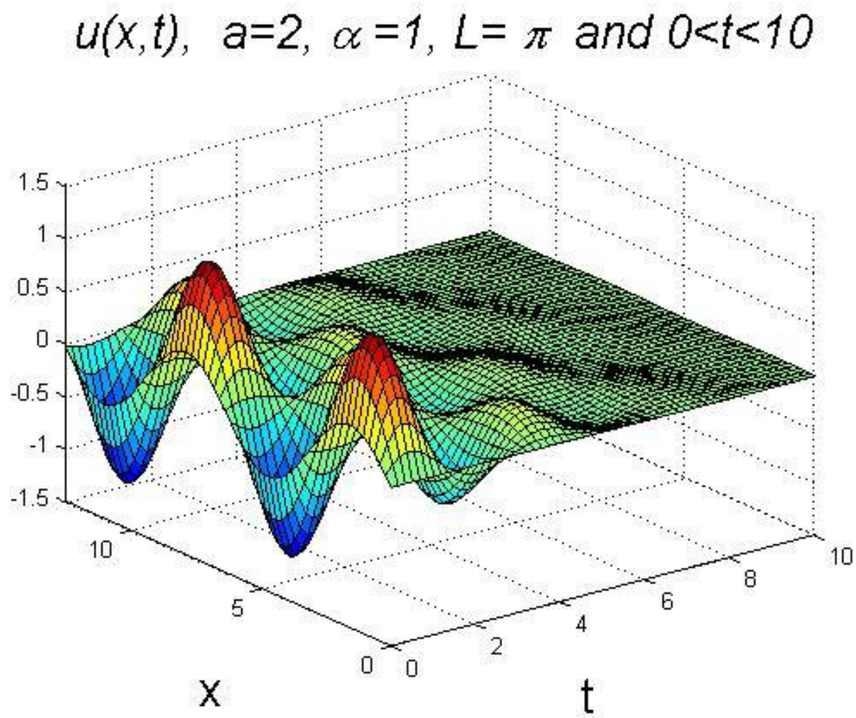

Figure 5: The two-dimensional plot of the vibration of a string u(x, t) over 2 period at various times 0<t<10 with strong damped effect, α=1.

RESULTS AND DISSCUSIONS:

The most typical results are presented graphically in the Figs 1 to 5. It can be seen that for values of the damping parameter $\alpha = 1$ (weak damping) and $\alpha = 10$ (strong damping) all solutions will tend to zero as time t tends to infinity. The figures 1 and 2 show that with the big value of damping parameter $\alpha = 10$ the string took the earlier time to oscillate, after that the string continues to vibrate in microscopic scale (unnoticeable) until it stops.

Whereas the figures 3, 4 and 5 show that with the small value of damping parameter $\alpha = 1$ the string continues to vibrate up to around 10 second with weak damping. As the results from this study, the damping took the earlier time to oscillate as the number of the non-negative damping coefficient is increased.

So the damping α can be used effectively to suppress the oscillation-amplitudes.

CONCLUSION

Throughout this paper, we have presented the analytical solution of the one-dimensional wave equation where the string is subject to a damping force using Laplace transform method. We solve partial differential equation and interpret the solution as representing a damped wave. The solutions show a good agreement with other techniques. The results show that method is powerful mathematical tools, very effective, convenient and well suited for study and solving the physics and engineering problems and can be easily extended to other nonlinear oscillations and it can be widely applicable in engineering.

Furthermore, it can be concluded that applying MatLab to plot the solutions of damped wave equation in one dimensions is very useful.

We show that with the big value of damping parameter $\alpha = 10$ the string took the earlier time to oscillate, after that the string continues to vibrate in microscopic scale (unnoticeable) until it stops, whereas with the small value of damping parameter $\alpha = 1$, the string continues to vibrate up to around 10 second with weak damping. As the results from this study, the damping took the earlier time to oscillate as the number of the non-negative damping coefficient is increase and can be used the damping α , effectively to suppress the oscillation-amplitudes.

REFERENCES

1. Donald C. Armstead and Michael A. Karls, (2006). " Does the Wave Equation Really Work?", *PRIMUS*, Vol. XVI, 2

2. Dean G. Duffy , , (2016), *"Advanced engineering mathematics with MATLAB"* Chapman & Hall_CRC).

3. Arfken, George B., Hans J. Weber and Frank E. Harris. (2013). *"Mathematical methods for physicists."* 7th edition, Academic Press is an imprint of Elsevier

4. Schiff J.L. (1999) *"The Laplace transform, theory and applications"*, Springer-verlag New York,.

5. Yener N. (2011) "A Simple Solution for the Damped Wave Equation with a Special Class of Boundary Conditions Using the Laplace Transform", *Progress In Electromagnetics Research B*, Vol. 33, 69-82, 2011.

6. Dyke. P.P.G., (2014) *"An Introduction to Laplace Transforms and Fourier Series* " , Springer.